

Conjecturing and Generalization Process on The Structural Development

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ABSTRACT. This study aims to describe how conjecturing process and generalization process of structural development to thirty children in middle school at grade 8 in solving problems of patterns. Processing of the data in this study uses qualitative data analysis techniques. The analyzed data is the data obtained through direct observation technique, documentation, and interviews. This study based on research studies Mulligan et al (2012) which resulted in a five - structural development stage, namely prestructural, emergent, partial, structural, and advance. From the analysis of the data in this study found there are two phenomena that is conjecturing and generalization process are related. During the conjecturing process, the childrens appropriately in making hypothesis of patterns problem through two phases, which are numerically and symbolically. Whereas during the generalization of process, the childrens able to related rule of pattern on conjecturing process to another context.

1. BACKGROUND

Patterning is critical to the abstraction of mathematical ideas and relationships, and the development of mathematical reasoning in young children (English, 2004; Mulligan, Prescott & Mitchelmore, 2004; Waters, 2004). The integration of patterning in early mathematics learning can promote the development of mathematical modelling, representation and abstraction of mathematical ideas. It seems advantageous then, that initiatives in mathematics curricula and assessment in Australia and at international level are promoting the development of early mathematical patterning and reasoning (Clements, 2004; Doig, 2005).

In mathematics education there is a general consensus that patterning involves “observing, representing and investigating patterns and relationships in social and physical phenomena and between mathematical objects themselves” (Australian Education Council, 1991, p 4). Mathematical patterns encountered in school range from number sequences and spatial arrays to algebraic generalisations and geometrical theorems. A pattern may be defined as a numerical or spatial regularity, and the relationship between the various components of a pattern constitute its structure. Pattern and structure are thus at the heart of school mathematics. Early algebraic thinking in the elementary school may involve the development of thinking skills where the letter symbol is assigned to describe patterns. Other activities, for example analysing relationships among quantities, noticing structure, studying change, generalising, problem solving, modelling, justifying, proving, and predicting, can be engaged without using the letter symbol (Keiran, 2004). The conceptual development of ‘pattern’ in a variety of situations engages these processes.

Pattern generalization is the activity to make a general rule pattern based on specific examples. Specific examples can be graphic, numeric, verbal and algebraic pattern (Janvier, 1987). In particular, patterns are seen by some researchers as path to transition to algebra because they are a fundamental step to build the generalization that is the essence of mathematics (Zazkis and Lijedahl, 2002). In generalizing patterns, it is not enough to declare a general rule and order patterns verbally but must also state the general rule of pattern with symbol.



International studies about the approach to algebra that involves the processes of generalization concern the study of: patterns, algebraically representable functional correspondences between pairs of variables, equations, structural aspects of arithmetic operations, simple numerical theorems (formulation of conjectures and their justification).

A conjecture is a logical statement, but whose truth has not been confirmed (Cañadas and Castro, 2005; Ontario Ministry of Education, 2005; Mason et al., 2010; Reid, 2002). The process of producing conjecture is called the conjecturing process. Conjecturing process is the mental activity used in building a conjecture based on one's knowledge (Sutarto et al., 2015). Mental activity in building conjecture is a process that occurs in the mind that can be seen through the behaviour of students in problem solving.

Associated with the conjecturing in the school mathematics is a process of building new knowledge for the students. This is in accordance with the statement by Lee and Sriraman (2010) that conjecturing in mathematical problem solving in learning process is to construct new knowledge for the students according to the existing knowledge that students already have.

Indicators of Conjecturing Process

The most common strategy used in organizing cases is registering or sorting the data; (3) Searching for and predicting patterns is one's activity when observing repetitive and regular situations, one naturally imagines that the pattern may apply to the next cases of unknown; (4) Formulating a conjecture is making a statement about all possible cases, based on empirical facts, but with an element of doubt or in other words conjecture is a statement that has not been validated; (5) Validating the conjecture is the activity performed to justify the conjecture generated based on specific cases but not in general; (6) Generalizing the conjecture is an activity on changing confidence related to the generated conjecture, that the conjecture is valid in general; (7) Justifying generalization is the activity performed to justify generalizations. Justifying the generalization involves giving reasons that explain conjecture with the intention of convincing others that the resulting conjecture is correct.

Table 1. Stages of Conjecturing Process

Stages of Conjecturing Process	Descriptions
Observing the case	Initial activities against specific cases of problems: <ul style="list-style-type: none"> • Observing and counting the number of green square, white square, green or white square • Observing and counting the number of green rectangle
Organizing the case	Activities that involve the use of strategies that facilitate work in certain cases: <ul style="list-style-type: none"> • Making a list or table to associate symbol 1x1 with the image number 1, symbol 2x2 with the image number 2, symbol 3x3 with the image number 3, and so on
Searching for and predicting the pattern	Activities of observing certain objects either organized or disorganized and thinking about the next object that is not yet known: <ul style="list-style-type: none"> • Calculating the difference between a square to the first, second, third, fourth and fifth • Calculating the difference between a white square and green square to the first, second, third, fourth and fifth
Formulating the conjecture	Making a statement about all possible cases, based on empirical facts but have not been validated: <ul style="list-style-type: none"> • Declaring the n formula of green square as $4n+4$, white square as n^2, and the formula for n green or white square as $(n+2)^2$

Validating the conjecture	Activities carried out to establish the truth of the conjecture produced by certain new cases but not in general: <ul style="list-style-type: none"> • To sketch the next object that represents the next pattern to establish the truth of the conjecture generated
Generalizing the conjecture	Changes related to the confidence of the generated conjecture, that the conjecture is valid in general: <ul style="list-style-type: none"> • Believing the formula of n green square as $4n+4$, white square as n^2, and the formula for n green or white square as $(n+2)^2$
Justifying the generalization	Giving reason that explain the generalization is correct: <ul style="list-style-type: none"> • Justifying generalization based on specific cases

Indicators of Generalization Process

Successful strategies seemed to proceed through the following sequence of stages: a direct modelling stage, the stage of identification of a pattern, the stage of proof testing of the pattern, and the final stage for finding a rule for the general case (Dindyal, 2007).

The direct modelling stage involved the focus students actually using strategies such as drawing, counting, and writing down the first few cases systematically. For example, drawing five white arrangement of square tiles measuring $n \times n$ ($n = 1, 2, 3, 4, 5$) which is surrounded by tiles green square measuring 1×1 tiles green-colored tiles are connected directly with its sides and its vertices. Counting the number of white square for the n formula, the number of green square for the n formula, the number of white or green square for the n formula, and the number of green rectangle for the n formula.

The second stage (identification of the pattern) was the stage during which the students were actually able to identify some useful pattern. Which pattern one chooses depends on the particular aspect of the pattern that one wishes to observe (Phillips, 1993), and this depended considerably on the students' systematic counting, drawing, or writing/recording from the first stage. For examples, identified the pattern of number of white square as 1, 4, 9, 16, 25. The pattern of number of green square as 8, 12, 16, 20, 24. The pattern of number of white or green square as 9, 16, 25, 36, 49. For the first pattern, the number of white square was square number. For the second pattern, the number of green square was multiple of 4. For the third pattern, the number of green or white square was square number.

In the third stage (proof testing of the pattern), the successful focus students tested their conjectures about the patterns by using a particular case beyond the range for them to model directly. For examples, the student were asked about larger grids where counting the number of green rectangle on strategies were not very practical that these students looked for alternative strategies. Counting the number of green rectangle where $1 \times 2, 1 \times 3, 1 \times 4, \dots, 1 \times (n+2)$ grid for every $1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4$, and 5×5 grid.

In the final stage (finding a rule for the general case), the students had to come up with a generalisation. Swafford and Langrall (2000) had claimed that the generalisation of a tiling problem situation might be presented symbolically. For the symbolic, the student wrote the number of green square as $4n+4$ which similar to 8, 12, 16, 20, 24 strategy for the $1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4$, and 5×5 grid. For the second pattern, the student wrote the number of green rectangle (where $1 \times 2, 1 \times 3, 1 \times 4, \dots, 1 \times (n+2)$ grid) as $4(n+2)\left(\frac{n+1}{2}\right)$ which similar to $4+8, 4+8+12, 4+8+12+16, 4+8+12+16+20, 4+8+12+16+20+24$ strategy for the $1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4$, and 5×5 grid.

2. METHODOLOGY

Subject

Researchers asked 30 students at junior high school (SMPN 1 Pace) to complete tiling problem. After experiencing saturation data in the subject, seven students able to make conjecture symbolically and then the only one student able to make generalization symbolically.

Instrument

There are two types of instruments used, main and auxiliary instruments. The main instrument is the researchers themselves who act as planners, data collectors, data analysts, interpreters, and reporters of research results. The auxiliary instrument used in this study is a Tiling Problem (TP) and interviews. The problem given aims to obtain a description of the process of conjecturing and generalization of students, while the interview used was unstructured interview. The TP is presented in Figure 1.

Procedure

In the first stage, students completed TP. In the second stage, the researchers conducted work-based interviews to understand the process of conjecturing and generalization and then the researchers recorded them by using a handy cam.

Data analysis

This study is a qualitative research with descriptive exploratory approach. At the data analysis stage, the activities conducted by researchers were (1) transcribing the data obtained from interviews, (2) data reduction, including explaining, choosing principal matters, focusing on important things, removing the unnecessary ones, and organizing raw data obtained from the field, (4) describing the conjecturing and generalization process in the solving of tiling problem, and (5) conclusion.

3. RESULTS AND DISCUSSION

Based on the results of the analysis of the TP answer sheets and interviews, we obtained data on the conjecturing and generalization process undertaken by students in solving tiling problem. Of the 10 students who did conjecturing process and produced a formula or symbolic general rule, 2 students did generalization process and produced a new combine formula which resulted from conjecturing process and other formula symbolically. The results of the process of conjecturing are presented in Table 1. After experiencing saturation in the process of data collection, there were 10 students that did the conjecturing and then the only 2 student that did the generalization. Of the 2 students, we chose one subject that did conjecturing and generalization process, that is the subject S1.

Table 2. Prosentase of Ability to Make Conjecturing and Generization of Pattern

Unable to make conjecturing or generalization of pattern	Able to make	
	Conjecturing Pattern	Generalization Pattern
18	10	2
60%	33,3%	6,7%

Conjecturing Process

At the stage of action, subject S1 realized that the images (first, second, third, fourth and fifth image) formed a pattern. S1 observed cases by observing and counting the number of square section with distinguishing green square and white square of the first, second, third, fourth and fifth image. Here are excerpts of the interview and the work of S1.

- I : What did you think when reading this issue?
 S1 : The pattern, off course.
 I : What kind of patterns?
 S1 : The pattern of addition, the pattern of multiples of 4, the pattern of exponential number

It is always 4, namely from to the first, second, third, fourth, and fifth and onwards (pointing to the results of the work). Based on the number of squares on the first, second, third, fourth and fifth images, subject S1 organized the case by writing a symbol indicating a similar pattern by circling the first, second, third, fourth and fifth images. This was confirmed by the transcript of the interview S1 and the work of S1 in Figure 1.

- S1 : I see a pattern. many green tiles are always multiples of 4, while many white tiles are squared numbers

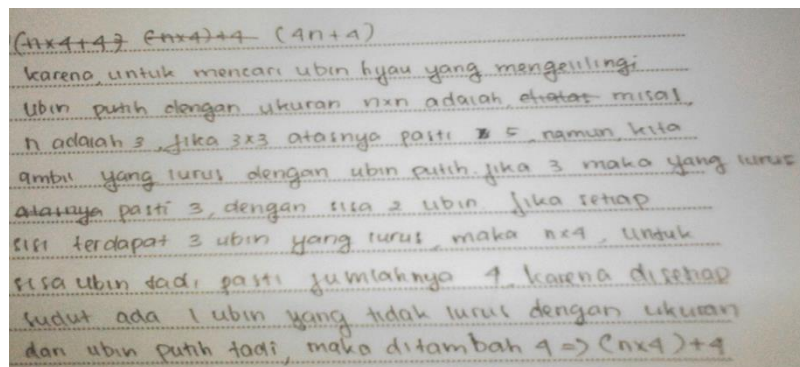


Figure 1. the work of S1 to seek green square tiles that surround the white tiles

Translate in English:

Because to find the green tiles that surround the white tiles with size $n \times n$ is suppose that n is 2, 3×3 if it would have been 5, but we take that straight to the white tile if 3 then straight definite 3, with the remaining 2 tiles. If each side there are three tiles straight then $n \times 4$. for the rest of the tiles definite number 4, because in every corner there is one tile that is not proportional to the size and white tiles just now, then increase $4(n \times 4) + 4$

At the process stage, S1 internalized the action by finding and predicting patterns. The activities were carried out by calculating number of green square of the first, second, third, fourth and fifth image, that was multiple of 4, as well as thinking about the fourth, fifth, sixth image and so on.

At the object stage, S1 encapsulated process to formulate a conjecture by connecting between the first, second, third, fourth and fifth image to the difference and the number of the green square of the image given. For example, for the first image, the student wrote eight for the first image, because there were four square which 1×1 grid, wrote 4 for the number of square which on its four corner points. For the second image, S1 wrote twelve, because there were four square which 1×2 grid, wrote 4 for the number of square which on its four corner points. The third image, S1 wrote sixteen, The fourth image, S1 wrote twenty and the fifth image, S1 wrote twenty. Furthermore, S1 formulated the general formula for n image, that was $4n+4$. This was confirmed by the results of the work of S1

(Figure 4) and interview transcript S1. S1 validated conjecture by calculating conformity of the formula with the number of green square of the sixth image. The formula obtained was correct as the number of green square at the fifth image plus 4 equal to 24. This is evidenced by the following interview transcript of S1 and the work of S1.

- S1** : So for the first image I created a formula; for example, the first image was multiplied by the addition, and I got 4, then right, up, left which parallel, down was fifth, and I added all, and the result was 4. Once multiplied, then added with four the green square which on its four corner points.
- S1** : I'm sure, I tried to count in the sixth image, the result is the same as for the fifth image, 24 plus 4 is 28.

At the scheme stage, S1 believed the general formula $4n+4$ produced was correct, based on the results of the validation. By believing that formula, S1 was generalizing. Next on justify generalization stage, S1 pinpointed the specific examples as done in validating the conjecture in order to convince others that the conjecture was generated correctly. This can be shown from the following interview excerpts.

- I** : Ok. How did you explain to others that the resulting formula is true (pointing to the work of S1).
- S1** : I'm going to show you an example.
- I** : Examples like what?
- S1** : Suppose the first image is true, the second image is true, the third image is true, fourth image is true, and fifth image is true, then is true and so on.

Generalization Process

In Tiling Problems focus students had to identify a general pattern starting from few specific cases. It was expected that reasoning inductively from a few cases the focus students would be able to generate a general rule or formula. Successful strategies seemed to proceed through the following sequence of stages: a direct modelling stage, the stage of identification of a pattern, the stage of proof testing of the pattern, and the final stage for finding a rule for the general case.

The direct modelling stage involved the focus students actually using strategies such as drawing and counting. For example, in Tiling Problem most of the students drew a white squares in the 1×1 grid, 2×2 grid, 3×3 grid, 4×4 grid and 5×5 grid which surrounded and connected directly with green tiles and measuring 1×1 grid, and then counted the number of white squares in the 1×1 grid, 2×2 grid, 3×3 grid, 4×4 grid and 5×5 grid and again counted the number of green rectangle which measured 1×2 , 1×3 , 1×4 , ... $1 \times (n+2)$ of white square in the 1×1 grid, 2×2 grid, 3×3 grid, 4×4 grid and 5×5 grid before identifying any pattern. For example, the number of green rectangle which measured 1×2 and 1×3 in the white square which 1×1 grid. The number of green rectangle which measured 1×2 , 1×3 and 1×4 in the white square which 2×2 grid.

The second stage was the stage during which the students were actually able to identify some useful pattern. Which pattern one chooses depends on the particular aspect of the pattern that one wishes to observe (Phillips, 1993), and this depended considerably on the students' systematic drawing and counting from the first stage. For example, in Tiling Problem for the number of white square in 5×5 grid, some students identified emergent some pattern: (i) the pattern of the number white square as 1, 4, 9, 16, 25 and (ii) the pattern of the number green square as 8, 12, 16, 20, 24, (iii) the pattern of the number green rectangle as $4 + 8 + 12 + 16 + \dots + 4(n+1)$. The generalisation was n^2 for the first pattern, multiple of 4 for the second pattern which was later simplified to $4n$ for $n = 2, 3, 4, 5, \dots$ etc. For the third pattern, can be modified with $4(1 + 2 + 3 + \dots + (n+1))$, and then

$4(n+2)\left(\frac{n+1}{2}\right)$, thus a systematic way of counting the number of white squares, green square and

green rectangle helped the students to generalise. The generalisation was fairly easy when there were sufficient examples to make the pattern quite evident. In this problem, the successful students were able to identify a connection between the number of white square and green square, connection between the number of green rectangle and green square. The systematic way of recording the number of green rectangle and green square in a table helped the successful students to identify a pattern in the results. In problems where this was not the case, the students had more difficulties in coming up with a useful pattern. For example, the problem of determining the number of rectangular tiles are colored green in the n pattern, the most students had to come up with a generalisation based on only one initial case. This proved to be hard for the students. Lee (1996) has pointed out that the problem for many students is not the inability to see a pattern but the inability to see an algebraically useful pattern.

In the third stage, the successful focus students tested their conjectures about the patterns by using a particular case beyond the range for them to model directly. For example, in the students were asked to find the number of green rectangle in the n pattern. Generally, the students who were able to attain this stage were able to get to the algebraic generalisation later. "Counting on" was a common strategy for some of the focus students to reach a solution for the pattern to $-n$, but this was not very helpful as an overall strategy. It was when these students were asked about larger grids such as 7×7 grid where counting on strategies were not very practical that these students looked for alternative strategies. So, they used their earlier patterns such as $4(1+2+3+ \dots + (7+1)) = 4(1+2+3+ \dots + 8)$, in this stage, the student identified there are four pairs which sum of the two number are the same numerically, there are $1+8 = 2+7 = 3+6 = 4+5$. It is differences if larger grids such as 10×10 grid where counting on strategies were not very practical that these students looked for alternative strategies. So, they used their earlier patterns such as $4(1+2+3+ \dots + (10+1)) = 4(1+2+3+ \dots + 11)$, in this stage, the student identified there are five pairs numerically which sum of the two number are the same, there are $1+11 = 2+10 = 3+9 = 4+8 = 5+7$ and then number 6 which reminder. They used the patterns they had identified to do so.

In the final stage, the students had to come up with a generalisation. Swafford and Langrall (2000) had claimed that the generalisation of a problem situation might be presented verbally or symbolically. In the problems that were used in this study, the focus students avoided a verbal generalisation and all of them tried to give symbolic generalisations. For the symbolic, this involved constructing an algebraic relation for the pattern they had noticed. Their success in the first three stages of the solution process helped them to come to the right conclusion. The students used the pattern that they had identified earlier to come up with the generalisation. For example, S1 from

school SMPN 1 Pace, grades 7 wrote $(4n+4) + 4(n+2)\left(\frac{n+1}{2}\right)$ which was similar to her $16+4(5)(2)$ strategy for $n = 3$.

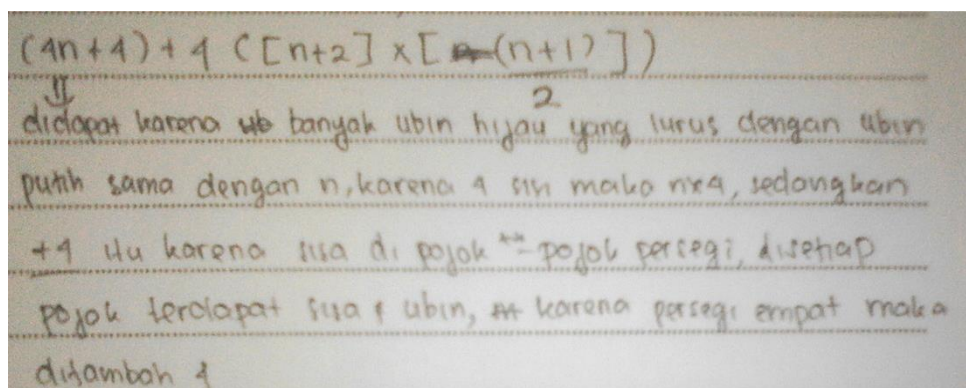


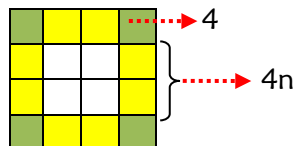
Figure 2. The work of S1 to seek green tiles are rectangular

Translate in English:

$$(4n+4) + 4(n+2) \left(\frac{n+1}{2} \right)$$

$(4n+4)$, obtained because many straight green tiles with white tiles equal to n , because four sides then $n \times 4$, while $+4$ it for the rest of the corners of the square at each corner are the remaining 4 tiles, as rectangular then plus 4.

Based on figure 3, S1 find a general rule of the number of green rectangle which 1×1 grid, $(4n+4)$. $4n$ is obtained from the number of rectangular tiles measuring 1×2 that is located right on the top, bottom, right and left white square tiles as much as 4, while n represents the size of the white square tiles. while coupled with 4, was obtained from the number of green tiles are located at the ends. More detail, description for answer S1 in generalization problem in Figure 3.

Figure 3. General rule of the number of green rectangle which 1×1 grid is $(4n+4)$

Some of the difficulties encountered by the students, such as producing variables on their own, and writing down the relations algebraically, hampered the students' progress. For instance, the students found it very difficult to come up with a symbolic generalisation for this problem. Generally, the students were able to fill up the table, but their search was for a linear symbolic relationship. Most of them were able to identify a recursive relationship in the table but only S1, from school SMPN 1

Pace, gave an explicit recursive formula. His formula was $(4n+4) + 4(n+2) \left(\frac{n+1}{2} \right)$, where

$(4n+4)$ stood for the number of green rectangle which 1×1 grid, and then $4(n+2) \left(\frac{n+1}{2} \right)$ stood

for the number of green rectangle which $1 \times 2, 1 \times 3, 1 \times 4, \dots, 1 \times (n+2)$ grid. This was confirmed by the transcript of the work of S1 to determine how she get this general formula in Figure 4.

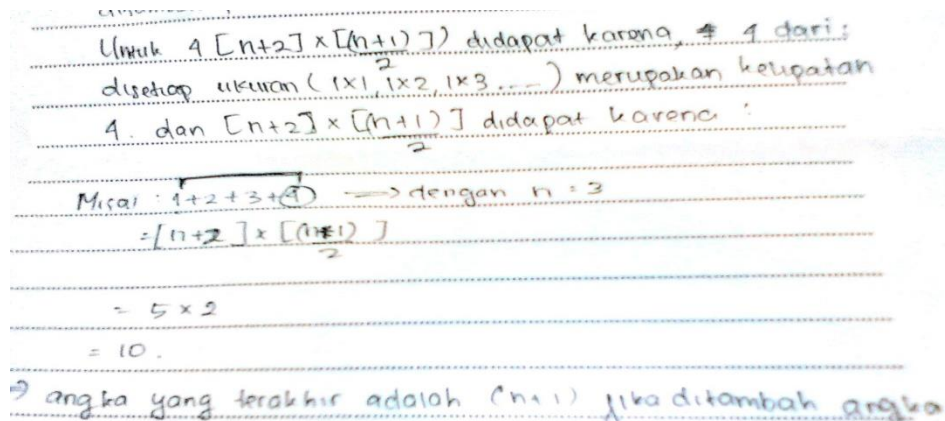


Figure 4. The work of S1 to seek green tiles are rectangular is $(n+2)\left(\frac{n+1}{2}\right)$

Translate in English:

For $4(n+2)\left(\frac{n+1}{2}\right)$ comes from because 4 from: every measured (1x1, 1x2, 1x3,...) which multiple by 4 and then $(n+2)\left(\frac{n+1}{2}\right)$ comes from because:

For example:

$$= 1 + 2 + 3 + 4 \text{ with } n = 3$$

$$= (n+2)\left(\frac{n+1}{2}\right)$$

$$= 5 \times 2$$

$$= 10$$

The last number is (n+2) if addition with number

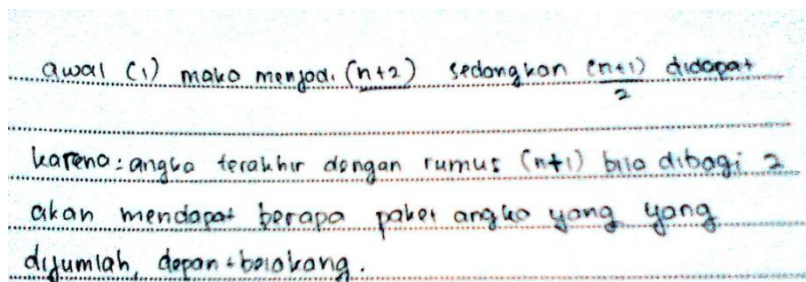


Figure 5. The work of S1 to seek green tiles are rectangular is $(n+2)\left(\frac{n+1}{2}\right)$

Translate in English:

The first (1) then (n+2). While $\left(\frac{n+1}{2}\right)$ comes from because the last number with formula (n+1) if divided y 2 will get the number which adding, the first number adding the last number.

Based on the Figure 4 and 5 above, S1 get formula $(n+2)\left(\frac{n+1}{2}\right)$, where (n+2) it is mean that how much the sum of pairs two number ($n+2 = 3+2 = 5$), and then S1 get general formula $\left(\frac{n+1}{2}\right)$ comes from how many pairs number (the first number adding to the last number, the second number adding to the third number) which result of adding the two pairs number is same. There are two pairs number $\left(\frac{3+1}{2} = 2\right)$ added, $1+4 = 2+3$ which the result is same.

However, she was able to give an explicit symbolic representation of the Tiling Problem in generalisation pattern. Some authors caution that, in their attempt to write symbolic representations, students often focus on inappropriate aspects of a number pattern – particularly the recursive relationship between successive terms in a sequence (MacGregor & Stacey, 1993; Orton & Orton, 1994). Thus, in this problem, it might be possible that the students' focus on the recursive relationship was responsible for their inability to produce an explicit generalisation. Even this problem was problematic for some of the students. The students had difficulties in coming up with the generalisation about the combine between number of green rectangle which 1×1 grid and number of green rectangle which $1 \times 2, 1 \times 3, \dots, 1 \times (n+2)$ grid because inability to determine the number of green rectangles. It seemed that a limited number of initial cases might not be enough for the students to find a pattern and hence a generalisation from the pattern, although Dreyfus (1991) had claimed that sometimes it is better to abstract from a single case.

4. CONCLUSIONS AND IMPLICATIONS

The conjecturing process in the solving of tiling problem is composed of conjecturing process based by contrast. The conjecturing process based on contrast happens at the action stage, in which the subject builds a conjecture by observing and counting the number of square separately based on differentiating a green square and a white square; and at the process stage, object and scheme is done perfectly.

The identification of a useful pattern by the students was a significant factor in their successful symbolic generalisation, which seemed to proceed in four sequential stages. However, the students had difficulties with symbolic generalisations. The students generally avoided verbalising their generalisations. Students with a weaker background in algebra, more difficulties generalising compared to the other students. Even the students with a stronger background in algebra displayed some misconceptions in handling algebraic expressions. In this study, the tiling problems had some connections to geometry, which may have added to the students' difficulties. In future studies, a broader range of problems with similar generalising activities may provide a more complete image.

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