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Super (a, d)-Cycle-Antimagic Total Labeling on Triangular Ladder Graph and Generalized Jahangir Graph

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Abstract. Let $G(V(G), E(G))$ be a finite simple graph with $|V(G)| = \nu_G$ and $|E(G)| = e_G$. Let H be a subgraph of G. The graph G is said to be (a, d)-H-antimagic covering if every edge in G belongs to at least one of the subgraphs G isomorphic to H and there is a bijective function $\xi : V \cup E \to \{1, 2, \ldots, \nu_G + e_G\}$ such that all subgraphs H' isomorphic to H, the H'-weights

$$w(H') = \Sigma_{v \in V(H')}\xi(v) + \Sigma_{e \in E(H')}\xi(e)$$

constitutes an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (t - 1)d\}$, where a and d are positive integers and t is the number of subgraphs G isomorphic to H. Such a labeling is called *super* if the vertices contain the smallest possible labels. This research provides super (a, d)-C_3-antimagic total labelng on triangular ladder TL_n for $n \geq 2$ and super (a, d)-C_{s+2}-antimagic total labeling on generalized Jahangir $J_{k,s}$ for $k \geq 2$ and $s \geq 2$.

1. Introduction

Wallis [13] defined a graph labeling as a mapping from the set of vertices, edges, or both vertices and edges to the positive or non-negative integers. The types of graph labeling which is still widely studied today are magic labelings and antimagic labelings. Magic labelings were introduced by Sedláček [12] in 1963. It was followed by Kotzig and Rosa [6] who developed magic labelings into an edge-magic total labeling. Furthermore, Gutiérrez and Lladó [2] generalized the concept of an edge-magic total labeling into an H-magic covering.

Let $G = (V, E)$ be a finite and simple graph. An edge-covering of G is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that every edge in G belongs to at least one of the subgraphs H_i for $1 \leq i \leq k$. If each H_i is isomorphic to H, then G admits an H-covering. A graph G is said to be an H-magic if it admits an H-covering and there is a bijective function $f : V \cup E \to \{1, 2, \ldots, \nu_G + e_G\}$ such that for any subgraph $H'(V', E')$ of G isomorphic to H, $\Sigma_{v \in V(H')}\xi(v) + \Sigma_{e \in E(H')}\xi(e) = m(f)$, where $m(f)$ is a magic sum. We call an H-supermagic if its vertices contain the smallest possible labels.

There are a large number of research problems on H-magic total labeling that have been applied on some classes of graphs. Lladó and Moragas [7] showed that a wheel graph, a windmill graph, and a book graph are cycle-magic. Ngurah et al. [10] studied cycle-supermagic covering on a triangular ladder graph, a book graph, and grids $P_m \times P_n$ for $m \geq 3$ and $n = 3, 4, 5$, then Roswitha et al. [11] proved that a generalized Jahangir graph $J_{k,s}$, a complete bipartite graph $K_{2,n}$ for $n \geq 2$, and a wheel graph W_n for n odd and $n \geq 4$, respectively, are C_{s+2}-supermagic, C_4-supermagic, and C_3-supermagic.

In 2009, Inayah et al. [3] introduced an (a, d)-H-antimagic covering. A graph G is said to be an (a, d)-H-antimagic covering if there exists a bijective function $\xi : V \cup E \to \{1, 2, \ldots, v_G + e_G\}$ such that for every subgraph H' isomorphic to H, $w(H') = \Sigma_{v \in V(H')} \xi(v) + \Sigma_{e \in E(H')} \xi(e)$ constitutes an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(t-1)d\}$ where a and d are positive integers and t is the number of subgraphs G isomorphic to H. Inayah et al. [3] studied an (a, d)-H-antimagic covering on fan graph for some d, then Inayah et al. [5] proved that shackles of a connected graph H is a super (a, d)-H-antimagic.

This research attempts to study super (a, d)-C_3-antimagic on triangular ladder graph TL_n for $n \geq 2$ and super (a, d)-C_{s+2}-antimagic on generalized Jahangir graph $J_{k,s}$ for $k \geq 2$ and $s \geq 2$.

2. Main Results

2.1. k-balanced multisets

Maryati [8] and Maryati et al. [9] defined a multiset as a set that allows the existence of the same elements in it. Let $k \in \mathbb{N}$ and Y be a multiset that contains positive integers. Y is said to be *k-balanced* if there exists k-subsets of Y, namely $Y_1, Y_2, \ldots, Y_k$, such that for every $i \in [1, k]$, $|Y_i| = \frac{|Y|}{k}$, $\sum Y_i = \frac{\sum Y}{k} \in \mathbb{N}$, and $\biguplus_{i=1}^{k} Y_i = Y$. If those are the cases for every $i \in [1, k]$, then Y_i is called a *balanced subset* of Y.

Lemma 2.1 [Roswitha et al. [11]] *Let x and k be nonnegative integers. Let $X = [x + 1, x + k]$ with $|X| = k$ and $Y = [x + k + 1, x + 2k]$ where $|Y| = k$. Then, the multiset $K = X \uplus Y$ is k-balanced for $j \in [1, k]$.*

Next, we have the following lemma.

Lemma 2.2 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $Y = [x + 1, x + \frac{k}{2}] \uplus [x + 2, x + 2k + 1] \uplus [x + \frac{3k}{2} + 2, x + 2k + 1]$, then Y is k-balanced for $i \in [1, k]$.*

Proof. For every $i \in [1, k]$ we define a multiset $Y_i = \{a_i, b_i, c_i\}$, where

$$a_i = \begin{cases} x + \lceil \frac{i}{2} \rceil, & \text{for } i \text{ odd;} \\ x + \lceil \frac{i+1}{2} \rceil, & \text{for } i \text{ even.} \end{cases}$$

$$b_i = x + \frac{3k}{2} + 2 - i$$

$$c_i = \begin{cases} x + \frac{3k}{2} + 2 + \lfloor \frac{i}{2} \rfloor, & \text{for } i \text{ odd;} \\ x + \frac{3k}{2} + 1 + \frac{i}{2}, & \text{for } i \text{ even.} \end{cases}$$

Furthermore, we defined

$$A = \{a_i \mid 1 \leq i \leq k\} = [x + 1, x + \frac{k}{2}] \uplus [x + 2, x + \frac{k}{2} + 1];$$

$$B = \{b_i \mid 1 \leq i \leq k\} = [x + \frac{k}{2} + 2, x + \frac{3k}{2} + 1];$$

$$C = \{c_i \mid 1 \leq i \leq k\} = [x + \frac{3k}{2} + 2, x + 2k + 1] \uplus [x + \frac{3k}{2} + 2, x + 2k + 1].$$

If $A \uplus B \uplus C = Y$ and $\biguplus_{i=1}^{k} Y_i = Y$, then for every $i \in [1, k]$ we obtained $|Y_i| = 3$ and $\sum Y_i = 3x + 3k + 4$. Therefore, Y is k-balanced.

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2.2. (k,δ)-anti balanced multisets

Inayah [4] defined (k,δ)-anti balanced multiset as follows. Let $k,\delta \in \mathbb{N}$ and X be a set containing the elements of positive integers. A multiset X is said to be (k,δ)-anti balanced if there exists k subsets from X, i.e. $X_1, X_2, \ldots, X_k$ such that for every $i \in [1,k]$, $|X_i| = \frac{|X|}{k}$, $\biguplus_{i=1}^{k} X_i = X$, and for $i \in [1,k-1]$, $\sum X_{i+1} - \sum X_i = \delta$ is hold. Here, we give several lemmas on (k,δ)-anti balanced multisets.

Lemma 2.3 *Let $k,s \geq 2$ be integers. If $X = [k+2, 2k+1], Y = [2, ks+1]$, and $Z = [ks+2, 2ks+1]$, then the multiset $K = (Y - X) \uplus Z$ is $(k,1)$-anti balanced.*

Proof. Let $k,s \geq 2$ be integers. We define a multiset $K_i = \{a_j^i \mid 1 \leq j \leq s\} \uplus \{b_j^i \mid 1 \leq j \leq s-1\}$ for $i \in [1,k]$, where

$$a_j^i = \begin{cases} (s+j)k + 2 - i, & \text{for } j \text{ odd;} \\ (s+j-1)k + 1 + i, & \text{for } j \text{ even.} \end{cases}$$

$$b_j^i = \begin{cases} j + i, & \text{for } j = 1; \\ jk + 1 + i, & \text{for } j \text{ even;} \\ (j+1)k + 2 - i, & \text{for } j \text{ odd and } j \geq 3. \end{cases}$$

It is obvious that for every $i \in [1,k]$, $|K_i| = 2s-1$, $K_i \subset K$, and $\biguplus_{i=1}^{k} K_i = K$. If s is odd, then the sum of elements in K_i is

$$\begin{aligned} \sum K_i =& \sum_{z=1}^{\frac{s-1}{2}}((s+2z-1)k+2-i) + \sum_{z=1}^{\frac{s-1}{2}}((s+2z-1)k+1+i) + 2ks + 2 - i + 1 + \\ & i + \sum_{z=1}^{\frac{s-1}{2}-1}((2z)k+1+i) + \sum_{z=1}^{\frac{s-1}{2}-1}(((2z+1)+1)k+2-i) + ks - k + 1 + i \\ =& 2k(s^2-1) + 3s - 2 + i. \end{aligned}$$

If s is even, then the sum of elements in K_i is

$$\begin{aligned} \sum K_i =& \sum_{z=1}^{\frac{s}{2}}((s+2z-1)k+2-i) + \sum_{z=1}^{\frac{s}{2}}((s+2z-1)k+1+i) + 1 + i + \sum_{z=1}^{\frac{s}{2}-1}((2z)k+ \\ & 1+i) + \sum_{z=1}^{\frac{s}{2}-1}(((2z+1)+1)k+2-i) \\ =& 2k(s^2-1) + 3s - 2 + i. \end{aligned}$$

Since $\sum K_i = 2k(s^2-1) + 3s - 2 + i$ for every $i \in [1,k]$ and for all $s \geq 2$ and $\sum K_{i+1} - \sum K_i = 1$ for every $i \in [1,k-1]$, then K is $(k,1)$-anti balanced.

Lemma 2.4 *Let $k,s \geq 2$ be integers. If $X = [k+2, ks+1]$ and $Y = [ks+2, 2ks+1]$, then the multiset $Z = X \uplus Y$ is $(k,3)$-anti balanced.*

Proof. Suppose $k,s \geq 2$ be integers. We defined a multiset $Z_i = \{a_j^i \mid 1 \leq j \leq s\} \uplus \{b_j^i \mid 1 \leq j \leq s-1\}$ for $i \in [1,k]$, where

$$a_j^i = \begin{cases} ks + k(j-1) + i + 1, & \text{for } j \text{ odd;} \\ k(s+j-1) + i + 1, & \text{for } j \text{ even.} \end{cases}$$

$$b_j^i = \begin{cases} k + j + i, & \text{for } j = 1; \\ 2 + 3k + k(j-2) - i, & \text{for } j > 1. \end{cases}$$

Hence, $|Z_i| = 2s-1$, $Z_i \subset Z$, and $\biguplus_{i=1}^{k} Z_i = Z$ for every $i \in [1,k]$. Since $\sum Z_i = 2k(s^2-1) + 3(s-1) + 3i$ for every $i \in [1,k]$ and for all $s \geq 2$ and $\sum Z_{i+1} - \sum Z_i = 3$ for every $i \in [1,k-1]$, then Z is $(k,3)$-anti balanced.

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Lemma 2.5 *Let $k, s \geq 2$ be integers. If $X = [1, k(s-1)]$ and $Y = [k(s+1)+2, k(2s+1)+1]$, then the multiset $Z = X \uplus Y$ is $(k,5)$-anti balanced.*

Proof. Let $k, s \geq 2$ be integers. We define the multiset $Z_i = \{a_j^i \mid 1 \leq j \leq s\} \uplus \{b_j^i \mid 1 \leq j \leq s-1\}$ for $i \in [1, k]$, where

$$a_j^i = \begin{cases} k(s+1)+j-1+2i, & \text{for } j = 1, 2; \\ k(s+1)+k(j-1)+1+i, & \text{for } j \text{ odd and } j \geq 3; \\ k(s+1)+kj+2-i, & \text{for } j \text{ even and } j \geq 4. \end{cases}$$

$$b_j^i = \begin{cases} k(j-1)+i, & \text{for } j \text{ odd;} \\ kj+1-i, & \text{for } j \text{ even.} \end{cases}$$

It is easy to verify that $|Z_i| = 2s-1$, $Z_i \subset Z$, and $\biguplus_{i=1}^{k} Z_i = Z$ for every $i \in [1, k]$. Since $\sum Z_i = 2k(s^2-1)+2s-3+5i$ for every $i \in [1, k]$ and for all $s \geq 2$ and $\sum Z_{i+1} - \sum Z_i = 5$ for every $i \in [1, k-1]$, then Z is $(k, 5)$-anti balanced.

Lemma 2.6 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $X = [x+1, x+k] \uplus [x+2, x+k+1] \uplus [x+3, x+k+2]$, then X is $(k,3)$-anti balanced.*

Proof. Let $k \geq 2$ be an even integer. For every $i \in [1, k]$, we define the multiset $X_i = \{x+i, x+1+i, x+2+i\}$. It can be verified that $|X_i| = 3$, $X_i \subset X$, and $\biguplus_{i=1}^{k} X_i = X$ for every $i \in [1, k]$. Since $\sum X_i = 3x+3+3i$ for every $i \in [1, k]$ and $\sum X_{i+1} - \sum X_i = 3$ for every $i \in [1, k-1]$, X is $(k, 3)$-anti balanced.

Lemma 2.7 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $Y = \{x+1+2j, j = 0, 1, \ldots, k\} \uplus \{x+2+j, j = 0, 1, 2, \ldots, 2(k-1)\}$, then Y is $(k, 2)$-anti balanced.*

Proof. Let $k \geq 2$ be an even integer. For every $r \in [1, k]$, we define the multiset $Y_r = \{a_r, b_r, c_r\}$, where $a_r = 2k+2+x-2r$, $b_r = x+1+2r$, $c_r = x-1+2r$. It can be verified that $|Y_r| = 3$, $Y_r \subset Y$, and $\biguplus_{r=1}^{k} Y_r = Y$ for every $r \in [1, k]$. Since $\sum Y_r = 3x+2k+2+2r$ for every $r \in [1, k]$ and $\sum Y_{r+1} - \sum Y_r = 2$ for every $r \in [1, k-1]$, Y is $(k, 2)$-anti balanced.

Lemma 2.8 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $Y = [x+1, x+k] \uplus [x+2, x+2k+1]$, then Y is $(k,1)$-anti balanced.*

Proof. Let $k \geq 2$ be an even integer. For every $r \in [1, k]$, we define the multiset $Y_r = \{a_r, b_r, c_r\}$, where $a_r = x+r$, $b_r = x+1+r$, $c_r = x+2k+2-r$. It can be verified that $|Y_r| = 3$, $Y_r \subset Y$, and $\biguplus_{r=1}^{k} Y_r = Y$ for every $r \in [1, k]$. Since $\sum Y_r = 3x+2k+3+r$ for every $r \in [1, k]$ and $\sum Y_{r+1} - \sum Y_r = 1$ for every $r \in [1, k-1]$, Y is $(k, 1)$-anti balanced.

Lemma 2.9 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $Y = [x+1, x+2k] \uplus [x+k+2, x+2k+1]$, then Y is $(k,1)$-anti balanced or $(k,3)$-anti balanced.*

Proof. Let $k \geq 2$ be an even integer. In this proof, we have two separated cases.
Case 1. For every $i \in [1, k]$, we define the multiset $Y_i = \{a_i, b_i, c_i\}$, where $a_i = x+k+1-i$, $b_i = x+k+i$, and $c_i = x+k+1+i$. It is easy to check that $|Y_i| = 3$, $Y_i \subset Y$, and $\biguplus_{i=1}^{k} Y_i = Y$ for every $i \in [1, k]$. Since $\sum Y_i = 3x+3k+2+i$ and $\sum Y_{i+1} - \sum Y_i = 1$ for every $i \in [1, k-1]$, Y is $(k, 1)$-anti balanced.
Case 2. For every $i \in [1, k]$, we define the multiset $Y_i = \{a_i, b_i, c_i\}$, where $a_i = x+i$, $b_i = x+k+i$, and $c_i = x+k+1+i$. It is easy to check that $|Y_i| = 3$, $Y_i \subset Y$, and $\biguplus_{i=1}^{k} Y_i = Y$ for every $i \in [1, k]$. Since $\sum Y_i = 3x+2k+1+3i$ and $\sum Y_{i+1} - \sum Y_i = 3$ for every $i \in [1, k-1]$, Y is $(k, 3)$-anti balanced.

Lemma 2.10 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $Y = [x+1, x+2k+1] \uplus \{x+1+2j, j = 1, 2, \dots, k-1\}$, then Y is $(k, 2)$-anti balanced.*

Proof. Let $k \geq 2$ be an even integer. For every $i \in [1, k]$, we define the multiset $Y_i = \{a_i, b_i, c_i\}$, where

$$a_i = \begin{cases} x - 1 + 2i, & i \in [1, \frac{k}{2}]; \\ x + 2k + 2 - 2i, & i \in [\frac{k}{2} + 1, k]. \end{cases}$$

$$b_i = \begin{cases} x + 1 + 2i, & i \in [1, \frac{k}{2}]; \\ x - 1 + 2i, & i \in [\frac{k}{2} + 1, k]. \end{cases}$$

$$c_i = \begin{cases} x + 2k + 2 - 2i, & i \in [1, \frac{k}{2}]; \\ x + 1 + 2i, & i \in [\frac{k}{2} + 1, k]. \end{cases}$$

Clearly, $|Y_i| = 3$, $Y_i \subset Y$, and $\biguplus_{i=1}^{k} Y_i = Y$ for every $i \in [1, k]$. Since $\sum Y_i = 3x + 2k + 2 + 2i$ for every $i \in [1, k]$ and $\sum Y_{i+1} - \sum Y_i = 2$ for every $i \in [1, k-1]$, Y is $(k, 2)$-anti balanced.

Lemma 2.11 *Let x be a nonnegative integer and $k \geq 2$ be an even integer. If $Y = [x+1, x+2k] \uplus \{x+1+2j, j = 1, 2, \dots, k\}$, then Y is $(k, 6)$-anti balanced.*

Proof. Let $k \geq 2$ be an even integers. For every $i \in [1, k]$, we define the multiset $Y_i = \{a_i, b_i, c_i\}$, where $a_i = x - 1 + 2i$, $b_i = x + 2i$, and $c_i = x + 1 + 2i$. It is easy to check that $|Y_i| = 3$, $Y_i \subset Y$, and $\biguplus_{i=1}^{k} Y_i = Y$ for every $i \in [1, k]$. Since $\sum Y_i = 3x + 6i$ for every $i \in [1, k]$ and $\sum Y_{i+1} - \sum Y_i = 6$ for every $i \in [1, k-1]$, Y is $(k, 6)$-anti balanced.

2.3. Triangular ladder graph TL_n

Jeyanthi and Maheswari (Gallian [1]) defined a triangular ladder graph TL_n as a graph obtained from the ladders $L_n = P_n \times P_2$ $(n \geq 2)$ with additional edges $u_i v_{i+1}$ for $1 \leq i \leq n-1$, where the consecutive vertices of two copies of P_n are $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ and the edges are $u_i v_i$ for $1 \leq i \leq n$. A triangular ladder graph TL_n has $|V(TL_n)| = 2n$ and $|E(TL_n)| = 4n - 3$.

Theorem 2.12 *For $n \geq 2$, a triangular ladder graph TL_n admits a super $(14n, 1)$-C_3-antimagic labeling.*

Proof. Let G be a triangular ladder graph, $G - TL_n$ and $V(G)$ and $E(G)$ be the sets of vertices and edges of of G, respectively. Here, we define a bijective function $\xi_1 : V(G) \cup E(G) \to \{1, 2, \dots, 6n-3\}$ and $\xi_1(V(G) = \{1, 2, \dots, 2n\}$. Let $W = [1, 6n-3]$ be the set of labels for all vertices and edges of G. Partition W into two sets, i.e. $K = [1, 2n]$ and $L = [2n+1, 6n-3]$. Let H_i be the arbitrary subgraphs C_3 of G with $V(H_i) = \{v_1, u_1, v_2, v_1\}$, $\{u_1, v_2, u_2, u_1\}$, $\{v_2, u_2, v_3, v_2\}$, $\{u_2, v_3, u_3, u_2\}$, …, $\{v_{n-1}, u_{n-1}, v_n, v_{n-1}\}$, $\{u_{n-1}, v_n, u_n, u_{n-1}\}$. The number of subgraphs C_3 of G is $(2n-2)$.

Next, we use the elements of K to label the entire vertices of G, namely $v_1, u_1, v_2, u_2, \dots, v_n, u_n$, respectively. Such labeling is applied based on Lemma 2.6 with $x = 0$ and $k = 2n - 2$. The elements of X_i are used to label all vertices in every subgraph H_i of G, i.e. $V(H_i)$. Since $\sum X_i = 3 + 3i$ for every $i \in [1, 2n-2]$ and $\sum X_{i+1} - \sum X_i = 3$ for every $i \in [1, 2n-3]$, so X is $((2n-2), 3)$-anti balanced.

Now, we label all edges of TL_n using the elements of L. This labeling is applied according to Lemma 2.7 with $x = 2n$ and $k = 2n - 2$. The elements of Y_r are used to label all edges in every subgraph H_i of G, i.e. $\{u_{n-1}u_n, u_{n-1}v_n, u_n v_n\}$, $\{v_{n-1}v_n, u_{n-1}v_{n-1}, u_{n-1}v_n\}$, …, $\{u_1u_2, u_1v_2, u_2v_2\}$, $\{v_1v_2, u_1v_1, u_1v_2\}$, respectively. Since $\sum Y_r = 10n - 2 + 2r$ for every $r \in [1, 2n-2]$ and $\sum Y_{r+1} - \sum Y_r = 2$ for every $r \in [1, 2n-3]$, so Y is $((2n-2), 2)$-anti balanced.

After all the vertices and edges of G are labeled, we obtain $w(H_i) = \sum X_i + \sum Y_r$ as the sum of labels from each subgraph H_i. If $r = 2n - 1 - i$, then $w(H_i) = \sum X_i + \sum Y_{2n-1-i} =$

$3 + 3i + 14n - 4 - 2i = 14n - 1 + i$ for all $i \in [1, 2n-2]$. Since $w(H_{i+1}) - w(H_i) = 1 = d$ and $w(H_1) = 14n = a$, we can deduce that triangular ladder graph TL_n is a super $(14n, 1)$-C_3-antimagic for $n \geq 2$.

As a consequence of this result, a triangular ladder graph TL_n is a super (a, d)-C_3-antimagic covering for $d = 2, 3, 4, 5, 6, 9$. It can be proved by using Lemma 2.8, Lemma 2.2, Lemma 2.9 ($(k, 1)$-anti balanced), Lemma 2.10, Lemma 2.9 ($(k, 3)$-anti balanced), and Lemma 2.11 to L, respectively.

Corollary 2.13 *For $n \geq 2$, a triangular ladder graph TL_n admits*

(i) a super $(12n + 3, 2)$-C_3-antimagic total labeling,
(ii) a super $(12n + 4, 3)$-C_3-antimagic total labeling,
(iii) a super $(12n + 3, 4)$-C_3-antimagic total labeling,
(iv) a super $(10n + 6, 5)$-C_3-antimagic total labeling,
(v) a super $(10n + 6, 6)$-C_3-antimagic total labeling,
(vi) a super $(6n + 12, 9)$-C_3-antimagic total labeling.

2.4. Generalized Jahangir graph $J_{k,s}$

Gallian [1] defined a generalized Jahangir graph $J_{k,s}$ as a graph contains $ks+1$ vertices consisting of a cycle C_{ks} and one additional vertex that is adjacent to k vertices of C_{ks} at distance s to each other on C_{ks}. A generalized Jahangir graph $J_{k,s}$ has $|V(J_{k,s})| = ks + 1$ and $|E(J_{k,s})| = ks + k$.

Theorem 2.14 *For $k, s \geq 2$, a generalized Jahangir graph $J_{k,s}$ is a super $(2ks^2 + 4ks + 2k + 3s + 6, 1)$-$C_{s+2}$-antimagic.*

Proof. Let G be a generalized Jahangir graph. Let the set of vertices of G, $V(G) = \{c, v_!, v_2, ..., v_s, v_{s+1}, ..., v_{2s}, v_{2s+1}, ..., v_{ks}\}$ and the set of edges of G, $E(G) = \{cv_1, cv_{s+1}, ..., cv_{ks}\}$. We define a bijective function $\xi_3 : V(G) \cup E(G) \to \{1, 2, \ldots, 2ks + k + 1\}$ and we set $\xi_3(V(G) = \{1, 2, \ldots, ks + 1\}$. Let $W = [1, 2ks + k + 1]$, the set of labels for all vertices and edges of G. Partition W into five sets, i.e. $P = \{1\}$, $X = [k + 2, 2k + 1]$, $Y - X = [2, ks + 1] \setminus X$, $Z = [ks+2, 2ks+1]$, and $L = [2ks+2, 2ks+k+1]$. Next, we define H_i as any subgraph C_{s+2} of G, where $V(H_i) = \{c, v_1, v_2, \ldots, v_{s+1}, c\}$, $\{c, v_{s+1}, v_{s+2}, \ldots, v_{2s+1}, c\}$, $\{c, v_{2s+1}, v_{2s+2}, \ldots, v_{3s+1}, c\}$, $\ldots$, $\{c, v_{(k-1)s+1}, v_{(k-1)s+2}, \ldots, v_1, c\}$. The number of subgraphs of G is kC_{s+2}.

Next, we label all of the vertices and edges from every subgraph H_i of G. First, we label the central vertex of G by the element of P, so $\xi_3(c) = 1$. Furthermore, we put the elements of X and L as the labels of vertices and edges that adjacent and incident to c, respectively. Let $k \geq 2$ and $M = X \uplus L$ with $|X| = k$ and $|L| = k$. Define $M_i = \{\{a_i, b_i\} \mid 1 \leq i \leq k\}$, where $a_i = 2(k + 1) - i$ and $b_i = 2ks + 1 + i$ for every $i \in [1, k]$. Furthermore, we define the sets

$$A = \{a_i \mid 1 \leq i \leq k\} \quad = [k + 2, 2k + 1];$$
$$B = \{b_i \mid 1 \leq i \leq k\} \quad = [2ks + 2, 2ks + k + 1].$$

If $A \uplus B = M$ and $\biguplus_{i=1}^{k} M_i = M$, then by Lemma 2.1, for $1 \leq i \leq k$, $|M_i| = 2$ and $\sum M_i = 2ks + 2k + 3$. Therefore, M is k-balanced.

Next, we put the elements of $Y - X$ and Z as the labels of vertices not adjacent to c and edges which not incident to c. If $(Y - X) \uplus Z = K$, then for $k \geq 2$ and $i \in [1, k]$, by Lemma 2.3, we set $K_i = \{a^i_j \mid 1 \leq j \leq s\} \uplus \{b^i_j \mid 1 \leq j \leq s - 1\}$. Use the elements b^i_j of K_i for every $i \in [1, k]$ and $j \in [1, s - 1]$ to label every vertex not adjacent to c. Meanwhile, the elements a^i_j of K_i for every $i \in [1, k]$ and $j \in [1, s]$ are used to label the edge not incident to c. Since $K_i = k(2s^2 - 2) + 3s - 2 + i$ for every $i \in [1, k]$ and $K_{i+1} - K_i = 1$ for every $i \in [1, k - 1]$, so K is $(k, 1)$-anti balanced.

We obtain the sum of labels from each subgraph H_i, $w(H_i) = 1 + 2(\sum M_i) + \sum K_i = 2ks^2 + 4ks + 2k + 3s + 5 + i$ for every $i \in [1, k]$. Since $w(H_{i+1}) - w(H_i) = 1 = d$ and $w(H_1) = 2ks^2 + 4ks + 2k + 3s + 6 = a$, it can be concluded that a generalized Jahangir graph $J_{k,s}$ is a super $(2ks^2 + 4ks + 2k + 3s + 6, 1)$-$C_{s+2}$-antimagic for $k, s \geq 2$.

The following corollary can be proved by using Lemma 2.1 to $K \uplus L = [2, k+1] \uplus [2ks + 2, 2ks + k + 1]$ and Lemma 2.4 to $X \uplus Y = [k + 2, ks + 1] \uplus [ks + 2, 2ks + 1]$ for Corollary 2.15(1), then Lemma 2.1 to $K \uplus L = [k(s-1) + 1, ks] \uplus [ks + 2, k(s+1) + 1]$ and Lemma 2.5 to $X \uplus Y = [1, k(s-1)] \uplus [k(s+1) + 2, k(2s+1) + 1]$ for Corollary 2.15(2).

Corollary 2.15 *For $k, s \geq 2$, a generalized Jahangir graph $J_{k,s}$ admits*

(i) a super $(2ks^2 + 4ks + 3s + 7, 3)$-$C_{s+2}$-antimagic covering,

(ii) a super $(2ks^2 + 5ks - 2k + 2s + 7, 5)$-$C_{s+2}$-antimagic covering.

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