

# On The Partition Dimension of a Lollipop Graph and a Generalized Jahangir Graph

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**Abstract.** Let  $G$  be a connected graph with vertex set  $V(G)$ , such that  $V(G)$  can be divided into any partition set  $S$ . The set  $\Pi$  with  $S \in \Pi$  is a resolving partition of  $G$  if each vertex in  $G$  has a distinct representation with respect to  $\Pi$ , and  $\Pi$  is an ordered  $k$ -partition. The minimum cardinality of resolving  $k$ -partitions of  $V(G)$  is called a partition dimension of  $G$ , denoted by  $pd(G)$ . The lollipop graph  $L_{m,n}$  is a graph obtained by joining a complete graph  $K_m$  to a path  $P_n$  with a bridge. A generalized Jahangir graph is a graph consisting of a cycle  $C_{mn}$  and one additional vertex which is adjacent to  $n$  vertices of  $C_{mn}$  at  $m$  distance to each other on  $C_{mn}$ . Many researchers have conducted research in determining the partition dimension for specific graph classes. They are as references to determine some of the graph classes that haven't been studied previously.

In this paper, we determine the partition dimension of a lollipop graph  $L_{m,n}$  and a generalized Jahangir graph  $J_{m,n}$ . The research methods in this paper is a book study.

The results of this paper are as follows. We obtain the partition dimension of a lollipop graph  $pd(L_{m,n}) = m$  for  $m \geq 3$  and  $n \geq 1$ . The partition dimension of a generalized Jahangir graph consists of two cases. We showed that  $pd(J_{m,n}) = 3$  for  $n = 3, 4, 5$  and we prove that  $pd(J_{m,n}) = \lfloor \frac{n}{2} \rfloor + 1$  for  $n \geq 6$ .

## 1. Introduction

The partition dimension of a graph was introduced by Chartrand et al. [1] in 1998. There is another concept of metric dimension form that was introduced by Slater in 1975 and Harary and Melter in 1976 (Darmaji [3]). We assume that  $G$  is a graph with a vertex set  $V(G)$ , such that  $V(G)$  can be divided into any partition set  $S$ . The set  $\Pi$  with  $S \in \Pi$  is a resolving partition of  $G$  if each vertex in  $G$  has a distinct representation with respect to  $\Pi$ , and  $\Pi$  is an ordered  $k$ -partition. The minimum cardinality of resolving  $k$ -partitions of  $V(G)$  is called a partition dimension of  $G$ , denoted by  $pd(G)$ .

Many researchers have conducted research in determining the partition dimension for specific graph classes. Tomescu et al. [5] in 2007 found the partition dimension of a wheel graph. In this paper, we determine the partition dimension of a lollipop graph and a generalized Jahangir graph.



## 2. Main Result

### 2.1. Partition Dimension

The following definition and lemma were given by Chartrand et al. [2]. Let  $G$  be a connected graph. For a subset  $S$  of  $V(G)$  and a vertex  $v$  of  $G$ , the distance  $d(v, S)$  between  $v$  and  $S$  is defined as  $d(v, S) = \min\{d(v, x) | x \in S\}$ . For an ordered  $k$ -partition  $\Pi = \{S_1, S_2, \dots, S_k\}$  of  $V(G)$  and a vertex  $v$  of  $G$ , the representation of  $v$  with respect to  $\Pi$  is defined as the  $k$ -vector  $r(v|\Pi) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k))$ . The partition  $\Pi$  is called a resolving partition if the  $k$ -vectors  $r(v|\Pi), v \in V(G)$  are distinct. The minimum  $k$  for which there is a resolving  $k$ -partition of  $V(G)$  is called the partition dimension of  $G$ , denoted by  $pd(G)$ .

**Lemma 2.1** [2] *Let  $G$  be a connected graph, then*

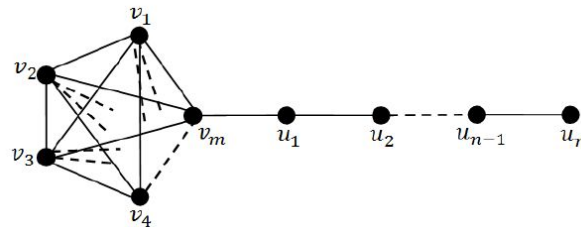
- (i)  $pd(G) = 2$  if and only if  $G = P_n$  for  $n \geq 2$ ,
- (ii)  $pd(G) = n$  if and only if  $G = K_n$ , and
- (iii)  $pd(G) = 3$  if  $G = n$ -cycle for  $n \geq 3$ .

**Lemma 2.2** [2] *Let  $\Pi$  be a resolving partition of  $V(G)$  and  $u, v \in V(G)$ . If  $d(u, w) = d(v, w)$  for all  $w \in V(G) - \{u, v\}$ , then  $u$  and  $v$  belong to distinct elements of  $\Pi$ .*

Let  $\Pi = \{S_1, S_2, \dots, S_k\}$ , where  $u$  and  $v$  belong to the same element, say  $S_i$ , of  $\Pi$ . Then  $d(u, S_i) = d(v, S_i) = 0$ . Since  $d(u, w) = d(v, w)$  for all  $w \in V(G) - \{u, v\}$ , we also have that  $d(u, S_j) = d(v, S_j)$  for all  $j$ , where  $1 \leq j \neq i \leq k$ . Therefore,  $r(u|\Pi) = r(v|\Pi)$  and  $\Pi$  is not a resolving partition.

### 2.2. The Partition Dimension of a Lollipop Graph

Weisstein [6] defined the lollipop graph  $L_{m,n}$  for  $m \geq 3$  as a graph obtained by joining a complete graph  $K_m$  to a path  $P_n$  with a bridge. See for example, the lollipop graph in Figure 1.



**Figure 1.** The Lollipop Graph

**Theorem 2.1** *Let  $L_{m,n}$  be a lollipop graph with  $m \geq 3$  and  $n \geq 1$ , then  $pd(L_{m,n}) = m$ .*

Let  $V(L_{m,n}) = V(K_m) \cup V(P_n)$ , where  $V(K_m) = \{v_1, v_2, \dots, v_{m-1}, v_m\}$  and  $V(P_n) = \{u_1, u_2, \dots, u_{n-1}, u_n\}$ . Take a vertex  $v \in V(K_m)$  such that  $v$  is adjacent to  $u \in V(P_n)$ . Based on Lemma 2.1, the partition dimension of  $L_{m,n}$  is  $pd(L_{m,n}) \geq m$ . Next, we show that the partition dimension of  $L_{m,n}$  is  $pd(L_{m,n}) = m$ . For each  $v, u \in V(L_{m,n})$ , let  $\Pi = \{S_1, S_2, \dots, S_m\}$  be the resolving partition where  $S_i = \{v_i\}$  for  $1 \leq i \leq m-1$  and  $S_m = \{v_m, u_1, \dots, u_n\}$ . We obtain the representation between all vertices of  $V(L_{m,n})$  with respect to  $\Pi$  as follows  $r(v_m|\Pi) = (d(v_1, S_i), \dots, d(v_m, S_i))$  and  $r(u_n|\Pi) = (n, n+1, n+1, \dots, n+1, 0)$  where

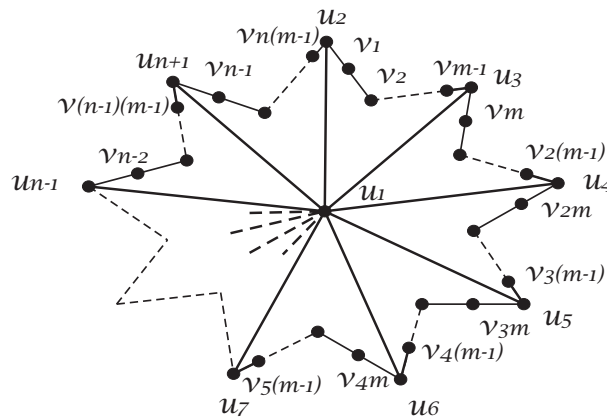
$$d(v_m, S_i) = \begin{cases} 0, & \text{for } i = m, \\ 1, & \text{for } i \neq m. \end{cases}$$

Therefore, each vertex  $v \in V(L_{m,n})$  has a distinct representation with respect to  $\Pi$ . Now, we

show that the cardinality of  $\Pi$  is  $m$ . The cardinality of  $\Pi$  is the number of  $k$ -partition in  $S_i$  and  $S_m$  for  $1 \leq i \leq m-1$ . The number of  $k$ -partition in  $S_i$  for  $1 \leq i \leq m-1$ , is  $m-1$ . The number of  $k$ -partition in  $S_m$  is 1, so we have  $|\Pi| = (m-1) + 1 = m$ . Furthermore,  $\Pi = \{S_1, S_2, \dots, S_m\}$  is a resolving partition of  $L_{m,n}$  with  $m$  elements. Hence, the partition dimension of lollipop graph is  $pd(L_{m,n}) = m$ .

### 2.3. The Partition Dimension of a Generalized Jahangir Graph

Mojdeh and Ghameshlou [4] defined a generalized Jahangir graph  $J_{m,n}$  with  $n \geq 3$  as a graph on  $nm+1$  vertices consisting of a cycle  $C_{mn}$  and one additional vertex which is adjacent to  $n$  vertices of  $C_{mn}$  at  $m$  distance to each other on  $C_{mn}$ . Let  $V(J_{m,n}) = V(C_{mn}) \cup \{u_1\}$  or  $V(J_{m,n}) = V(P_m) \cup V(W_n)$  where  $V(P_m) = \{v_1, v_2, \dots, v_{n(m-1)}\}$  and  $V(W_n) = \{u_1, u_2, \dots, u_{n+1}\}$ , we have  $V(J_{m,n}) = \{u_1, u_2, \dots, u_{n+1}, v_1, v_2, \dots, v_{n(m-1)}\}$ . Example of a generalized Jahangir graph is in Figure 2.



**Figure 2.** The Generalized Jahangir Graph

**Lemma 2.3** Let  $G$  be a subgraph of a generalized Jahangir graph  $J_{m,n}$  for  $m \geq 3$ ,  $n \geq 3$  and  $\Pi$  is a resolving partition of  $G$ . The distance of vertex  $u_1$  to the other vertices are  $d(u_1, u_i) = 1$  and  $d(u_1, v_k) = 2$  for  $1 < i \leq n+1$  and  $1 \leq k \leq n(m-1)$ . The distance of vertex  $u$  to the other vertex  $u$  are the same,  $d(u_i, u_j) = 2$  for  $1 < i \leq n+1$ ,  $1 < j \leq n+1$  and  $i \neq j$ .

Let  $u_1$  be a central vertex of  $V(G) = V(J_{m,n})$ . Since each vertex  $u_i$  is connected with central vertex, so  $d(u_1, u_i) = 1$  and  $d(u_i, u_j) = 2$  with  $1 < i \leq n+1$ ,  $1 < j \leq n+1$ , and  $i \neq j$ . If vertex  $v_k$  is a vertex of path  $P_m$  and has a minimum distance to vertex  $u_i$ , then  $d(u_1, v_k) = 2$  for  $1 \leq k \leq n(m-1)$ .

**Theorem 2.2** Let  $J_{m,n}$  be a generalized Jahangir graph for  $m \geq 3$  and  $n \geq 3$  then

$$pd(J_{m,n}) = \begin{cases} 3, & \text{for } n = 3, 4, 5; \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{for } n \geq 6. \end{cases}$$

Let  $J_{m,n}$  be a generalized Jahangir graph for  $m \geq 3$ ,  $n \geq 3$  and a vertex set  $V(J_{m,n})$ . The proof can be divided into two cases according to the values of  $n$ .

(i) Case  $n = 3, 4, 5$

(a) For  $n = 3$

Let  $a = \{v_1, \dots, v_m\}$ ,  $b = \{v_1, \dots, v_{m-1}\}$ ,  $c = \{v_{m+1}, \dots, v_{n(m-1)}\}$ , and  $d = \{v_m, \dots, v_{2(m-1)}\}$ , then

$$S_1 = \{u_1, \dots, u_{n+1}, v_{2m-1}, \dots, v_{mn-3}\},$$

$$S_2 = \begin{cases} a, & \text{for } m = 3; \\ b, & \text{for } m \geq 4; \end{cases} \quad S_3 = \begin{cases} c, & \text{for } m = 3; \\ d, & \text{for } m \geq 4. \end{cases}$$

(b) For  $n = 4$

Let  $a = \{u_1, \dots, u_{n+2}, v_{(n(m-1))-1}\}$ ,

$b = \{u_1, \dots, u_{n+2}, v_{3m-2}, \dots, v_{n(m-1)}\}$ ,

$c = \{u_1, \dots, u_{n+2}, v_{2m-1}, v_{3m-2}, \dots, v_{n(m-1)}\}$ ,

$d = \{v_1, \dots, v_m\}$ ,

$e = \{v_1, \dots, v_{m-1}, v_{2(m+1)}, v_{m(n-1)-3}\}$ ,

$f = \{v_1, \dots, v_{m-1}, v_{m(n-1)-3}\}$ ,

$g = \{v_{m+1}, \dots, v_{n(m-1)}\}$ ,

$h = \{v_m, \dots, v_{2(m-1)}, v_{2m-1}, \dots, v_{(m-1)(n-1)-1}\}$ ,

$i = \{v_m, \dots, v_{2(m-1)}, v_{2m}, \dots, v_{(m-1)(n-1)-1}\}$ , and

$j = \{v_m, \dots, v_{2(m-1)}, v_{2m}, v_{2m+1}, v_{2m+3}, \dots, v_{(m-1)(n-1)-1}\}$  then

$$S_1 = \begin{cases} a, & \text{for } m = 3; \\ b, & \text{for } m = 5; \\ c, & \text{for } m \geq 4, m \neq 5; \end{cases} \quad S_2 = \begin{cases} d, & \text{for } m = 3; \\ e, & \text{for } m = 9; \\ f, & \text{for } m \geq 4, m \neq 9; \end{cases}$$

$$S_3 = \begin{cases} g, & \text{for } m = 3; \\ h, & \text{for } m = 5; \\ i, & \text{for } m \geq 4, m \neq 5, 9; \\ j, & \text{for } m = 9. \end{cases}$$

For  $n = 5$  is the same as above. The representations of  $r(u|\Pi)$  and  $r(v|\Pi)$ , with  $u, v \in V(J_{m,n})$  are

$$\begin{aligned} r(u_1|\Pi) &= (0, 2, 2), & r(u_2|\Pi) &= (0, 1, 3), & r(u_3|\Pi) &= (0, 1, 1), \\ r(u_n|\Pi) &= (0, 3, 1), & r(v_1|\Pi) &= (1, 0, 4), & r(v_2|\Pi) &= (2, 0, 5), \\ &\dots, & r(v_{n(m-1)-1}|\Pi) &= (0, 3, 5), & r(v_{n(m-1)}|\Pi) &= (0, 2, 4). \end{aligned}$$

So the representation of  $r(u|\Pi)$  and  $r(v|\Pi)$  are distinct and we conclude that  $\Pi$  is a resolving partition of  $J_{m,n}$  with 3 elements. Furthermore, we prove that there is no resolving partition with 2 elements. Suppose that  $J_{m,n}$  has a resolving partition with 2 elements,  $\Pi = \{S_1, S_2\}$ . According to Lemma 2.3, it is a contradiction. Therefore,  $J_{m,n}$  has no a resolving partition with 2 elements. Thus, the partition dimension of  $J_{m,n}$  is  $pd(J_{m,n}) = 3$  for  $n = 3, 4, 5$ .

(ii) Case  $n \geq 6$ .

Let  $\Pi = \{S_1, S_i\}$  be a resolving partition of  $J_{m,n}$  where  $S_1 = \{u_i, v_j\}$ , for  $1 \leq i \leq n+1$ ,  $1 \leq j \leq n(m-1)$  and  $S_i = \{v_k\}$ , for  $1 < i \leq \lfloor \frac{n}{2} \rfloor + 1$ ,  $1 \leq k \leq n(m-1)$ ,  $k \neq j$ . Since each vertex of  $J_{m,n}$  has a distinct representation with respect to  $\Pi$ , then  $\Pi$  is a resolving partition of  $J_{m,n}$  with  $\lfloor \frac{n}{2} \rfloor + 1$  elements.

Next, we show that the cardinality of  $\Pi$  is  $\lfloor \frac{n}{2} \rfloor + 1$ . The cardinality of  $\Pi$  is the number of  $k$ -partition in  $S_1$  and  $S_i$  for  $1 < i \leq \lfloor \frac{n}{2} \rfloor$ . The number of  $k$ -partition in  $S_1$  is 1. The number of  $k$ -partition in  $S_i$  for  $1 < i \leq \lfloor \frac{n}{2} \rfloor$  is  $\lfloor \frac{n}{2} \rfloor$ . We obtain  $|\Pi| = 1 + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$ . Therefore,  $\Pi = \{S_1, S_2, \dots, S_{\lfloor \frac{n}{2} \rfloor + 1}\}$  is a resolving partition of  $J_{m,n}$ .

Now, we show that  $J_{m,n}$  has no  $pd(J_{m,n}) < \lfloor \frac{n}{2} \rfloor + 1$ . We may assume that  $\Pi$  is a resolving partition of  $J_{m,n}$  where  $pd(J_{m,n}) < \lfloor \frac{n}{2} \rfloor + 1$ . So, each vertex  $v \in V(J_{m,n})$  has a distinct representation  $r(v_i|\Pi)$  and  $r(v_j|\Pi)$ . If we select  $\Pi = \{S_1, S_2, S_3, \dots, S_{\lfloor \frac{n}{2} \rfloor}\}$  as a resolving partition then there is a partition class containing any 2 vertices of  $v_j$ . By using Lemma 2.3, the distance of vertex  $u_1$  to the others have the same distance, and  $r(v_i|\Pi) = r(v_j|\Pi)$ . It is a contradiction. Hence, the partition dimension of  $J_{m,n}$  is  $pd(J_{m,n}) = \lfloor \frac{n}{2} \rfloor + 1$  for  $n \geq 6$ .

### 3. Conclusion

According to the discussion above it can be concluded that the partition dimension of a lollipop graph  $L_{m,n}$  and a generalized Jahangir graph  $J_{m,n}$  are as stated in Theorem 2.1 and Theorem 2.2.

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