

# Identification and estimation of state variables on reduced model using balanced truncation method

**Trifena Punana Lesnussa, Didik Khusnul Arif, Dieky Adzkiya and Erna Apriliani**

Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya 60111, Indonesia

E-mail: [trivelesnussa@gmail.com](mailto:trivelesnussa@gmail.com), [didik@matematika.its.ac.id](mailto:didik@matematika.its.ac.id),  
[dieky@matematika.its.ac.id](mailto:dieky@matematika.its.ac.id), [april@matematika.its.ac.id](mailto:april@matematika.its.ac.id)

**Abstract.** In this paper, we study the identification of variables on a model reduction process and estimation of variables on reduced system. We aim to relate variables on reduced and original system, so that we can compare the estimation accuracy of the original system and reduced system. As such, the objective of this paper is to discuss identification and estimation of variables on reduced model. First, model order reduction is done by using balanced truncation method. This process begins with the construction of balanced system. After that, we identify the relationship between variables of the balanced system and the original system. Then, we eliminate variables of the balanced system that have a small influence on the system. Furthermore, we estimate state variables on the original system and reduced system using a Kalman Filter algorithm. Finally, we compare the estimation result of the identified reduced and original system.

## 1. Introduction

Generally, a system in nature has a great order. This is because the system has many state variables. If the number of state variables in a system is increased, the system is closer to the real phenomenon. However, it caused difficulties in analyzing the system. Thus the computational time is increased significantly. Therefore, we need a method to simplify the order of such systems. In this case, we reduce the order of the original system. One way is using model order reduction [1]. There are many methods in model order reduction, such as balanced truncation method, Hankel norm approximation and singular perturbation approximation [1]. Among those methods, balanced truncation method is the frequently used method because of the simplicity of the method [2, 3].

Estimation is a method to compute the state variables of a discrete-time stochastic system based on measurement data [4]. Kalman filter is an optimal estimation method for linear systems. For nonlinear systems, there are many methods that can be applied such as extended Kalman filter, ensemble Kalman filter, fuzzy Kalman filter. We have studied the application of extended Kalman filter and fuzzy Kalman filter on Nonisothermal Continuous Stirred Tank Reactor systems in [5].

We have investigated the application of Kalman filter to the original system and its reduced system in [6, 7]. Then in [8], the authors have developed Kalman filter algorithm on reduced system. However there are some difficulties in comparing the accuracy of estimation between the original system and reduced system because the order is different. More precisely, we want to



compare the estimation of state variables. Since the dimension of state variables in the reduced and original models are different, we cannot compare them. Therefore, we need an identification of state variables on reduced model such that we can determine the corresponding state variables on the original model. In this paper, we want to determine a relationship between state variables on the reduced and original models: given a state variable in the reduced model, we can compute the corresponding state variable in the original model.

## 2. Preliminaries

In this section, we describe some definitions that will be used in this paper.

### 2.1. Discrete-Time Linear-Time-Invariant Systems

A discrete-time linear-time-invariant system  $(A, B, C, D)$  is defined as [4]

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k \\z_k &= Cx_k + Du_k\end{aligned}\tag{1}$$

where  $x_k \in \mathbb{R}^n$ ,  $u_k \in \mathbb{R}^m$ ,  $z_k \in \mathbb{R}^p$  and  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$  are constant matrices. There are many properties that a system may have. Stability, controllability and observability are some important properties of a system. As it will be clear later, we focus on the previous three properties.

**Definition 1 (Stability)** *A discrete-time linear-time-invariant system (1) is asymptotically stable if and only if  $|\lambda_i(A)| < 1$  for  $i = 1, 2, \dots, n$  with  $\lambda_i(A)$  denotes eigenvalues of  $A$ . If  $|\lambda_i(A)| \leq 1$  for  $i = 1, 2, \dots, n$  then the system is stable.*

**Definition 2 (Controllability)** *The discrete-time linear-time-invariant system (1) is controllable if rank of the controllability matrix  $M_c = [B|AB|\dots|A^{n-1}B]$  equals  $n$ .*

**Definition 3 (Observability)** *The discrete-time linear-time-invariant system (1) is observable if rank of the observability matrix  $M_o = [C|CA|\dots|CA^{n-1}]^T$  equals  $n$ .*

**Definition 4 (Controllability and Observability Gramians)** *The controllability gramian of the discrete-time linear-time-invariant system (1) is  $W = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$ . The observability gramian of the discrete-time linear-time-invariant system (1) is  $M = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$ .*

The controllability and observability gramians are respectively a unique positive definite solution of the following Lyapunov equation:

$$\begin{aligned}AW + WA^T + BB^T &= 0 \\A^T M + MA + C^T C &= 0\end{aligned}$$

### 2.2. Model Reduction of Discrete-Time Linear-Time-Invariant Systems

In this section, we describe the process of model order reduction using balanced truncation method. This process begins with the construction of balanced system (or balanced realization). On balanced system, we identify the relationship between variables of the original and balanced systems. After all variables have been identified, we eliminate the variables that have small influence to the system.

*2.2.1. Balanced Realization* Balanced system is a system such that controllability and observability gramians are equal. Furthermore the gramian is a diagonal matrix. Balanced system is obtained by applying a transformation to the original system  $(A, B, C, D)$ . Suppose that defined of gramian like  $diag(\sigma_1, \sigma_2, \dots, \sigma_n)$  with  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$  and  $\sigma_1, \sigma_2, \dots, \sigma_n$  is singular Hankel value.

From the original system  $(A, B, C, D)$ , we construct a new system  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ . The new system can be constructed if and only if realization of the system is asymptotically stable, have controllability gramian  $\tilde{W}$  and observability gramian  $\tilde{M}$  is equal and it is a diagonal matrix.

Let us describe a procedure to determine transformation matrix  $T$ . First, we check whether realization  $(A, B, C, D)$  is asymptotically stable, controllable and observable. Second, compute the controllability gramian  $W$  and observability gramian  $M$  by using Definition 4. Third, we use Cholesky factorization on  $W$  to determine  $\phi$  such that  $W = \phi\phi^T$ . Next, we use singular value decomposition on  $\phi^T M \phi$  to compute  $U$  and  $\Sigma$  such that  $\phi^T M \phi = U\Sigma^2 U^T$ . The last step, compute transformation matrix  $T = \Sigma^{1/2} U^T \phi^{-1}$ .

By applying the transformation matrix to the original realization,  $x_k = T\tilde{x}_k$  we obtain an ordered balanced realization  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T^{-1}AT, T^{-1}B, CT, D)$ .

*2.2.2. Identification of Variables* Let a transformation matrix  $T$  be given such that it meets the following equation:

$$x_k = T\tilde{x}_k \quad (2)$$

In this work, we focus on the identification of variables in the balanced system and the original system.

*2.2.3. Balanced Truncation Methods* Model reduction by using balanced truncation simply applies the truncation operation to a balanced realization. Let  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  be a balanced realization, where the gramian is denoted by  $\Sigma$ . Remember that the controllability gramian is equal to the observability gramian and the gramian is a diagonal matrix. First we partition the gramian as follows

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \quad (3)$$

where  $\Sigma_1 = diag(\sigma_1, \sigma_2, \dots, \sigma_r)$ ,  $\Sigma_2 = diag(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n)$  and  $\sigma_r > \sigma_{r+1}$ .

By using the preceding partition, the corresponding partition of balanced system  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is given by

$$G = \left( \begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{C}_2 & D \end{array} \right) \quad (4)$$

where  $\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$ ,  $\tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}$ , and  $\tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2)$ . The reduced model is defined as the subsystem corresponding to  $\Sigma_1$ . Thus the reduced model can be written as  $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r)$  where  $\tilde{A}_r = \tilde{A}_{11}$ ,  $\tilde{B}_r = \tilde{B}_1$ ,  $\tilde{C}_r = \tilde{C}_1$  and  $\tilde{D}_r = D$ .

Next, we present a proposition saying that the truncation of a balanced realization is again a balanced realization. Furthermore, the proposition also states that the properties of the original realization are the same with the truncated one.

**Proposition 5 ([1, Lemma 9.4.1])** *Suppose  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is a balanced realization and assumed it is stable, controllable, observable and balanced with gramian*

$$\tilde{M} = \tilde{N} = diag(\sigma_1, \sigma_2, \dots, \sigma_n), \quad \sigma_1 \geq \dots \geq \sigma_r \geq \dots \geq \sigma_n > 0$$

If  $\sigma_r > \sigma_{r+1}$  then reduced system of order  $r$  will be stable, controllable, observable, balanced and satisfies  $\|G_s - G_{s_r}\|_\infty \leq 2(\sigma_{r+1} + \dots + \sigma_n)$  with  $G_s$  and  $G_{s_r}$  is a transfer function of  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  and the reduced system  $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r)$ , respectively.

### 2.3. Kalman Filter on Discrete-Time Linear-Time-Invariant System

In this section, we describe the process of Kalman filter algorithm over the original system and reduced system. Firstly, we analyze an estimation process of state variables in discrete-time linear-time-invariant system (1). Here, we analyze the estimation process and construct the best estimator for discrete-time linear-time-invariant system (1) based on the Kalman filter algorithm [4]. Furthermore, we apply the Kalman filter procedure to original model and reduced model.

In Kalman filter, the model is described as a stochastic discrete-time linear-time-invariant system, as follows

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + Gw_k, \\z_k &= Cx_k + Du_k + v_k,\end{aligned}\tag{5}$$

where  $x_k \in \mathbb{R}^n$  is a state vector at time  $k$ ,  $u_k \in \mathbb{R}^m$  is an input vector at time  $k$ ,  $z_k \in \mathbb{R}^p$  is a measurement vector or output vector at time  $k$ ,  $w_k$  is a system noise and  $v_k$  is a measurement noise. Variables  $w_k$  and  $v_k$  are normally distributed random variables, i.e.  $w_k \sim N(0, Q)$  and  $v_k \sim N(0, R)$ . Matrices  $A$ ,  $B$ ,  $C$  and  $D$  are appropriately dimensioned real constant matrices.

Estimation of state variables in the system (5) can be done by using Kalman filter. First, we estimate the state variables based on a dynamic system (prediction step) and then we improve the estimation results by using the measurement data (correction step). The Kalman filter algorithm for stochastic discrete-time system is as follows [4]:

#### (i) Initialization

- Define the estimation of initial state  $\hat{x}_0$
- Define the covariance of initial state's estimation  $P_0$
- Define the variance of system noise  $Q$
- Define the variance of measurement noise  $R$

#### (ii) Prediction step

(a) Error Covariance:  $P_{k+1}^- = AP_k A^T + GQG^T$

(b) Estimation:  $\hat{x}_{k+1}^- = A\hat{x}_k + Bu_k$

#### (iii) Correction step

(a) Error Covariance:  $P_{k+1} = P_{k+1}^- - P_{k+1}^- C^T (CP_{k+1}^- C^T + R)^{-1} CP_{k+1}^-$

(b) Estimation:  $\hat{x}_{k+1} = \hat{x}_{k+1}^- + P_{k+1} C^T R^{-1} (z_{k+1} - C\hat{x}_{k+1}^-)$

Initially, we define the estimation of initial state and its covariance. In this step, we also define the variance of system and measurement noises. After that, in each time step, we execute the prediction and correction steps. Notice that in the prediction step, we do not need any measurement data, whereas in the measurement step, we need a measurement data at that time step.

## 3. Main Results

In this section, we discuss the steps in the identification of state variables on the reduced model. Firstly, we construct a discrete-time linear-time-invariant system  $(A, B, C, D)$ . Then, we examine the properties of the system, such as stability, controllability and observability. If the system is asymptotically stable, controllable and observable, we continue to the next step. If the system is not asymptotically stable or the system is not controllable or the system is not observable then we have to construct another system until the three properties are satisfied.

Then, we construct the balanced system. The balanced system is a transformation of the original system using a transformation matrix. On balanced system, we identify the relationship between state variables of the original and balanced systems. After all state variables in balanced system have been identified, we remove state variables of the balanced system that have small influences on the system using balanced truncation method.

The last step is estimation of reduced and original systems by using Kalman filter algorithm. Finally, we compare the estimation result of the reduced and original system. More precisely, first we identify the states corresponding to the estimation results of reduced system, then we compare the identification results against the estimation results over the original system. In this study, we use a reduced system of order 6.

### 3.1. Construction of The Original System

Given the following discrete-time linear-time-invariant system:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}_{k+1} = \begin{pmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -0.5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & -0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -0.81 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0.9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}_k + \begin{pmatrix} 4 \\ 1 \\ 3 \\ 3 \\ -2 \\ 1 \\ 4 \\ 1 \end{pmatrix} u_k \quad (6)$$

$$y_k = (1 \quad 5 \quad 3 \quad 7 \quad -3 \quad 2 \quad 4 \quad 6) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}_k + u_k$$

This system (6) is asymptotically stable, controllable and observable. Since the system in (6) satisfies the three properties, we can use the system in the numerical example.

3.2. Model Order Reduction

3.2.1. *Balanced System* We obtain the following balanced system

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \end{pmatrix}_{k+1} = \begin{pmatrix} 0.9630 & 0.1108 & -0.0024 & -0.0087 & -0.0148 & -0.0011 & -0.0001 & -0.0000 \\ -0.1108 & 0.5633 & -0.4411 & 0.0630 & 0.1315 & 0.0017 & 0.0006 & 0.0003 \\ 0.0024 & -0.4411 & -0.8034 & -0.0835 & -0.1420 & -0.0096 & -0.0006 & -0.0003 \\ -0.0087 & -0.0630 & 0.0835 & 0.8972 & 0.3007 & -0.0489 & 0.0008 & 0.0006 \\ 0.0148 & 0.1315 & -0.1420 & -0.3007 & -0.0347 & 0.3154 & 0.0004 & -0.0007 \\ 0.0011 & 0.0017 & -0.0095 & 0.0489 & 0.3154 & -0.0100 & -0.1268 & -0.0702 \\ 0.0001 & 0.0006 & -0.0006 & 0.0008 & 0.0004 & -0.1268 & 0.7454 & -0.2847 \\ -0.000 & -0.0003 & 0.0003 & 0.0006 & 0.0007 & 0.0702 & 0.2847 & -0.3364 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \end{pmatrix}_k + \begin{pmatrix} -9.2733 & -12.2348 & -1.1801 & -1.3131 & 2.3704 & 0.1318 & 0.0109 & -0.0056 \end{pmatrix}^T \tilde{u}_k$$

$$\tilde{y}_k = \begin{pmatrix} 9.2733 & -12.2348 & -1.1801 & 1.3131 & 2.3704 & 0.1318 & 0.0109 & 0.0056 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \end{pmatrix}_k + \tilde{u}_k \tag{7}$$

The transformation matrix used to obtain the balanced system (7) is as follows

$$T = \begin{pmatrix} -2.4836 & 2.7670 & -1.0787 & -0.7136 & 5.0536 & 4.6166 & 0.9975 & -2.4005 \\ -0.0078 & -0.0038 & 0.2493 & -0.1043 & 0.4635 & -0.4272 & -0.1333 & 0.2741 \\ -0.1688 & -0.1837 & 0.3369 & 0.2326 & -0.0481 & 0.0043 & 0.1288 & -0.3290 \\ -0.0529 & -0.1288 & 0.1588 & -0.1614 & 0.4081 & -0.2571 & -0.6292 & 0.4543 \\ 0.0083 & 0.0887 & 0.1864 & 0.2261 & -0.1375 & 0.0517 & -0.0523 & 0.2043 \\ -0.0030 & -0.0482 & -0.1442 & -0.1395 & 0.0139 & -0.0243 & -0.0338 & 0.1123 \\ -0.0010 & -0.4052 & -2.0874 & 0.6429 & -1.1149 & 0.4837 & -0.1793 & 0.8229 \\ 2.1179 & -1.9245 & 0.9545 & 0.2277 & -0.6083 & -0.3822 & 0.7209 & -0.6768 \end{pmatrix}$$

3.2.2. *Identification of State Variables in the Balanced System* In the following equation, we describe the relationship between state variables of the original system and the balanced system:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} -2.4836 & 2.7670 & -1.0787 & -0.7136 & 5.0536 & 4.6166 & 0.9975 & -2.4005 \\ -0.0078 & -0.0038 & 0.2493 & -0.1043 & 0.4635 & -0.4272 & -0.1333 & 0.2741 \\ -0.1688 & -0.1837 & 0.3369 & 0.2326 & -0.0481 & 0.0043 & 0.1288 & -0.3290 \\ -0.0529 & -0.1288 & 0.1588 & -0.1614 & 0.4081 & -0.2571 & -0.6292 & 0.4543 \\ 0.0083 & 0.0887 & 0.1864 & 0.2261 & -0.1375 & 0.0517 & -0.0523 & 0.2043 \\ -0.0030 & -0.0482 & -0.1442 & -0.1395 & 0.0139 & -0.0243 & -0.0338 & 0.1123 \\ -0.0010 & -0.4052 & -2.0874 & 0.6429 & -1.1149 & 0.4837 & -0.1793 & 0.8229 \\ 2.1179 & -1.9245 & 0.9545 & 0.2277 & -0.6083 & -0.3822 & 0.7209 & -0.6768 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \\ \tilde{x}_8 \end{pmatrix}$$

3.2.3. *Balanced Truncation Method* Since the reduced model is of order 6, the reduced model is as follows. The reduced model is obtained by removing the state variables  $\tilde{x}_7$  and  $\tilde{x}_8$  from the

balanced system.

$$\begin{pmatrix} \tilde{x}_1^r \\ \tilde{x}_2^r \\ \tilde{x}_3^r \\ \tilde{x}_4^r \\ \tilde{x}_5^r \\ \tilde{x}_6^r \end{pmatrix}_{k+1} = \begin{pmatrix} 0.9630 & 0.1108 & -0.0024 & -0.0087 & -0.0148 & -0.0011 \\ -0.1108 & 0.5633 & -0.4411 & 0.0630 & 0.1315 & 0.0017 \\ 0.0024 & -0.4411 & -0.8034 & -0.0835 & -0.1420 & -0.0096 \\ -0.0087 & -0.0630 & 0.0835 & 0.8972 & 0.3007 & -0.0489 \\ 0.0148 & 0.1315 & -0.1420 & -0.3007 & -0.0347 & 0.3154 \\ 0.0011 & 0.0017 & -0.0095 & 0.0489 & 0.3154 & -0.0100 \end{pmatrix} \begin{pmatrix} \tilde{x}_1^r \\ \tilde{x}_2^r \\ \tilde{x}_3^r \\ \tilde{x}_4^r \\ \tilde{x}_5^r \\ \tilde{x}_6^r \end{pmatrix}_k + \begin{pmatrix} -9.2733 \\ -12.2348 \\ -1.1801 \\ -1.3131 \\ 2.3704 \\ 0.1318 \end{pmatrix} \tilde{u}_k^r$$

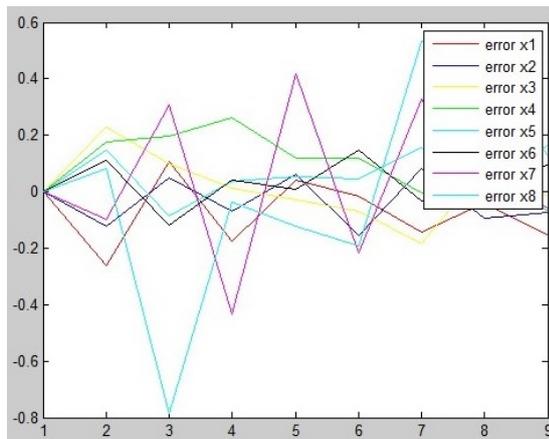
$$\tilde{y}_k^r = (9.2733 \quad -12.2348 \quad -1.1801 \quad 1.3131 \quad 2.3704 \quad 0.1318) \begin{pmatrix} \tilde{x}_1^r \\ \tilde{x}_2^r \\ \tilde{x}_3^r \\ \tilde{x}_4^r \\ \tilde{x}_5^r \\ \tilde{x}_6^r \end{pmatrix}_k + \tilde{u}_k^r$$

*3.2.4. Identification of State Variables in the Reduced System* In this section, we describe the relationship between state variables of the original system and the reduced system. This relationship is derived from the identification of state variables in the balanced system. More precisely, we remove the columns of transformation matrix  $T$  that corresponds to the removed state variables. The relationship between state variables of the original system and the reduced system is described in the following equation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} -2.4836 & 2.7670 & -1.0787 & -0.7136 & 5.0536 & 4.6166 \\ -0.0078 & -0.0038 & 0.2493 & -0.1043 & 0.4635 & -0.4272 \\ -0.1688 & -0.1837 & 0.3369 & 0.2326 & -0.0481 & 0.0043 \\ -0.0529 & -0.1288 & 0.1588 & -0.1614 & 0.4081 & -0.2571 \\ 0.0083 & 0.0887 & 0.1864 & 0.2261 & -0.1375 & 0.0517 \\ -0.0030 & -0.0482 & -0.1442 & -0.1395 & 0.0139 & -0.0243 \\ -0.0010 & -0.4052 & -2.0874 & 0.6429 & -1.1149 & 0.4837 \\ 2.1179 & -1.9245 & 0.9545 & 0.2277 & -0.6083 & -0.3822 \end{pmatrix} \begin{pmatrix} \tilde{x}_1^r \\ \tilde{x}_2^r \\ \tilde{x}_3^r \\ \tilde{x}_4^r \\ \tilde{x}_5^r \\ \tilde{x}_6^r \end{pmatrix}$$

*3.2.5. Simulation* In this simulation, we compare state variables on the original system and identified state estimation on reduced systems. The estimation of state variables on the reduced system is done by using Kalman filter algorithm. The procedure for estimation of identified state variables on the reduced system is as follows. First, we apply balanced truncation method to the original system. In this step, we obtain a reduced system. Then we apply Kalman filter algorithm on the reduced system. In this step, we obtain estimation of state variables on the reduced system. Finally, we use the identification result. In this step, we obtain the identified state estimation on the reduced system that has the same dimension with the original system. This allows us to compare the identified state estimation on the reduced system and the state on the original system. In this case, we determine the estimation error on the identified state variables. The estimation error is defined as the difference between the estimation results of identified state variables on the reduced system and real states on the original system. The comparison results are shown in Table 1 and the graphics are shown in Figure 1. From Table 1, the estimation error on the identified state estimation is quite close to the real values.

Furthermore we want to compare the estimation result in Table 1 and the estimation error of the original system. In Table 2, we display the estimation error of the original system. The estimation error is defined as the difference between the estimation results of state variables on the original system and real states on the original system. According to Tables 1 and 2,



**Figure 1.** Error of estimation results on the identified state estimation of the reduced system and state on the original system. The  $x$ -axis represents the time index, whereas the  $y$ -axis represents the error at each time. Each line plot represents a state component. Since dimension of the state space is 8, there are 8 plots in the figure.

**Table 1.** Estimation error of the identified reduced system.

time step	Estimation error of the identified reduced system							
	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
1	-0.508	-0.029	-0.041	-0.002	-0.069	0.0198	0.1637	0.4973
2	1.1773	0.0873	0.0631	0.0423	0.042	-0.059	-0.805	-0.778
3	0.8762	-0.204	-0.18	-0.056	0.0319	0.0646	0.9103	-0.799
4	-0.293	0.1614	0.1214	0.2225	0.136	-0.099	-0.233	0.7497
5	-0.46	-0.039	0.1279	0.1228	0.1459	0.0046	-0.299	0.601
6	-0.245	0.007	0.1883	0.1205	-0.078	0.0842	0.1452	-0.027
7	-0.154	-0.03	0.3448	0.1116	0.0751	-0.093	-0.126	-0.449
8	0.4889	-0.014	0.1985	0.2061	-0.049	-0.007	-0.448	-1.169

**Table 2.** Estimation Error of the Original system of all state variables for the first 8 time steps.

time step	Estimation error of the original system							
	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
1	-0.263	0.1048	-0.174	0.0414	-0.015	-0.141	-0.039	-0.157
2	-0.121	0.0486	-0.068	0.0603	-0.156	0.0841	-0.096	-0.074
3	0.2297	0.1002	0.0131	-0.029	-0.07	-0.186	0.0829	-0.018
4	0.1753	0.1961	0.2639	0.1207	0.1192	-0.005	0.0998	-0.071
5	0.1488	-0.088	0.0391	0.0549	0.0446	0.1556	-0.03	0.1687
6	0.1108	-0.118	0.0428	0.0105	0.1477	-0.033	-0.025	0.1005
7	-0.098	0.3075	-0.434	0.4199	-0.216	0.3272	0.0301	-0.056
8	0.0831	-0.787	-0.036	-0.123	-0.191	0.5306	0.4268	0.1269

the estimation error of both cases is pretty close. Thus we conclude that the performance of estimation on the identified state variables is acceptable.

In Table 3, we discuss the computational times of both cases. In the first case, we measure the computational time for estimation of state variables on the original system. In the second case, we measure the computational time for estimation of identified state variables. According

to Table 3, the computational time for estimation in the reduced system is faster than in the original system. This is because the number of state variables in the reduced system is smaller than the number of state variables in the original system.

**Table 3.** Computational times for the estimation process in the original and reduced systems.

Computational time (sec)	
Original system	5.21722
Identified reduced system	3.164235

#### 4. Conclusions

In this work, we have designed a procedure for identification and estimation of state variables on reduced models of discrete-time linear-time-invariant systems. From the identification process, we obtain the corresponding state variables in the original model from state variables in the reduced model via a linear transformation. Thus we can compare the state variables in the identified reduced model and original model. The estimation error in the original system is quite close to the error in the identified reduced system. However, the computational time for estimation in the reduced system is faster than the computational time for estimation in the original model.

#### Acknowledgments

This work has been supported by:

- Penelitian Doktor Baru number 31040/IT2.11/PN.08/2016 with the title of Konstruksi Algoritma Filter Kalman Tereduksi Pada Sistem Tidak Stabil Dan Aplikasinya Pada Aliran Air Sungai;
- Penelitian Unggulan Perguruan Tinggi year 2017 with the title of Estimasi Ketinggian dan Kecepatan Aliran Sungai dengan Filter Kalman Tereduksi Metode Residual.

#### References

- [1] Green M and Limebeer D 2012 *Linear Robust Control* (Dover Publications)
- [2] Sari Y, Arif D and Adzkiya D 2016 *International Conference on Mathematics: Pure, Applied and Computation*
- [3] Kartika D, Arif D and Adzkiya D 2016 *International Conference on Mathematics: Pure, Applied and Computation*
- [4] Lewis F 1986 *Optimal Estimation* (John Wiley and Sons)
- [5] Fitria R and Arif D 2017 *International Journal of Computing Science and Applied Mathematics* **3** 16–20
- [6] Arif D, Widodo, Apriliani E and Salmah 2011 *3rd International Conferences and Workshops on Basic and Applied Sciences* p M007
- [7] Arif D, Widodo, Salmah and Apriliani E 2012 *Proceedings of the 8th International Conference on Mathematics, Statistics and Its Applications*
- [8] Arif D, Widodo, Salmah and Apriliani E 2014 *International Journal of Control and Automation* **7** 257–270