The fundamental properties of quasi-semigroups

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ABSTRACT. In this paper we focus on the abstract Cauchy problems of the time-dependent evolution equation $\dot{x} = A(t)x(t)$. If the operator $A(t) = A$ is time-independent, we can use the C_0-semigroups theory to solve the abstract Cauchy problem. In this case A is a infinitesimal generator of the C_0-semigroups. However, if $A(t)$ is time-dependent, we can not apply directly the C_0-semigroups theory to solve the problem. In this situation we can use the quasi semigroups theory as development of the two parameters semigroups. This semigroups is induced by bounded evolution operators $U(t,s)$ that satisfy some assumptions. In this paper we determine the fundamental properties of the quasi semigroups included its generator related to the time-dependent evolution equation.

1. Introduction

Let consider the time-independent abstract Cauchy problems

$$\dot{x}(t) = Ax(t), \quad t \geq 0 \tag{1}$$

on a Banach space X and A is a densely defined operator on $\mathcal{D}(A) \subseteq X$. Fattorini [6] gave necessary and sufficient conditions in order to (1) be well-posed in $t \geq 0$. Especially, if A is the generator of C_0-semigroup of bounded linear operators on X, then the semigroups theory is a powerful tool for solving (1), (see [3], [5], [15], and [17]). We can get the comprehensively properties of A and its application therein.

Next, we consider the time-dependent abstract Cauchy problems [7]

$$\dot{x}(t) = A(t)x(t) + f(t), \quad 0 \leq t \leq T \tag{2}$$

and the associated homogeneous equation

$$\dot{x}(t) = A(t)x(t). \tag{3}$$

Here x is an unknown function from the real interval $[0,T]$ into a Banach space X, f is a given function from $[0,T]$ into X, and $A(t)$ is a given, closed, linear operator in X with domain $\mathcal{D}(A(t)) = \mathcal{D}$, independent of t and dense in X. The higher order of parabolic type of (2) also was investigated by Obrecht [14].

The solution of (2) formally given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)ds, \quad x_0 = x(0) \tag{4}$$

where $U(t,s)$ is a linear operator on X depending on s and t, with $s \leq t$, [9]. The main problem is to find some sufficient conditions for the existence of $U(t,s)$.

If $A(t) = A$ is independent of t, then $U(t,s)$ is given formally by $U(t,s) = e^{(t-s)A}$. In this case A is an infinitesimal generator of C_0-semigroup of bounded linear operators on X, and equation (2) and (3) admit a unique solution respectively. These suggest to generalize the results as $A(t)$ depend on t. It is a natural to take over some assumption on the infinitesimal generator A for the $A(t)$ for each t. However, Ladas and Lashmikantham [10], Masuda [12] and Fattorini [6] gave the assumptions for $A(t)$ in order to each equation (2) and (3) admits a unique solution.

The family of operators $U(t,s)$ is also called evolution operator, propagator, solution operator or Green's function associated to (2) or (3). Let $B(X)$ be the set of all bounded linear operators from X into X. Then each operator $U(t,s)$ belong to $B(X)$ for all $s,t \geq 0$, and satisfies the following conditions:

(a) $U(r,r) = I$, identity of $B(X)$,
(b) $U(t,r)U(r,s) = U(t,s)$, $0 \leq s \leq r \leq t$,
(c) $U(\cdot\,,\,\cdot)$ is strongly continuous,

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(d) The operator $\frac{\partial U(t,r)}{\partial t}x$ exists, and continuous for each $x \in X$.
For detail properties of the evolution operator see [2], [3], [5], [6], [11] and [15].

The appearance of this the family $U(t,s)$ that urges the new semigroups theory of two parameters, is called quasi-semigroups. Leiva and Barcenas [11] have introduced the quasi-semigroups $R(t,s),\ s,t \geq 0$. They prove several properties of $R(t,s)$ and its generator. They also give the certain conditions in order to (2) has a unique solution. Furthermore, they can construct the necessary and sufficient conditions for exact and approximate controllability for the non-autonomous control systems. The sequel works about dual quasi-semigroups and controllability also was done by Barcenas *et.al* [1].

Megan and Cuc [13] study some stability concepts for linear systems the evolution which can be described by C_0-quasi-semigroup. The results obtained may be regarded as generalizations of well known results of Datko, Pazy, Littman and Neerven about exponential stability of C_0-semigroups. Moreover, the C-quasi-semigroups as generalization of C-semigroups also can be constructed [8]. Therein, some examples are given and the properties of C-quasi-semigroups are verified. Also some applications of C-quasi-semigroups in abstract evolution equations is considered.

In this paper we shall investigate the properties of C_0-quasi-semigroups which have not yet discussed in Leiva and Bercenas [11] inline to the properties of C-semigroups. Especially, we consider the Hille-Yosida theorem version in C_0-quasi-semigroups. The finally, we shall consider some applications of C_0-quasi-semigroups for solving the evolution equations that are described by PDE's.

2. Quasi-semigroups

The theory of quasi-semigroups of bounded linear operators, as a generalization of semigroups of operators, was introduced by Leiva and Barcenas [11].

Definition 2.1. Let X be a Banach space. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0} \subseteq B(X)$ is called a strongly continuous quasi-semigroup (or C_0-quasi-semigroup) of operators if for every $r,s,t \geq 0$ and $x \in X$:
$(q1)$ $R(t,0) = I$, the identity operator on X,
$(q2)$ $R(t,s+r) = R(t+r,s)R(t,r)$,
$(q3)$ $\lim_{s\to 0}\|R(t,s)x - x\| = 0$,
$(q4)$ there exists a continuous increasing mapping $M : [0,\infty) \to [0,\infty)$ such that
$$\|R(t,s)\| \leq M(t+s).$$

For a C_0-quasi-semigroup $\{R(t,s)\}_{s,t\geq 0}$ on a Banach space X, let $\mathcal{D}$ be the set of all $x \in X$ for which the following limits exist
$$\lim_{s\to 0^+}\frac{R(0,s)x - x}{s} \text{ and } \lim_{s\to 0^+}\frac{R(t,s)x - x}{s} = \lim_{s\to 0^+}\frac{R(t-s,s)x - x}{s}, t > 0.$$
For $t \geq 0$ we define an operator $A(t)$ on $\mathcal{D}$ as
$$A(t)x = \lim_{s\to 0^+}\frac{R(t,s)x - x}{s}.$$
The family $\{A(t)\}_{t\geq 0}$ is called infinitesimal generator of the C_0-quasi-semigroups $\{R(t,s)\}_{s,t\geq 0}$.

Next, we say generator in short instead of infinitesimal generator. Throughout this paper we denote $T(t)$ and $R(t,s)$ as semigroups $\{T(t)\}_{t\geq 0}$ and quasi-semigroups $\{R(t,s)\}_{s,t\geq 0}$, respectively. We also denote $\mathcal{D}$ as domain of $A(t),\ t \geq 0$. We give some useful examples of C_0-quasi-semigroups that are summarized from [8] and [11].

Example 2.2 *If $T(t)$ is a C_0-semigroup on a Banach space X with its generator A, then $R(t,s)$, with*
$$R(t,s) = T(s), \quad s,t \geq 0,$$
defines a C_0-quasi-semigroup on X with its generator $A(t) = A,\ t \geq 0$, and $\mathcal{D} = \mathcal{D}(A)$.

Example 2.3 *Let X be a Banach space of all bounded uniformly continuous functions on $[0,\infty)$ with the supremum-norm. The family of operators $R(t,s)$ in $B(X)$ defined by*
$$(R(t,s)x)(\xi) = x(s^2 + 2ts + \xi), \quad s,t \geq 0$$

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is a C_0-quasi-semigroup on X with generator $A(t) = 2tx'$, $t \geq 0$, where $\mathcal{D} = \{x \in X : x' \in X\}$.

We can verify easily the conditions $(q1)$ – $(q4)$ of the Definition 2.1. For $t \geq 0$ fixed, and $x \in \mathcal{D}$ we have

$$\left(\frac{R(t,s)x - x}{s}\right)(\xi) = \frac{x(s^2 + 2ts + \xi) - x(\xi)}{s} = \frac{f(s) - f(0)}{s}, s > 0,$$

where $f(s) = x(s^2 + 2ts + \xi)$, for some $\xi > 0$. Consequently,

$$(A(t)x)(\xi) = \lim_{s \to 0^+} \left(\frac{R(t,s)x - x}{s}\right)(\xi) = f'(0) = 2tx'(\xi).$$

So, $R(t,s)$ is a C_0-quasi-semigroup on X with its generator $A(t)x = 2tx'$ and $\mathcal{D} = \{x \in X : x' \in X\}$. Here x' denotes the derivative of x respect to ξ.

Example 2.4 *If $T(t)$ be a C_0-semigroup on a Banach space X with its generator A, then $R(t,s)$ with*

$$R(t,s) = e^{T(t+s) - T(t)}, \quad s,t \geq 0,$$

is a C_0-quasi-semigroup on X with its generator $A(t) = AT(t)$, $t \geq 0$, and $\mathcal{D} = \mathcal{D}(A)$.

In fact, for $x \in \mathcal{D}$ and $T(t)$ is a C_0-semigroup we have

$$\begin{aligned} A(t)x = \lim_{s \to 0^+} \frac{R(t,s)x - x}{s} &= \lim_{s \to 0^+} \left(\frac{e^{T(t+s)-T(t)} - I}{s}\right) x \\ &= \frac{d}{ds}[T(t)]T(s)e^{T(s+t)-T(s)}\Big|_{s=0} x \\ &= AT(t)T(s)|_{s=0} x = AT(t)x. \end{aligned}$$

Example 2.5 *Let $T(t)$ be a C_0-semigroup on a Banach space X with its generator A. For $s,t \geq 0$,*

$$R(t,s) = T(g(t+s) - g(t)),$$

where $g(t) = \int_0^t a(u)\, du$ and $a \in C[0,\infty)$ with $a(t) > 0$. The $R(t,s)$ is a C_0-quasi- semigroup on X with its generator $A(t) = a(t)A$.

For each $r,s,t \geq 0$ and $x \in X$ we have

$(q1)$ $R(t,0) = T(g(t) - g(t)) = I$,

$(q2)$
$$\begin{aligned} R(t,s+r) &= T\big(g(t+s+r) - g(t)\big) \\ &= T\big(g(t+s+r) - g(t+r) + g(t+r) - g(t)\big) \\ &= T\big(g(t+s+r) - g(t+r)\big)T\big(g(t+r) - g(t)\big) \\ &= R(t+r,s)R(t,r), \end{aligned}$$

$(q3)$ $\lim_{s \to 0^+} \|R(t,s)x - x\| = \lim_{s \to 0^+} \big\|T\big(g(t+s) - g(t)\big)x - x\big\| = 0$, since g is continuous.

$(q4)$ Since T is strongly continuous on X, there exists $\omega, M_\omega > 0$ such that $\|T(s)\| \leq M_\omega e^{\omega s}$. Therefore, $\|R(t,s)\| = \|T(g(t+s) - g(t))\| \leq M(t+s)$, where $M(t+s) = M_\omega e^{\omega(g(t+s)-g(t))}$.

Moreover,

$$\begin{aligned} A(t)x = \lim_{s \to 0^+} \frac{R(t,s)x - x}{s} &= \lim_{s \to 0^+} \left(\frac{T(g(t+s) - g(t)) - I}{s}\right) x \\ &= g'(t+s)\frac{d}{ds}\left[T\big(g(t+s) - g(t)\big)x\right]\Big|_{s=0} = a(t)Ax. \end{aligned}$$

Thus, $R(t,s)$ is a C_0-quasi-semigroup on X with generator $A(t) = a(t)A$ and $\mathcal{D} = \mathcal{D}(A)$.

3. Main Results

In the following results we use in outline of results of [3], and some results because of reformulation from [1], [8], and [11].

Theorem 3.1 *If $R(t,s)$ is a C_0-quasi-semigroup on a Banach space X, then*

(a) *For each $t \geq 0$, $R(t,.)$ is strongly continuous on $[0,\infty)$;*

(b) *For each $t \geq 0$ and $x \in X$ hold*

$$\lim_{s \to 0^+} \frac{1}{s}\int_0^s R(t,u)\, x\, du = x.$$

Proof. (a) For $s, t \geq 0$ fixed, by commutativeness and boundedness of $R(t,s)$, then for $r \geq 0$ and $x \in X$ we have

$$\begin{aligned}\|R(t,s+r)x - R(t,s)x\| &= \|R(t+s,r)R(t,s)x - R(t,s)x\| \\ &\leq \|R(t,s)\|\|R(t+s,r)x - x\|.\end{aligned}$$

By (q3) we conclude that

$$\lim_{r\to 0^+} \|R(t,s+r)x - R(t,s)x\| = 0,$$

i.e $R(t,.)$ is strongly continuous on $[0,\infty)$.

(b) Let $x \in X$, $t \geq 0$, and $\varepsilon > 0$ be given. By strongly continuity of $R(t,.)$ there exists $\tau > 0$ such that for $s \in [0,\tau]$ we have

$$\|R(t,s)x - x\| < \varepsilon.$$

Therefore, for $s \in (0,\tau]$

$$\begin{aligned}\left\|\frac{1}{s}\int_0^s R(s,u)\, x\, du - x\right\| &= \left\|\frac{1}{s}\int_0^s [R(t,u)\, x - x]\, du\right\| \\ &\leq \frac{1}{s}\int_0^s \|R(t,u)x - x\|\; du < \frac{1}{s}\int_0^s \varepsilon\, du = \varepsilon.\end{aligned}$$

□

Theorem 3.2 *If $R(t,s)$ is a C_0-quasi-semigroup on a Banach spce X with its generator $A(t)$, then*

(a) *If $x \in \mathcal{D}$, $t \geq 0$, and $s_0, t_0 \geq 0$, then $R(t_0,s_0)x \in \mathcal{D}$ and*

$$R(t_0,s_0)A(t)x = A(t)R(t_0,s_0)x;$$

(b) *For each $s > 0$,*

$$\frac{\partial}{\partial s}(R(t,s)x) = A(t+s)R(t,s)x = R(t,s)A(t+s)x, \qquad x \in \mathcal{D};$$

(c) *If $A(\cdot)$ is locally integrable, then for every $x \in \mathcal{D}$ and $s \geq 0$*

$$R(t,s)x = x + \int_0^s A(t+u)R(t,u)\, x\, du;$$

(d) *If $f : [0,\infty) \to X$ is a continuous, then for every $t \in [0,\infty)$*

$$\lim_{h\to 0}\frac{1}{h}\int_s^{s+h} R(t,u)f(u)du = R(t,s)f(s).$$

Proof. (a) For $x \in \mathcal{D}$ states that

$$\lim_{s\to 0^+}\frac{R(t,s)x - x}{s} = A(t)x.$$

By strongly continuity of $R(t_0,s_0)$,

$$\begin{aligned}A(t)R(t_0,s_0)x &= \lim_{s\to 0^+}\frac{R(t,s)R(t_0,s_0)x - R(t_0,s_0)x}{s} \\ &= R(t_0,s_0)\lim_{s\to 0^+}\frac{R(t,s)x - x}{s} = R(t_0,s_0)\, A(t)x.\end{aligned}$$

So, $R(t_0,s_0)x \in \mathcal{D}$, and $R(t_0,s_0)A(t)x = A(t)R(t_0,s_0)x$.

(b) For $r > 0$, the commutativeness of $R(t,s)$ gives

$$\frac{R(t,s+r)x - R(t,s)x}{r} = R(t,s)\frac{(R(t+s,r) - I)x}{r}.$$

If $x \in \mathcal{D}$, then for $r \to 0^+$ the limit in the right hand exists. This implies the existence of limit in the left hand, i.e

$$\lim_{r\to 0^+}\frac{R(t,s+r)x - R(t,s)x}{r} = R(t,s)A(t+s)x. \tag{5}$$

For $r < 0$ such that $s + r \geq 0$,

$$\begin{aligned}\frac{R(t,s+r)x - R(t,s)x}{r} &= \frac{R(t,s)x - R(t,s+r)x}{-r} \\ &= \frac{R(t+s+r,-r)R(t,s+r)x - R(t,s+r)x}{-r} \\ &= R(t,s+r)\frac{R(t+s+r,-r)x - x}{-r} \\ &= R(t,s+r)\frac{R(t+s+r,-r)x - R(t+s,-r)x + R(t+s,-r)x - x}{-r}.\end{aligned}$$

The strongly continuity of $R(t,s)$ implies
$$\lim_{r\to 0^-} R(t+s+r,-r)x - R(t+s,-r)x = 0.$$
Therefore,
$$\lim_{r\to 0^-} \frac{R(t+s+r,-r)x - x}{-r} = \lim_{r\to 0^-} \frac{R(t+s,-r)x - x}{-r} = A(t+s)x.$$
Again, the strongly continuity of $R(t,s)$ implies
$$\lim_{r\to 0^-} \frac{R(t,s+r)x - R(t,s)x}{r} = R(t,s)A(t+s)x. \tag{6}$$
From (5) and (6) we have
$$\frac{\partial}{\partial s}(R(t,s)x) = A(t+s)R(t,s)x = R(t,s)A(t+s)x.$$

(c) By integrating the last equation from 0 to s, we have
$$R(t,s)x = x + \int_0^s R(t,u)A(t+u)x du.$$

(d) For $t > 0$, by continuity of f and the strongly continuity of $R(t,.)$ on $[0,\infty)$, then $y(.) = R(t,.)f(.)$ is continuous in s. If we define
$$F(\xi) = \int_s^{s+\xi} y(u)\ du,$$
then
$$F'(0) = \lim_{h\to 0} \frac{1}{h}\int_s^{s+h} y(u)du = \lim_{h\to 0} \frac{1}{h}\int_s^{s+h} R(t,u)f(u)du. \tag{7}$$
On the other hand, since
$$F'(\xi) = y(s+\xi)$$
then
$$F'(0) = y(s) = R(t,s)f(s). \tag{8}$$
From (7) and (8) we have
$$\lim_{h\to 0} \frac{1}{h}\int_s^{s+h} R(t,u)f(u)du = R(t,s)f(s).$$
□

In the semigroups theory, if A is an infinitesimal generator of C_0-semigroup with domain $\mathcal{D}$, then A is a closed operator and $\mathcal{D}$ is dense in X. These are is not always true for any C_0-quasi-semigroups.

We return to the C_0-quasi-semigroup $R(t,s)$ in Example 2.3, with the generator $A(t)x = 2tx'$ and its domain $\mathcal{D} = \{x \in X : x' \in X\}$. We can show that for every $t > 0$, $A(t)$ is not closed. In fact, we can choose a sequence (x_n) in $\mathcal{D}$ with $x_n(\xi) = e^{-n\xi}/n$, for $\xi \geq 0$ and $n \in \mathbb{N}$. Obvious that
$$x_n \to x \quad \text{where} \quad x(\xi) = 0 \ \text{ for all } \xi \geq 0.$$
However,
$$A(t)x_n \to y$$
where $(A(t)x_n)(\xi) = -2te^{-n\xi}, \xi \geq 0$ and $y(\xi) = \begin{cases} -2t, & \xi = 0 \\ 0, & \xi > 0 \end{cases}$. We have $A(t)x \neq y$. Therefore, $A(t)$ is not closed.

Example 3.3 *Let X be as in Example 2.3. For $s > 0$ given an initial value problem*
$$\frac{\partial x}{\partial t}(t,s) = 2(t+s)\frac{\partial x}{\partial s}(t,s)$$
$$x(0,s) = 0.$$

This initial value problem can be rewritten as
$$\dot{x}(t) = A(s+t)x(t)$$
$$x(0) = 0,$$
where $A(t)x = 2tx'$. By Theorem 3.2 (c), for each $x_0 \in \mathcal{D}$ the initial value problem has a solution $x(t) = R(s,t)x_0$ where $R(t,s)$ is a C_0-quasi-semigroup with generator $A(t)$ that satisfies $A(t)x(\xi) = 2tx'(\xi)$ on its domain
$$\mathcal{D} = \{x \in X : x' \in X, x(0) = 0\}.$$

The set $\mathcal{D}$ is not dense in X, since $\overline{\mathcal{D}} = \{x \in X \,:\, x(0) = 0\} \neq X$, [16]. This result also concludes that for $t \geq 0$ the operator $A(t)$ is not a generator for any C_0-semigroup.

□

The two examples previously state that the operator $A(t)$, $t \geq 0$, can be a generator for some C_0-quasi-semigroup but it must not satisfy the the necessary condition of the Hille-Yosida theorem [3]. Next, we define the resolvent operator of $A(t)$ similar to one in operator theory. For each $t \geq 0$ we define the resolvent operator of $A(t)$ as

$$\mathcal{R}_\lambda = \mathcal{R}\big(\lambda, A(t)\big) = (\lambda I - A(t))^{-1},$$

with its resolvent set $\rho(A(t))$.

Throughout this paper, we put S as a set of all complex numbers λ such that $-\theta \leq \arg\lambda \leq \theta$ with $\pi/2 < \theta < \pi$, see [10].

Theorem 3.4 *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $R(t,s)$ on a Banach space X and resolvent $\mathcal{R}\big(\lambda, A(t)\big)$ exists in S. If $\lambda \in \rho(A(t))$, then $\mathcal{R}\big(\lambda, A(t)\big)R(t,s) = R(t,s)\mathcal{R}\big(\lambda, A(t)\big)$ for all $s \geq 0$.*

Proof. For $\lambda \in \rho(A(t))$ and $x \in X$, we set $y = \mathcal{R}\big(\lambda, A(t)\big)x$. Since $A(t)^{-1}$ exists, then we can write y as

$$y = \mathcal{R}\big(\lambda, A(t)\big)x = A(t)^{-1}(\lambda A(t)^{-1} - I)^{-1}x \quad \text{or} \quad A(t)y = (\lambda A(t)^{-1} - I)^{-1}x,$$

i.e $y \in \mathcal{D}$. From Theorem 3.2 (a) for $s \geq 0$ we have

$$R(t,s)\big(\lambda - A(t)\big)y = \big(\lambda I - A(t)\big)R(t,s)y$$

By applying $\mathcal{R}\big(\lambda, A(t)\big)$ to the both sides, we conclude that $\mathcal{R}\big(\lambda, A(t)\big)R(t,s)x = R(t,s)\mathcal{R}\big(\lambda, A(t)\big)x$ as asserted.

□

Theorem 3.5 *If $A(t)$ is a generator of a C_0-quasi-semigroup $R(t,s)$ on a Banach space X, then for $\lambda \in \mathbb{C}$, $A(t) - \lambda I$ is a generator of C_0-quasi-semigroup*

$$K(t,s) = e^{-\lambda s}R(t,s) \text{ for all } s,t \geq 0.$$

Proof. It is easy to show that $K(t,s)$ is a C_0-quasi-semigroup. Furthermore, for $t \geq 0$ and $x \in \mathcal{D}$,

$$\begin{aligned}\lim_{s\to 0^+} \frac{K(t,s) - I}{s}x &= \lim_{s\to 0^+} \frac{e^{-\lambda s}R(t,s) - I}{s}x \\ &= e^{-\lambda s}\left[-\lambda R(t,s) + \frac{\partial}{\partial s}R(t,s)\right]\Big|_{s=0} \\ &= e^{-\lambda s}[-\lambda R(t,s) + A(t+s)R(t,s)]\big|_{s=0} \\ &= A(t) - \lambda I.\end{aligned}$$

So, $A(t) - \lambda I$ is the generator of a C_0-quasi-semigroup $K(t,s)$.

□

The following we shall analyze the necessary and sufficient conditions of the Hille-Yosida theorem in order to an operator generates a quasi-semigroup. To achieve this aim we introduce the so called Yosida approximation, (see [3], [5], [17], and [18]). For $\lambda \in \rho(A)$ we set

$$A_\lambda = \lambda A\mathcal{R}(\lambda, A) = \lambda^2\mathcal{R}(\lambda, A) - \lambda I \in \mathcal{B}(X). \qquad (9)$$

We note that

$$A\mathcal{R}(\lambda, A) = \lambda\mathcal{R}(\lambda, A) - I \quad \text{and} \quad A\mathcal{R}(\lambda, A)x = \mathcal{R}(\lambda, A)Ax \qquad (10)$$

hold for all $x \in \mathcal{D}(A)$.

Lemma 3.6 *Let $A(t)$ be a closed, densely defined and $M, \omega \geq 0$ such that $[\omega, \infty) \subseteq \rho(A(t))$ and*

$$\big\|\mathcal{R}\big(\lambda, A(t)\big)\big\| \leq \frac{M}{\lambda}$$

for all $\lambda \geq \omega$. Then $\lambda\mathcal{R}\big(\lambda, A(t)\big)x \to x$ as $\lambda \to \infty$ for all $x \in X$ and $\lambda\mathcal{R}\big(\lambda, A(t)\big)y \to A(t)y$ as $\lambda \to \infty$ for all $y \in \mathcal{D}$.

Proof. Let $x \in \mathcal{D}$. Equation (10) and the assumptions yield that

$$\left\|\lambda\mathcal{R}\big(\lambda,A(t)\big)x-x\right\|=\left\|\mathcal{R}\big(\lambda,A(t)\big)A(t)x\right\|\le\frac{M}{\lambda}\|A(t)x\|\to 0$$
as $\lambda\to\infty$. Because $\lambda\mathcal{R}\big(\lambda,A(t)\big)$ is uniformly bounded, by M, the first assertion follows. The second one is a consequence of the first assertion, taking $x=A(t)y$ and using (10).

□

Theorem 3.7 *If $A(t)$ is a closed, densely defined on $\mathcal{D}$ and $M,\omega\ge 0$ such that $[\omega,\infty)\subseteq\rho(A(t))$ and*
$$\left\|\mathcal{R}\big(\lambda,A(t)\big)^r\right\|\le\frac{M}{(\lambda-\omega)^r},\qquad r\ge 1,$$
for all $\lambda>\omega$, then $A(t)$ generates a C_0-quasi-semigroup $R(t,s)$. Moreover, $\|R(t,s)\|\le Me^{\omega s}$ for all $s,t\ge 0$.

Proof. By using Yosida approximation we set
$$A_n(t)\coloneqq nA(t)\mathcal{R}(n,A(t))=n^2\mathcal{R}(n,A(t))-nI$$
for all $n>\omega$, $n\in\mathbb{N}$. Lemma 3.6 and assumptions imply that
$$\lim_{n\to\infty}A_n(t)y=A(t)y \text{ for all } x\in\mathcal{D}.$$
Let $s\ge 0$. We define
$$R_n(s)\coloneqq e^{sA_n(t)}=e^{-sn}\sum_{k=0}^{\infty}\frac{(n^2s)^k}{k!}\mathcal{R}(n,A(t))^k.$$
Hence
$$\|R_n(s)\|\le e^{-sn}\sum_{k=0}^{\infty}\frac{(n^2s)^k}{k!}\frac{M}{(n-\omega)^k}$$
$$=Me^{-ns}e^{\left(\frac{n^2}{n-\omega}\right)s}=Me^{\left(\frac{n\omega}{n-\omega}\right)s}. \tag{11}$$
Next, take $m,n\in\mathbb{N}$, $s_0>0$, $y\in\mathcal{D}$ and $s\in[0,s_0]$. Obvious, it holds that $A_m(t)A_n(t)=A_n(t)A_m(t)$ and
$$A_n(t)e^{sA_m(t)}=A_n(t)\sum_{k=0}^{\infty}\frac{s^k}{k!}A_m^k(t)=\sum_{k=0}^{\infty}\frac{s^k}{k!}A_m^k(t)A_n(t)=e^{sA_m(t)}A_n(t).$$
Consequently, we can compute
$$e^{sA_n(t)}y-e^{sA_m(t)}y=\int_0^s\frac{d}{ds}e^{(s-u)A_m(t)}\,e^{uA_n(t)}y\,du=\int_0^s e^{(s-u)A_m(t)}e^{uA_n(t)}\,(A_n(t)-A_m(t))y\,du.$$
Equation (7) implies that
$$\|R_n(s)y-R_m(s)y\|=\left\|e^{sA_n(t)}y-e^{sA_m(t)}y\right\|\le s_0\|A_n(t)y-A_m(t)y\|\to 0\quad\text{as}\quad m,n\to\infty. \tag{12}$$
Therefore, $(R_n(s)y)$ is a Cauchy sequence, and so it converges in $\mathcal{D}$. Furthermore, we have that $\big(R_n(s)\big)$ is equicontinuous on X. Since $\mathcal{D}$ is dense in X, then we can extend this convergence to every $x\in X$, (see Lemma 3.8 of [16]). Hence, we may define
$$R(t,s)x\coloneqq\lim_{n\to\infty}R_n(s)x$$
for all $s,t\ge 0$ and $x\in X$. In facts, $R(t,0)=I$ and
$$R(t,s+r)x=\lim_{n\to\infty}e^{(s+r)A_n}=\lim_{n\to\infty}e^{sA_n}.\lim_{n\to\infty}e^{rA_n}=R(t+r,s)R(t,r)x$$
for all $r,s,t\ge 0$. By letting $m\to\infty$ in (12), we deduce that
$$\|R_n(s)y-R(t,s)y\|\le s_0\|A_n(t)y-A(t)y\|$$
for all $s\in[0,s_0]$. This states that $(R_n(s)y)$ uniformly converges to $R(t,s)y$ on $[0,s_0]$. Therefore, $R(t,\cdot)y$ is continuous for all $y\in\mathcal{D}$. Finally, by (11) there exists continuous increasing function M_1 on $[0,\infty)$ such that
$$\|R(t,s)\|\le M_1(t+s)$$
where $M_1(t+s)=Me^{\omega s}$. Thus $R(t,s)$ is a C_0-quasi-semigroup.

Let $B(t)$ be the generator of $R(t,s)$. For $s>0$ and $y\in\mathcal{D}$, we have
$$\frac{1}{s}(R(t,s)y-y)=\lim_{n\to\infty}\frac{1}{s}\left(e^{sA_n(t)}y-y\right)=\lim_{n\to\infty}\frac{1}{s}\int_0^s e^{uA_n(t)}A_ny\ du=\frac{1}{s}\int_0^s e^{uA(t)}A(t)y\ du.$$
By letting $s\to 0^+$, it follows that $y\in\mathcal{D}(B(t))$ and $B(t)y=A(t)y$.
Now, if $\lambda>\omega$, then by proof of Theorem 3.4 we have
$$(\lambda I-A(t))\mathcal{D}=X.$$
Since we assume that $B(t)$ is a generator, then Theorem 3.2 (a) gives
$$\big(\lambda I-B(t)\big)\mathcal{D}\big(B(t)\big)=X.$$
However, $A(t)\mathcal{D}=B(t)\mathcal{D}$, and hence
$$(\lambda I-B(t))\mathcal{D}=\big(\lambda I-B(t)\big)\mathcal{D}\big(B(t)\big).$$

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Thus, $\mathcal{D} = \mathcal{D}(B(t))$, and this proves that $B(t) = A(t)$.

□

We call Theorem 3.7 as Hille-Yosida theorem in quasi-semigroups version. In fact, this is just the sufficient condition of Hille-Yosida theorem. The necessary condition of the theorem is not valid for any quasi-semigroups. Example 3.3 and comments thereon explain this condition. As consequence, Theorem 3.7 gives sufficient condition that guarantees the existence and uniqueness solution of the time-independent abstract Cauchy problems (2) and (3), see [8] and [11].

Corollary 3.8 [8, 11] *If $A(t)$ is the generator of C_0-quasi-semigroup $R(t,s)$ on a Banach space X, then for each $x_0 \in \mathcal{D}$ and $r \geq 0$ the initial value problem*

$$\dot{x}(t) = A(r+t)x(t),\ x(0) = x_0, \tag{13}$$

admits a unique solution.

4. Applications

In this section we give some examples of application of C_0-quasi-semigroups to solve the partial differential equations completed by initial value or boundary conditions.

Example 4.1 *For $r \geq 0$ consider the boundary condition problem*

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\xi) &= a(r+t)\frac{\partial^2 x}{\partial \xi^2}(t,\xi), \qquad 0 < \xi < 1, \qquad t \geq 0 \\ x(t,0) &= x(t,1) = 0. \end{aligned} \tag{14}$$

where a is a continuous function with $a(t) > 0$ for $t \geq 0$.

Let X be a Hilbert space of $L_2[0,1]$ and the operator $A : \mathcal{D}(A) \to X$ given by

$$Ax = \frac{d^2x}{d\xi^2}$$

and

$$\mathcal{D}(A) = \{x \in X : x,\ x' \text{ absolutely continuous and } x'' \in X \text{ with } x(0) = x(1) = 0\}.$$

The boundary condition problem (14) can be written as

$$\dot{x}(t) = a(r+t)Ax(t),\ \ t \geq 0 \tag{15}$$

where A is a generator of C_0-semigroup $T(t)$ that given

$$T(t)x = \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle \phi_n, x\rangle x$$

where $\lambda_n = n^2\pi^2$ and $\phi_n(\xi) = \sin(n\pi\xi)$. From Example 2.5, the family of bounded operators

$$R(t,s) = T\big(g(t+s) - g(t)\big), \qquad t,s \geq 0$$

is a C_0-quasi-semigroup with the generator $A(t) = a(t)A$ and $\mathcal{D} = \mathcal{D}(A)$ which is dense in X. According to Theorem 3.2 (c) and Corollary 3.8, then for each $x_0 \in \mathcal{D}$ the problem (15) admits a unique solution

$$x(t) = R(r,t)x_0, \qquad x(0) = x_0.$$

Therefore, the initial value problem (14) has a unique solution

$$x(t,\xi) = T\big(g(t+r) - g(r)\big)\, x_0(\xi)$$

where $g(t) = \int_0^t a(s)\, ds$.

Example 4.2 *Given a boundary condition problem*

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\xi) &= \frac{1}{w(\xi)}\left(\frac{\partial}{\partial \xi}\left(p(\xi)\frac{\partial x}{\partial \xi}(t,\xi)\right) - q(\xi)x(t,\xi)\right), \qquad a < \xi < b, t \geq 0, \\ \frac{\partial x}{\partial \xi}(a,t) &= \frac{\partial x}{\partial \xi}(b,t) = 0. \end{aligned} \tag{16}$$

where $w,\ p, q, x' = \frac{\partial x}{\partial \xi}$ are continuous functions on the interval $[a,b]$ with $p(\xi) > 0$ and $w(\xi) > 0$.

Let X be a Hilbert space of $L_2[a,b]$ and A_{SL} is the Sturm-Liouville operator that defined as

$$A_{SL}x = \frac{1}{w}\left(-\frac{d}{d\xi}\left(p\frac{dx}{d\xi}\right) + qx\right)$$

on a domain

$$\mathcal{D}(A_{SL}) = \{x \in X: x, x' \text{ are absolutely continuous}, x'' \in X, x'(a) = 0, x'(b) = 0\}.$$

The boundary condition problem (16) can be written as

$$\dot{x}(t) = Ax(t), \ \ t \geq 0 \tag{17}$$

where $A = -A_{SL}$ is the generator of a C_0-semigroup $T(t)$ with $\mathcal{D}(A) = \mathcal{D}(A_{SL})$, see [3] and [4].

Example 2.2 states that the family of bounded operators

$$R(t,s) = T(s), \qquad t, s \geq 0$$

is a C_0-quasi-semigroup with its generator $A(t) = A$ and $\mathcal{D} = \mathcal{D}(A)$ which is dense in X. Furthermore, for each $x_0 \in \mathcal{D}$ (17) admits a unique solution

$$x(t) = R(s,t)x_0, \qquad x(0) = x_0.$$

Thus, the boundary condition problem (16) has a unique solution

$$x(t,\xi) = T(t)x_0(\xi).$$

5. Conclusions

In this paper we have discussed the fundamental properties of quasi-semigroups. This investigation has be done based on the existing properties of semigroups. The results show that some properties inline to the properties of semigroups, although the others do not. Especially, the Hille-Yosida theorem does not hold in C_0-quasi-semigroup except for the sufficient condition. On the final part, we have showed the powerfulness of C_0-quasi-semigroup to solve the problems related to PDE's. The next works, we shall investigate the contraction and Riesz-spectral in quasi-semigroups version.

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