Model reduction of unstable systems using balanced truncation method and its application to shallow water equations

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Abstract. In this paper, we discuss the reduction of unstable systems. Suppose we have a large-scale unstable system. The reduction of such systems are done in order to obtain a simpler model that has a similar behavior with the original model. In reality, a higher order model is not preferred because the computation takes more time. Thus we need a method to reduce the order of a model that is known as model reduction. Model reduction of a system is the method of approximation of a system with a lower order but the dynamic behavior is equal or almost equal to the original model. In this paper, we discuss model reduction of unstable systems. Finally this method is applied to shallow water equation that describes the flow of water in rivers: we obtain a procedure for model reduction of unstable systems and the reduced model of shallow water equations.

1. Introduction

Studying systems' behavior can be done through mathematical model. If we want to construct a model that is close to the real phenomenon, we need a large number of variables. As a consequence, the mathematical model has many state variables. Existing systems in the universe often has a big order. On the other hand, this affects the computational time to simulate the model, due to the huge number of variables in the system. Hence, we need a method to simplify the order of a system.

There are many methods in the literature for model order reduction, such as balanced truncation, modal analysis, krylov method, Hankel norm approximation, singular value decomposition [1, 2, 3]. The application of balanced truncation methods to heat conduction distribution has been discussed in [4, 5]. Those methods are suitable for stable, controllable and observable systems. However, in reality, there are many unstable systems. Thus we need to develop a model reduction method for unstable systems. In this paper, we discuss a model reduction procedure for unstable systems using balanced truncation method. In the model order reduction of an unstable system, first the system is decomposed into stable subsystem and unstable subsystems using decomposition algorithms. Then we reduce the stable subsystem by using balanced truncation method, i.e. constructing a balanced system, partition the systems and truncating the least controllable or least observable state. Finally, we add the unstable

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system into the reduced stable subsystems. Then, we apply the model reduction of unstable systems to shallow water equations.

The paper is structured as follows. Section 2 discusses the shallow water equations that will be reduced later. Then Section 3 discretizes the original model into a discrete-time system. In Section 4, we discuss the model reduction of unstable systems and its application to shallow water equations.

2. Shallow Water Equations

In this section, we discuss the shallow water equation that describes the flow of water in rivers [6]:

$$\frac{\partial h}{\partial t} + D\frac{\partial v}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial x} + C_f u = 0 \tag{2}$$

with initial and boundary conditions:

$$h(x,0) = 1, v(x,0) = 0, h(0,t) = \psi_b(t), v(L,t) = v_N(t) \tag{3}$$

where $h(x,t)$ is the water level above the reference plane at position x and time t, $v(x,t)$ is the average current velocity at position x and time t, t is the time variable, x is the position along the river, D is the water depth, g is the gravitational acceleration and c is a friction constant.

3. Discretization

In this section, we discretize the Shallow Water Equation so that we can obtain a discrete-time system that is suitable for model reduction. We use implicit scheme called Preissman discretization [7] as follows:

$$\frac{\partial v}{\partial x} = \theta\left(\frac{v_{i+1}^{k+1} - v_i^{k+1}}{\Delta x}\right) + (1-\theta)\left(\frac{v_{i+1}^{k} - v_i^{k}}{\Delta x}\right) \tag{4}$$

$$\frac{\partial h}{\partial t} = \theta\left(\frac{h_{i+1}^{k+1} - h_i^{k+1}}{2\Delta t}\right) + (1-\theta)\left(\frac{h_{i+1}^{k} - h_i^{k}}{2\Delta t}\right) \tag{5}$$

then, from equations (1) and (2), we obtain

$$\frac{1}{2\Delta t}h_i^{k+1} - \frac{D\theta}{\Delta x}v_i^{k+1} + \frac{1}{2\Delta t}h_{i+1}^{k+1} + \frac{D\theta}{\Delta x}v_{i+1}^{k+1} = \frac{1}{2\Delta t}h_i^{k} + \frac{D(1-\theta)}{\Delta x}v_i^{k} + \frac{1}{2\Delta t}h_{i+1}^{k} - \frac{D(1-\theta)}{\Delta x}v_{i+1}^{k} \tag{6}$$

$$\frac{-g\theta}{\Delta x}h_i^{k+1} + \left(\frac{1}{2\Delta t} + \frac{C_f\theta}{2}\right)v_i^{k+1} + \frac{g\theta}{\Delta x}h_{i+1}^{k+1} + \left(\frac{1}{2\Delta t} + \frac{C_f\theta}{2}\right)v_{i+1}^{k+1} = \frac{g(1-\theta)}{\Delta x}h_i^{k} + \left(\frac{1}{2\Delta t} - \frac{C_f(1-\theta)}{2}\right)v_i^{k} - \frac{g(1-\theta)}{\Delta x}h_{i+1}^{k} + \left(\frac{1}{2\Delta t} - \frac{C_f(1-\theta)}{2}\right)v_{i+1}^{k} \tag{7}$$

Using the initial and boundary condition (3) and generate terms (6) and (7) for $i = 0$ to $N = 5$, then, the following values for the parameters are assumed

$$L = 30km, D = 10m, C_f = 0.0002, \Delta t = 10, \theta = 0.3, g = 9.8m/s^2$$

Thus, we can write the equations (6) and (7) in matrix notation as follows

$$A_1 x_{k+1} = A_2 x_k + B u_k \tag{8}$$

$$x_{k+1} = A_1^{-1} A_2 x_k + A_1^{-1} B_1 u_k \tag{9}$$

The above system is a discrete-time linear-time-invariant system

$$x_{k+1} = A x_k + B u_k$$

where $A = A_1^{-1} A_2$ *and* $B = A_1^{-1} B_1$. By using the parameters described above, those matrices are given by:

$$A = \begin{pmatrix} 0.8465 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0311 & 0.9980 & -0.0653 & -0.0007 & 0.0653 & 0.0020 & -0.0654 & -0.0033 & 0.0655 & 0.0046 & -0.0327 & 0.9972 \\ 0 & 0 & 1.0003 & 0.0333 & -0.0013 & -0.0666 & 0.0026 & 0.0667 & -0.0039 & -0.0668 & 0.0023 & -0.0231 \\ 0 & 0 & 0.0327 & 0.9983 & -0.0653 & -0.0013 & 0.0654 & 0.0026 & -0.0654 & -0.0039 & 0.0327 & -0.9970 \\ 0 & 0 & 0 & 0 & 1.0003 & 0.0333 & -0.0013 & -0.0666 & 0.0026 & 0.0667 & -0.0016 & 0.0032 \\ 0 & 0 & 0 & 0 & 0.0327 & 0.9983 & -0.0653 & -0.0013 & 0.0654 & 0.0026 & -0.0327 & 0.9968 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0003 & 0.0333 & -0.0013 & -0.0666 & 0.0010 & 0.0167 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0327 & 0.9983 & -0.0653 & -0.0013 & 0.0327 & -0.9969 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0003 & 0.0333 & -0.0003 & -0.0367 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0327 & 0.9983 & -0.0327 & 0.9974 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0.0233 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0005 \\ 0.0500 \\ -0.0005 \\ 1.0000 \end{pmatrix}, x_{k+1} = \begin{pmatrix} u_0^{k+1} \\ h_0^{k+1} \\ u_1^{k+1} \\ h_1^{k+1} \\ u_2^{k+1} \\ h_2^{k+1} \\ u_3^{k+1} \\ h_3^{k+1} \\ u_4^{k+1} \\ h_4^{k+1} \\ u_5^{k+1} \\ h_5^{k+1} \end{pmatrix}, x_k = \begin{pmatrix} u_0^k \\ h_0^k \\ u_1^k \\ h_1^k \\ u_2^k \\ h_2^k \\ u_3^k \\ h_3^k \\ u_4^k \\ h_4^k \\ u_5^k \\ h_5^k \end{pmatrix}, u_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{11}$$

then, we construct a measurement equation at time k, as follows

$$y_k = C_k x_k \tag{12}$$

where

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad D = 0 \tag{13}$$

Thus, we have system $G = (A, B, C, D)$ as the original system in the model order reduction. The properties of the preceding system are unstable, uncontrollable and unobservable. In the next section, we discuss a model reduction method for such unstable systems.

4. Model Reduction of Unstable Systems

In this section, we discuss model reduction of unstable systems and its applications to shallow water equations. The steps in model order reduction of unstable systems are as follows. First, we decompose the original system into stable and unstable subsystems (Section 4.1). Then, we reduce the stable subsystem using balanced truncation method (Section 4.2). Finally, we combine the reduced stable subsystem and the unstable subsystem (Section 4.3).

4.1. Unstable System Decomposition

In this section, the unstable system G is decomposed into stable subsystem and unstable subsystem using a decomposition algorithm [8, 9]. The decomposition algorithm consists of two steps:

Step 1 Transform the system G using a unitary matrix U in block diagonal upper Schur form [10]. Thus, in the first step, the transformed system becomes:

$$G_t = \left(\begin{array}{c|c} U'AU & U'B \\ --- & --- \\ CU & D \end{array}\right) = \left(\begin{array}{cc|c} A_{t11} & A_{t12} & B_{t1} \\ 0 & A_{t22} & B_{t2} \\ --- & --- & --- \\ C_{t1} & C_{t2} & D \end{array}\right) \tag{14}$$

Step 2 The transformed system G_t in the first step, contains a coupling term A_{t12}. Using transformation $X_t = WX$, we obtain completely decoupled system G_d as follows:

$$G_d = \left(\begin{array}{c|c} W^{-1}A_tW & W^{-1}B_t \\ --- & --- \\ C_tW & D \end{array}\right) = \left(\begin{array}{cc|c} A_{11} & 0 & B_1 \\ 0 & A_{22} & B_t \\ --- & --- & --- \\ C_1 & C_2 & D \end{array}\right) \tag{15}$$

The transformed model can be decomposed into stable and unstable subsystems as follows

$$\begin{aligned} G_d &= \left(\begin{array}{c|c} A_{11} & B_1 \\ --- & --- \\ C_1 & D \end{array}\right) + \left(\begin{array}{c|c} A_{22} & B_2 \\ --- & --- \\ C_2 & 0 \end{array}\right) \\ &= G_s(Stable\ subsystem) + G_u(Unstable\ subsystem) \end{aligned} \tag{16}$$

By using the parameters defined in the preceding section, the stable and unstable subsystems are given by

$$Gs = \left(\begin{array}{ccccccc|c} 0.9980 & -0.0000 & -0.0022 & 0.0014 & 0.0002 & -0.0009 & -0.0044 & -0.0041 \\ 0 & 0.8465 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9663 & -0.0353 & -0.0172 & -0.0155 & -0.0071 & 0.0002 \\ 0 & 0 & 0 & 0.9663 & -0.0325 & -0.0294 & -0.0147 & -0.0004 \\ 0 & 0 & 0 & 0 & 0.9663 & -0.0593 & -0.0272 & 0.0008 \\ 0 & 0 & 0 & 0 & 0 & 0.9663 & -0.0330 & -0.0245 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16.000 \\ ---- & ---- & ---- & ---- & ---- & ---- & ---- & ---- \\ -17.3644 & -0.0000 & -7.0040 & 2.4502 & -0.9750 & 1.4097 & 0.0156 & 0 \end{array}\right) \tag{17}$$

$$G_u = \left(\begin{array}{ccccc|c} 1.0323 & -0.0620 & 0.0321 & -0.0174 & 0.0009 & -0.2121 \\ 0 & 1.0323 & -0.0342 & 0.0185 & -0.0006 & 0.2363 \\ 0 & 0 & 1.0323 & -0.0368 & -0.0009 & -0.5164 \\ 0 & 0 & 0 & 1.0323 & -0.0014 & 0.9656 \\ 0 & 0 & 0 & 0 & 1.0000 & 0.0950 \\ ---- & ---- & ---- & ---- & ---- & ---- \\ -3.7625 & -3.2642 & -1.5663 & -0.8370 & -3.2687 & 0 \end{array}\right) \tag{18}$$

4.2. Reduction Stable Subsystem

4.2.1. Non-minimal system If the system G_s is stable and non-minimal, we can determine the minimal order of system G_s by using "minreal.m" command of Matlab [11]. It eliminates uncontrollable or unobservable state in state-space models, The output has minimal order (m) and has the same response characteristics as the stable subsystem. The order of minimal system G_s is 6^{th}.

4.2.2. Balanced Realized Model Reduction The steps of reduction [10] are as follows:

Step 1 Determining controllability and observability gramians. We solve the Lyapunov equations for discrete-time system, and obtain the controllability Gramian, P and Observability Gramian, Q:

$$APA - P + BB = 0 \text{ and } A^T QA - Q + CC = 0$$

where in our case

$$P = \begin{pmatrix} 1.3940 & -0.2355 & 0.4021 & -0.8253 & 1.0620 & -0.0660 \\ -0.2355 & 0.2550 & -0.0109 & 0.0735 & -0.0881 & 0.0035 \\ 0.4021 & -0.0109 & 0.8939 & -0.0909 & 0.1600 & -0.0063 \\ -0.8253 & 0.0735 & -0.0909 & 3.7160 & -0.3654 & 0.0122 \\ 1.0620 & -0.0881 & 0.1600 & -0.3654 & 4.5936 & -0.3927 \\ -0.0660 & 0.0035 & -0.0063 & 0.0122 & -0.3927 & 256.0000 \end{pmatrix} \tag{19}$$

$$Q = 10^4 \begin{pmatrix} 7.5501 & -0.1140 & 0.2893 & -0.1134 & -0.2496 & -0.0255 \\ -0.1140 & 0.0818 & -0.0799 & 0.0310 & -0.0164 & 0.0008 \\ 0.2893 & -0.0799 & 0.1052 & -0.0514 & 0.0226 & -0.0016 \\ -0.1134 & 0.0310 & -0.0514 & 0.0345 & -0.0231 & 0.0009 \\ -0.2496 & -0.0164 & 0.0226 & -0.0231 & 0.0379 & 0.0003 \\ -0.0255 & 0.0008 & -0.0016 & 0.0009 & 0.0003 & 0.0001 \end{pmatrix} \tag{20}$$

Step 2 Determination of hankel singular values (HSV). We obtain HSV of the system by finding the square root of the eigen values of product of P and Q:

$$HSV = \sqrt{\lambda_i(PQ)} = \begin{pmatrix} 327.1102 & 45.1700 & 20.2872 & 9.9500 & 3.4579 & 0.8132 \end{pmatrix}$$

Step 3 Performing singular value decomposition (SVD) of the gramians

$$P = U_p \Sigma_p V_p^T \text{ and } Q = U_q \Sigma_q V_q^T \tag{21}$$

Step 4 Finding Truncation Matrices (S_L and S_R). Let $V_R = U_p \sqrt{\Sigma_p}$, $V_L = U_q \sqrt{\Sigma_q}$ and $E = V_L^T V_R$, Perform singular value decomposition of E such that

$$E = U_E \Sigma_E V_E^T \tag{22}$$

Then, we get the truncation matrices are obtained as $S_L = V_L\ U_E\ V_E^{-1/2}$ In our problem, the matrices S_L and S_R are given by

$$S_L = \begin{pmatrix} 15.0779 & 4.9136 & -1.3724 & 0.7625 & -0.3531 & 0.0465 \\ -0.3677 & 2.3781 & -4.0138 & 4.0882 & -2.6846 & 0.3729 \\ 0.7671 & -3.5190 & 3.7912 & -0.7231 & -0.9586 & 0.1790 \\ -0.3449 & 2.5361 & -0.4330 & -1.0507 & -0.3756 & 0.1085 \\ -0.3961 & -2.5627 & -1.1332 & -0.7168 & -0.0245 & 0.0957 \\ -0.0529 & 0.0289 & 0.0272 & 0.0317 & 0.0422 & 0.0512 \end{pmatrix}$$

and $S_R = V_R\ U_E\ V_E^{-1/2}$

$$S_R = \begin{pmatrix} 0.0651 & 0.0013 & -0.0144 & -0.0087 & 0.0103 & 0.0709 \\ -0.0111 & -0.0022 & -0.0332 & 0.0861 & -0.1782 & 0.1007 \\ 0.0205 & -0.0407 & 0.1350 & -0.0406 & -0.2717 & 0.2210 \\ -0.0418 & 0.1506 & -0.0346 & -0.3925 & -0.3485 & 0.4200 \\ 0.0443 & -0.1830 & -0.2738 & -0.2600 & -0.0821 & 0.5227 \\ -0.0440 & 0.1801 & 0.3677 & 0.8396 & 3.1280 & 16.0818 \end{pmatrix}$$

Step 5 Obtaining Balanced Realized Minimal Model. BT model of G_s using the truncation matrices S_L and S_R is as follows,

$$G_d = \left(\begin{array}{c|c} A_{bal} & B_{bal} \\ --- & --- \\ C_{bal} & D_{bal} \end{array} \right) + \left(\begin{array}{c|c} S'_L A S_R & S'_L B \\ --- & --- \\ C S_R & D \end{array} \right) \tag{23}$$

In our case, the balanced realized minimal model is given by:

$$G_{bal} = \left(\begin{array}{cccccc|c} 0.9988 & -0.0016 & 0.0012 & -0.0015 & 0.0018 & -0.0022 & -0.8988 \\ 0.0016 & 0.9971 & 0.0095 & -0.0048 & 0.0082 & -0.0091 & 0.5083 \\ 0.0012 & -0.0095 & 0.9944 & 0.0234 & -0.0133 & 0.0190 & 0.4663 \\ 0.0015 & -0.0048 & -0.0234 & 0.9850 & 0.0538 & -0.0414 & 0.5221 \\ 0.0018 & -0.0082 & -0.0133 & -0.0538 & 0.9217 & 0.1819 & 0.6760 \\ 0.0022 & -0.0091 & -0.0190 & -0.0414 & -0.1819 & -0.0335 & 0.8174 \\ ---- & ---- & ---- & ---- & ---- & ---- & ---- \\ -0.8988 & -0.5083 & 0.4663 & -0.5221 & 0.6760 & -0.8174 & 0 \end{array} \right)$$

Step 6 Obtaining r^{th} order BT Model. We select the reduced order number $r = 1$ of the system G_s on the basis of higher magnitudes of HSV. So, we get 1^{st} order BT model as

$$G_{sr-bt} = \left(\begin{array}{c|c} 0.9988 & -0.8988 \\ ----- & ----- \\ -0.8988 & 0 \end{array} \right) \tag{24}$$

4.3. Reduced order model for original unstable system

The balanced truncated reduced model of unstable system is obtained by adding the unstable subsystem into the reduced stable subsystem:

$$G_{r-bt} = G_{sr-bt} + G_u$$

In our case, it is given by

$$G_{r-bt} = \left(\begin{array}{cccccc|c} 0.9988 & 0 & 0 & 0 & 0 & 0 & -0.8988 \\ 0 & 1.0323 & -0.0620 & 0.0321 & -0.0174 & 0.0009 & -0.2121 \\ 0 & 0 & 1.0323 & -0.0342 & 0.0185 & -0.0006 & 0.2363 \\ 0 & 0 & 0 & 1.0323 & -0.0368 & -0.0009 & -0.5164 \\ 0 & 0 & 0 & 0 & 1.0323 & -0.0014 & 0.9656 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0.0950 \\ ---- & ---- & ---- & ---- & ---- & ---- & ---- \\ -0.8988 & -3.7625 & -3.2642 & -1.5663 & -0.8370 & -3.2687 & 0 \end{array} \right) \tag{25}$$

5. Conclusion

In this work, we have developed a model reduction for unstable systems. Even our procedure works for non-minimal systems. Furthermore we have successfully obtained a reduced model for shallow water equations. We can use the results in this paper to design a low-dimensional controller, observer and some other applications. In this approach, we have not analyzed the procedure formally. We are planning to work on such issue in the future.

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