The Construction of Labeling and Total Irregularity Strength of Specified Caterpillar Graph

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Abstract. Let G be a simple, connected and undirected graph with vertex set V and edge set E. A total k-labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ is defined as totally irregular total k-labeling if the weights of any two different both vertices and edges are distinct. The weight of vertex x is defined as $wt(x) = f(x) + \sum_{xy \in E} f(xy)$, while the weight of edge xy is $wt(xy) = f(x) + f(xy) + f(y)$. A minimum k for which G has totally irregular total k-labeling is mentioned as total irregularity strength of G and denoted by $\mathrm{ts}(G)$. This paper contains investigation of totally irregular total k-labeling for caterpillar graphs $S_{n,2,m}$ and determination of their total irregularity strengths. In addition, the total vertex and total edge irregularity strength of this graph also be determined. The results are $tvs(S_{n,2,m}) = \lceil \frac{n+m-1}{2} \rceil$, $tes(S_{n,2,m}) = \lceil \frac{n+m+2}{3} \rceil$, and $ts(S_{n,2,m}) = \lceil \frac{n+m-1}{2} \rceil$ for $n, m \geq 3$.

1. Introduction

Let us consider a simple, connected and undirected graph G with a vertex set $(V(G))$ and an edge set $(E(G))$. A labeling of a graph G is a mapping that carries a set of graph elements into a set of integers, called labels (see Wallis [10]). If the domain of mapping is a vertex set, or an edge set, or a union of vertex and edge sets, then the labeling is called *vertex labeling*, *edge labeling*, or *total labeling*, respectively. In his survey, Gallian [2] shows that there are various kinds of labelings on graphs, and one of them is an irregular total labeling.

For a graph G, Bača *et al.* [1] defined a labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be a *vertex irregular total k-labeling* if for every two different vertices x and y the *vertex-weights* $wt_f(x) \neq wt_f(y)$, where the vertex-weight $wt_f(x) = f(x) + \sum_{xz \in E} f(xz)$. A minimum k for which G has a vertex irregular total k-labeling is defined as the *total vertex irregularity strength* of G and denoted by $tvs(G)$. They obtained the exact values of the total vertex irregularity strength for cycle, star, complete graphs and prisms. Moreover, Nurdin *et al.* [7] proved the exact value of the total vertex irregularity strength for any tree T with n pendant vertices and no vertex of degree two, that is

$$tvs(T) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{1}$$

For a graph G, Bača *et al.* [1] also define a labeling $g : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G the *edge-weights* $wt_g(xy) = g(x) + g(xy) + g(y)$ and $wt_g(x'y') = g(x') + g(x'y') + g(y')$ are different. The *total edge irregularity strength* denoted by $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. They also obtained the exact values of the *tes* for path, cycle, star, wheel and friendship graphs. The *tes* of generalized web graphs have been determined by Indriati *et al.* [3]. Moreover, Ivančo and Jendrol [5] proved that for any tree T, satisfy

$$tes(T) = \max\left\{\lceil(|E(T)|+2)/3\rceil, \lceil(\Delta(T)+1)/2\rceil\right\}. \tag{2}$$

Combining the ideas of vertex irregular total k-labeling and edge irregular total k-labeling, Marzuki *et al.* [6] introduced another irregular total k-labeling called the totally irregular total k-labeling. A labeling $h : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be a *totally irregular total k-labeling* of the graph G if for every two different vertices x and y the *vertex-weights* $wt_h(x) \neq wt_h(y)$, where the vertex-weight $wt_h(x) = h(x) + \sum_{xz\in E} h(xz)$ and also for every two different edges xy and $x'y'$ of G the *edge-weights* $wt_h(xy) = h(x) + h(xy) + h(y)$ and $wt_h(x'y') = h(x') + h(x'y') + h(y')$ are different. The *total irregularity strength*, $ts(G)$, is defined as the minimum k for which G has a totally irregular total k-labeling. For the total irregularity strength of a graph G, they observed that

$$ts(G) \geq \max\{tes(G), tvs(G)\}. \tag{3}$$

They determined the total irregularity strength of cycle and path. Ramdani and Salman [8] obtained the total irregularity strength of some cartesian product graphs, namely $K_{1,n}\Box P_2$, $P_n\Box P_2$, $(P_n + P_1)\Box P_2$, and $C_n\Box P_2$. In [9], Ramdani *et al.* determined the total irregularity strength of Gear graphs $G_n, n \geq 3$, fungus graphs $F_{g_n}, n \geq 3$ and disjoint union of stars $mS_n, n, m \geq 2$. In [4], the total irregularity strength of double stars $S_{n,m}$ and caterpillar $S_{n,2,n}$ has been determined. In this paper, we continue to investigate the total irregularity strength of caterpillar $S_{n,2,m}$ for $n, m \geq 3$.

2. Caterpillar $S_{n,2,m}$

A caterpillar $S_{n,2,m}$ is a class of graph constructed from the double-star $S_{n,m}$ by inserting one vertex on the bridge connecting of the two centers of two stars. Therefore, this caterpillar contains three stars with the center of the two end-stars have degree n and m respectively, while the center of the middle star has degree two. This graph is a tree with $n + m + 1$ vertices, $n + m$ edges and $n + m - 2$ pendant vertices. Assume $n \leq m$. If $n > m$, then change n by m, for example $S_{7,2,5}$ can be written by $S_{5,2,7}$. Maximal degree of the graph is $\Delta = m$.
According to (3), the lower bound of its total irregularity strength is the maximum value between its total edge irregularity strength and its total vertex irregularity strength. The total edge irregularity strength of graph $S_{n,2,m}$ can be found by (2), that is

$$tes(S_{n,2,m}) = \max\left\{\left\lceil\frac{\Delta+1}{2}\right\rceil, \left\lceil\frac{|E|+2}{3}\right\rceil\right\} = \max\left\{\left\lceil\frac{m+1}{2}\right\rceil, \left\lceil\frac{n+m+2}{3}\right\rceil\right\} = \left\lceil\frac{n+m+2}{3}\right\rceil. \tag{4}$$

The next theorem gives the total vertex irregularity strength of $S_{n,2,m}$.

Theorem 2.1. *Let consider the caterpillar $S_{n,2,m},\ n, m \geq 3$. Its total vertex irregularity strength is*

$$tvs(S_{n,2,m}) = \left\lceil\frac{n+m-1}{2}\right\rceil.$$

Proof. $S_{n,2,m}$ is a tree with $n + m + 1$ vertices, $n + m$ edges and $n + m - 2$ pendant vertices. There is a vertex (that is a center of middle star) has degree two, therefore (1) can not be used for determining the total vertex irregularity strength of the graph. $S_{n,2,m}$ has $n + m - 2$ pendant vertices, one vertex of degree two and two vertices of degree $n, m \geq 3$ respectively. In order to obtain as small as possible label, start the labeling from the vertex with smallest degree (in this situation, pendant vertices have a smallest degree). After that, we label vertices with greater degree and continue until all vertices are labeled. The smallest vertex-weight is two, then with a consecutive weights, the smallest weight of $n+m-2$ pendant vertices is not smaller than $n+m-2+1 = n+m-1$ which is a sum of two labels, namely the label of pendant vertex and the label of edge incident to this vertex. Then, the greatest label of pendant vertices is not smaller than $\lceil \frac{n+m-1}{2} \rceil$. There is a vertex of degree two, therefore the greatest vertex-label is not smaller than $\lceil \frac{n+m}{3} \rceil$. For vertex of degree n, the greatest label is not smaller than $\left\lceil \frac{n+m+1}{n+1} \right\rceil$. The same idea, for vertex of the largest degree, m, the greatest label is not smaller than $\left\lceil \frac{n+m+2}{m+1} \right\rceil$. Therefore, the greatest label of all vertices is not smaller than

$$\max\left\{ \left\lceil \frac{n+m-1}{2} \right\rceil, \left\lceil \frac{n+m}{3} \right\rceil, \left\lceil \frac{n+m+1}{n+1} \right\rceil, \left\lceil \frac{n+m+2}{m+1} \right\rceil \right\} = \left\lceil \frac{n+m-1}{2} \right\rceil.$$

Let the vertex set of this graph be $V(S_{n,2,m}) = \{v_i^1 : 1 \le i \le m-1\} \cup \{v_i^3 : 1 \le i \le n-1\} \cup \{v^j : j =$

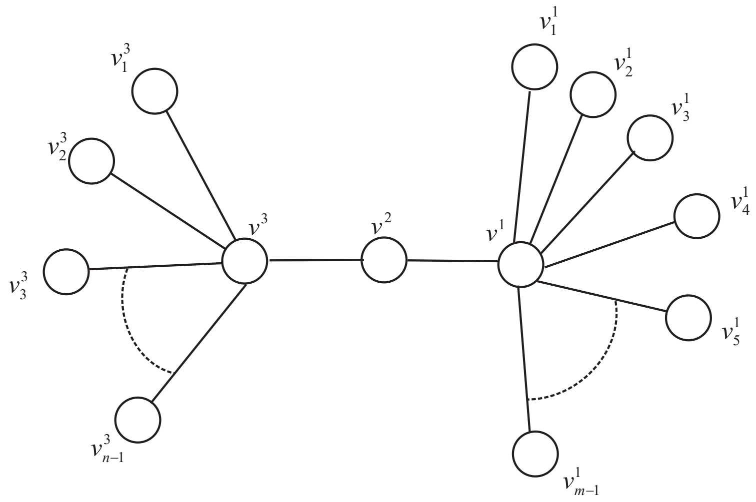

Figure 1. *The caterpillar* $S_{n,2,m}$

$1, 2, 3\}$ and the edge set be $E(S_{n,2,m}) = \{v^1 v_i^1 : 1 \le i \le m-1\} \cup \{v^3 v_i^3 : 1 \le i \le n-1\} \cup \{v^j v^{j+1} : j = 1, 2\}$. Figure 1 shows the illustration of this graph. To determine the exact value of tvs, define the vertex irregular total k-labeling h as follows.
Case 1: For both n and m are odd or even.
Assume $k = \lceil \frac{n+m-1}{2} \rceil$.

$$h(v_i^1) = \begin{cases} 1, & \text{for } 1 \le i \le k, \\ i-k+1, & \text{for } k+1 \le i \le m-1. \end{cases}$$

$$h(v_i^3) = k+i-n, \text{for } 1 \le i \le n-1.$$

$$h(v^j) = \begin{cases} k-1, & \text{for } j = 1, \\ k, & \text{for } j = 2, 3. \end{cases}$$

$$h(v^1 v_i^1) = \begin{cases} i, & \text{for } 1 \le i \le k, \\ k, & \text{for } k+1 \le i \le m-1. \end{cases}$$

$$h(v^3 v_i^3) = k, \text{for } 1 \le i \le n-1.$$

$$h(v^j v^{j+1}) = \begin{cases} k-n+1, & \text{for } j = 1, \\ k, & \text{for } j = 2. \end{cases}$$

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Under the total labeling h, it is shown that the greatest label for all vertices is $k = \lceil \frac{n+m-1}{2} \rceil$. It means that h is a total k-labeling with $k = \lceil \frac{n+m-1}{2} \rceil$. The weight of vertices are

$$wt_h(v_i^j) = \begin{cases} i+1, & \text{for } 1 \le i \le m-1,\ j=1, \\ m+i, & \text{for } 1 \le i \le n-1,\ j=3. \end{cases}$$

$$wt_h(v^j) = \begin{cases} \frac{k(k+1)}{2} + (m+1-k)k - n, & \text{for } j=1, \\ 3k-n+1, & \text{for } j=2, \\ (n+1)k, & \text{for } j=3. \end{cases}$$

The weight of vertices v_i^j for j=1, 3 form a consecutive integers from 2 up to m and for $m+1$ until $m+n-1$, respectively. The weights among vertices v^1, v^2, and v^3 are distinct. Then, it indicates that the weights of every pair of vertices are distinct. Therefore, we conclude that h is a vertex irregular total k-labeling and the total vertex irregularity strength is $tvs(S_{n,2,m}) = k = \lceil \frac{n+m-1}{2} \rceil$.

Case 2: For n and m have a different parity and odd number less than even number.
Assume $k = \lceil \frac{n+m-1}{2} \rceil$. Define the vertex irregular total k-labeling h as follows.

$$h(v_i^1) = \begin{cases} 1, & \text{for } 1 \le i \le k, \\ i-k+1, & \text{for } k+1 \le i \le m-1. \end{cases}$$

$$h(v_i^3) = k+i-n+1, \text{for } 1 \le i \le n-1.$$

$$h(v^j) = \begin{cases} k-2, & \text{for } j=1, \\ k, & \text{for } j=3. \end{cases}$$

$$h(v^2) = \begin{cases} k-1, & \text{for odd number} = 3, \\ 2k-2n+2, & \text{for odd number} \neq 3. \end{cases}$$

$$h(v^1v_i^1) = \begin{cases} i, & \text{for } 1 \le i \le k, \\ k, & \text{for } k+1 \le i \le m-1. \end{cases}$$

$$h(v^3v_i^3) = k, \text{for } 1 \le i \le n-1.$$

$$h(v^1v^2) = \begin{cases} k, & \text{for odd number} = 3, \\ n, & \text{for odd number} \neq 3. \end{cases}$$

$$h(v^2v^3) = \begin{cases} k-1, & \text{for odd number} = 3, \\ n-1, & \text{for odd number} \neq 3. \end{cases}$$

Under total labeling h, it is shown that the greatest label for all vertices is $k = \lceil \frac{n+m-1}{2} \rceil$. It means that

h is a total k-labeling with $k = \lceil \frac{n+m-1}{2} \rceil$. The weights of vertices are

$$wt_h(v_i^j) = \begin{cases} i+1, & \text{for } 1 \le i \le m-1,\ j=1, \\ 2k+i-n+1, & \text{for } 1 \le i \le n-1,\ j=3. \end{cases}$$

$$wt_h(v^1) = \begin{cases} \frac{k(k+1)}{2} + (m+1-k)k-2, & \text{for odd number} = 3, \\ \frac{k(k+1)}{2} + (m-k)k+n-2, & \text{for odd number} \neq 3. \end{cases}$$

$$wt_h(v^2) = \begin{cases} 3k-2, & \text{for odd number} = 3, \\ 2k+1, & \text{for odd number} \neq 3. \end{cases}$$

$$wt_h(v^3) = \begin{cases} 4k-1, & \text{for odd number} = 3, \\ (k+1)n-1 & \text{for odd number} \neq 3. \end{cases}$$

With the same reason as in Case 1, the weights of vertices v_i^j for j=1, 3 form a consecutive integers from 2 up to m and for $m+1$ until $m+n-1$, respectively and the weights among vertices v^1, v^2, and v^3 are distinct. Therefore, the weight of every pair of vertices are distinct. We conclude that h is a vertex irregular total k-labeling and the total vertex irregularity strength is $tvs(S_{n,2,m}) = k = \lceil \frac{n+m-1}{2} \rceil$.

Case 3: For n and m have a different parity and odd number more than even number.
Assume $k = \lceil \frac{n+m-1}{2} \rceil$. The vertex irregular total k-labeling h is defined as follows.

$$h(v_i^1) = \begin{cases} 1, & \text{for } 1 \le i \le k, \\ i-k+1, & \text{for } k+1 \le i \le m-1. \end{cases}$$

$$h(v_i^3) = k+i-n+1, \text{for } 1 \le i \le n-1.$$

$$h(v^j) = \begin{cases} k-2, & \text{for } j=1, \\ k, & \text{for } j=3. \end{cases}$$

$$h(v^2) = \begin{cases} |m-n|+1, & \text{for } 3n-m \ge 3, \\ k, & \text{for } 3n-m < 3. \end{cases}$$

$$h(v^1v_i^1) = \begin{cases} i, & \text{for } 1 \le i \le k, \\ k, & \text{for } k+1 \le i \le m-1. \end{cases}$$

$$h(v^3v_i^3) = k, \text{for } 1 \le i \le n-1.$$

$$h(v^1v^2) = \begin{cases} n, & \text{for } 3n-m \ge 3, \\ k-n+2, & \text{for } 3n-m < 3. \end{cases}$$

$$h(v^2v^3) = \begin{cases} n-1, & \text{for } 3n-m \ge 3, \\ k-n+1, & \text{for } 3n-m < 3. \end{cases}$$

Under total labeling h, it is shown that the greatest label for all vertices is $k = \lceil \frac{n+m-1}{2} \rceil$. It means that

h is a total k-labeling with $k = \lceil \frac{n+m-1}{2} \rceil$. The weights of vertices are

$$wt_h(v_i^j) = \begin{cases} i+1, & \text{for } 1 \le i \le m-1,\ j=1, \\ 2k+i-n+1, & \text{for } 1 \le i \le n-1,\ j=3. \end{cases}$$

$$wt_h(v^1) = \begin{cases} \frac{k(k+1)}{2} + (m-k)k + n - 2, & \text{for } 3n-m \ge 3, \\ \frac{k(k+1)}{2} + (m+1-k)k - n, & \text{for } 3n-m < 3. \end{cases}$$

$$wt_h(v^2) = \begin{cases} 2n + |m-n|, & \text{for } 3n-m \ge 3, \\ 3k-2n+3, & \text{for } 3n-m < 3. \end{cases}$$

$$wt_h(v^3) = \begin{cases} nk+n-1, & \text{for } 3n-m \ge 3, \\ (n+1)k-n+1, & \text{for } 3n-m < 3. \end{cases}$$

With the same reason as in Case 2, the weight of vertices v_i^j for j=1, 3 form a consecutive integers from 2 up to m and for $m+1$ until $m+n-1$, respectively and the weight among vertices v^1, v^2, and v^3 are distinct. It means that the weight of every pair of vertices are distinct. Therefore, h is a vertex irregular total k-labeling. The total vertex irregularity strength of this graph is $tvs(S_{n,2,m}) = k = \lceil \frac{n+m-1}{2} \rceil$.
From the three cases, it is shown that the total vertex irregularity strength of caterpillar $S_{n,2,m}$ is $\lceil \frac{n+m-1}{2} \rceil$. □

The next theorem proved the total irregularity strength of graph $S_{n,2,m}$ as follows.

Theorem 2.2. *Let $S_{n,2,m},\ n,m \ge 3$ be a caterpillar graph. Then the total irregularity strength of this graph is*

$$ts(S_{n,2,m}) = \left\lceil \frac{n+m-1}{2} \right\rceil.$$

Proof. With the same statement as in Theorem 2.1, $S_{n,2,m}$ is a tree with $n+m+1$ vertices, $n+m$ edges and $n+m-2$ pendant vertices. Theorem 2.1 proved that $tvs(S_{n,2,m}) = \lceil \frac{n+m-1}{2} \rceil$. Total edge irregularity strength of this graph is in (4). According to (3), the lower bound of the total irregularity strength is

$$ts(S_{n,2,m}) \ge \max\{tes(S_{n,2,m}), tvs(S_{n,2,m})\} = \max\left\{ \left\lceil \frac{n+m+2}{3} \right\rceil, \left\lceil \frac{n+m-1}{2} \right\rceil \right\} = \left\lceil \frac{n+m-1}{2} \right\rceil.$$

To prove the exact value of the total irregularity strength of this graph, $ts(G)$, we will show the existence of the totally irregular total k-labeling as follows. The similar definition of vertex and edge set of $S_{n,2,m}$ is presented in Theorem 2.1. In fact, the vertex irregular total k-labeling h which is obtained in Theorem 2.1 also satisfies the condition of totally irregular total k-labeling. The vertex weight distinction has been shown in the proof of Theorem 2.1. Now, we only show the distinction of the edge weight of this graph. By assume $k = \lceil \frac{n+m-1}{2} \rceil$, from the vertex irregular total k-labeling h in Theorem 2.1, the weight of the edges are as follows.

$$wt_h(v^1 v_i^1) = \begin{cases} k+i, & \text{for } 1 \le i \le m-1, \text{ both } n \text{ and } m \text{ are odd or even,} \\ k-1+i, & \text{for } 1 \le i \le m-1, \ n \text{ and } m \text{ have a different parity} \end{cases}$$

$$wt_h(v^3 v_i^3) = \begin{cases} 3k-n+i, & \text{for } 1 \le i \le n-1, \text{ both } n \text{ and } m \text{ are odd or even,} \\ 3k-n+i+1, & \text{for } 1 \le i \le n-1, \ n \text{ and } m \text{ have a different parity} \end{cases}$$

For the case 1: both n and m are odd or even.

$$wt(v^j v^{j+1}) = \begin{cases} 3k-n, & \text{for } j=1, \\ 3k, & \text{for } j=2. \end{cases}$$

For the case 2: n and m have a different parity, odd number $<$ even number.

$$wt(v^j v^{j+1}) = \begin{cases} 3k-n, & \text{for } j=1, \\ 3k-n+1, & \text{for } j=2. \end{cases}$$

For the case 3: n and m have a different parity, odd number $>$ even number.

$$wt(v^1 v^2) = \begin{cases} k+n+|m-n|-1, & \text{for } 3n-m \geq 3, \\ 3k-n, & \text{for } 3n-m < 3. \end{cases}$$

$$wt(v^2 v^3) = \begin{cases} k+n+|m-n|, & \text{for } 3n-m \geq 3, \\ 3k-n+1, & \text{for } 3n-m < 3. \end{cases}$$

It can be seen that for the case 1, that is both n and m are odd or even, the weights of $v^1v_i^1$ and $v^3v_i^3$ form a consecutive integers from $k+1$ up to $k+m-1$ and for $k+m+1$ until $k+m+n-1$, respectively. While for the case 2 and 3, that is n and m have a different parity, the weights of $v^1v_i^1$ and $v^3v_i^3$ form a consecutive integers from k up to $k+m-2$ and for $k+m+1$ until $k+m+n-1$, respectively. For the case 1, the weight of v^1v^2 is $k+m$ and the weight of v^2v^3 is $k+m+n$. While for the case 2, the weight of v^1v^2 is $3k-n$ and the weight of v^2v^3 is $3k-n+1$. Same as another cases, the weights of v^1v^2 and v^2v^3 for the case 3 also different. Therefore, for all cases, the weight of each pair of edges are distinct. Then we conclude that h is also totally irregular total k-labeling. Therefore, we determine that the total irregularity strength of this graph is $\left\lceil \frac{n+m-1}{2} \right\rceil$. $\square$

Furthermore, we conclude this paper with the following open problem for the direction of further research which is still in progress.
Open problem: What is the total irregularity strength of caterpillar $S_{n,2,2,\ldots,m}$ for t-times of 2's.

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