On the total H-irregularity strength of graphs: A new notion

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Abstract. A total edge irregularity strength of G has been already widely studied in many papers. The total α-labeling is said to be a total edge irregular α-labeling of the graph G if for every two different edges e_1 and e_2, it holds $w(e_1) \neq w(e_2)$, where $w(uv) = f(u) + f(uv) + f(v)$, for $e = uv$. The minimum α for which the graph G has a total edge irregular α-labeling is called the total edge irregularity strength of G, denoted by $tes(G)$. A natural extension of this concept is by considering the evaluation of the weight is not only for each edge but we consider the weight on each subgraph $H \subseteq G$. We extend the notion of the total α-labeling into a total H-irregular α-labeling. The total α-labeling is said to be a total H-irregular α-labeling of the graph G if for $H \subseteq G$, the total H-weights $W(H) = \sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)$ are distinct. The minimum α for which the graph G has a total H-irregular α-labeling is called the total H-irregularity strength of G, denoted by $tHs(G)$. In this paper we initiate to study the tHs of shackle and amalgamation of any graphs and their bound.

Keywords: Total α-labeling, Total H-irregularity strength, shackle of any graph, amalgamation of any graph.

1. Introduction

All graphs in this paper are simple, nontrivial and undirected graphs. A total labeling $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, \alpha\}$ is called a total α-labeling of a graph G. The weight of an edge uv of G, denoted by $w(uv)$, is the sum of the labels of end vertices u and v and also edge uv, i.e. $w(uv) = f(u)+f(uv)+f(v)$. The total α-labeling is said to be a total edge irregular α-labeling of the graph G if for every two different edges e_1 and e_2, it holds $w(e_1) \neq w(e_2)$. The minimum α for which the graph G has a total edge irregular α-labeling is called the total edge irregularity strength of G, denoted by $tes(G)$. A natural extension of this concept is by considering the evaluation of the weight is not only for each edge but we consider the weight on each subgraph $H \subseteq G$. Thus, we extend the notion of the total α-labeling into a total H-irregular α-labeling. The total α-labeling is said to be a total H-irregular α-labeling of the graph G if for $H \subseteq G$, the total H-weights $W(H) = \sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)$ are distinct. The minimum α for which the graph G has a total H-irregular α-labeling is called the total H-irregularity strength of G, denoted by $tHs(G)$. The minimum α for which the graph G has a subgraph irregular total $\alpha-$labeling is called *the total $H-$irregularity strength of* G, $tHs(G)$.

The beginning of the study of the irregularity strength is introduced by Togni et al. [10] and Frieze et al. [4]. By then, there are some result related to the total H-irregularity strength study. Jendrol et al. [6] determined the total edge irregularity strength of complete and bipartite complete graph, Jeyanthi et al. [7] studied about total edge irregularity strength of disjoin union wheel graph, and Baca et al. [2], [3] studied about total edge irregularity strength of generelized of prism graph and any graphs. Furthermore Ahmad et al. [1] found total edge irregularity strength of zigzag graph, as well as Pfender [8] studied about total edge irregularity strength of large graph, and the last Rajasingh et al. [9] also studied total edge irregularity strength of series parallel graph.

In this paper, we study the existence of the total H-irregularity α-labeling of some graph operations, namely shackle and amalgamation of graph G. A shackle of $G_1, G_2, \ldots, G_k$, denoted by $Shack(G_1, G_2, \ldots, G_k)$, is any graph constructed from non-trivial connected and ordered graphs $G_1, G_2, \ldots, G_k$ such that for every $1 \le i, j \le k$ with $|i-j| \ge 2, G_i$ and G_j have no common vertex and for every $1 \le i \le k-1, G_i$ and $G_i + 1$ share exactly one common vertex, called a linkage vertex, where the $k-1$ linkage vertices are all distinct. Meanwhile, let $\{G_1, G_2, \ldots, G_n\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{0_i} or edge e_{0_i} called a terminal *vertex* or *edge*, respectively [5]. The *vertex-amalgamation* of $G_1, G_2, \ldots, G_n$ denoted by *Amal* $\{G_i, v_{0_i}\}$, is formed by taking all the G_i's and identifying their terminal vertices. Similarly, the *edge-amalgamation* of $G_1, G_2, \ldots, G_n$, denoted by $Amal\{G_i, e_{0_i}\}$, is formed by taking all the G_i's and identifying their terminal edges. Furthermore, if G_i' are isomorphic graphs then we denote such graphs as *Shack* $\{G, v, n\}$ and *Amal* $\{G, v, n\}$ for vertex, or *Shack* $\{G, e, n\}$ and *Amal* $\{G, e, n\}$ for edge. In this paper we will study the tHs of shackle and amalgamation of any graphs and as well as determine their bound.

2. The Results

Prior to show the values of tHs of those graphs, we will show the lower bound of tHs in general graph by the following lemma.

Lemma 2.1 *Given a graph $H \subset G$. Let p_H, q_H be respectively be number of vertices and edges of H and $|H|$ be the number of subgraphs. The total H-irregularity strength satisfies*

$$tHs(G) \ge \lceil \tfrac{p_H+q_H+|H|-1}{p_H+q_H} \rceil$$

Proof. A total α-labeling is a labeling $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, \alpha\}$. The H- irregularity total α-labeling of graph G is a total α-labeling such that for each subgraph $H \subseteq G$, the weight $W(H) = \sum_{v \in V(K)} f(v) + \sum_{e \in E(K)} f(e)$ are all distinct. Furthermore, since we require the minimum α for which the graph G has a total H-irregular α-labeling, the set of the total H-weight should be consecutive, otherwise it will not give a minimum tHs. Thus, the set of total H weight is $W(H) = \{p_H + q_H, p_H + q_H + 1, p_H + q_H + 2, \ldots, p_H + q_H + (|H| - 1)\}$. On the other hand the maximum possible H weight of graph G is at most $tHs(G)(p_H + q_H)$. It implies

$$\begin{aligned} tHs(G)(p_H + q_H) &\ge p_H + q_H + |H| - 1 \\ tHs(G) &\ge \tfrac{p_H+q_H+|H|-1}{p_H+q_H} \end{aligned}$$

Since $tHs(G)$ should be integer, and we need a sharpest lower bound, it implies

$$tHs(G) \ge \lceil \frac{p_H + q_H + |H| - 1}{p_H + q_H} \rceil.$$

It completes the proof. □

Now, we are ready to show our main results.

Theorem 2.1 *Let $G = \mathrm{Shack}(H, v, n)$ be a shackle of any graph H. Then the total H-irregularity strength satisfies*

$$tHs(\mathrm{Shack}(H, v, n)) = \lceil \frac{m + n + 1}{m + 2} \rceil$$

where p_H and q_H are respectively the number of vertices and edges in subgraph $H \subseteq G$ and $m = p_H + q_H - 2$ and $n = |H|$.

Proof. The vertex set and edge set of the graph $\mathrm{Shack}(H, v, n)$ can be split into two following sets: $V(\mathrm{Shack}(H, v, n)) = \{v_{ij}; 1 \le i \le p_H - 2, 1 \le j \le n\} \cup \{x_k; 1 \le k \le n + 1\}$ and $E(\mathrm{Shack}(H, v, n)) = \{e_{lj}; 1 \le l \le q_H, 1 \le j \le n\}$. Thus, the graph $\mathrm{Shack}(H, v, n)$ has $|V(\mathrm{Shack}(H, v, n))| = (n - 1)p_H + 1$, $|E(\mathrm{Shack}(H, v, n))| = nq_H$. Since $m = p_H + q_H - 2$, then by Lemma 2.1, we have $tHs(\mathrm{Shack}(H, v, n)) \ge \lceil \frac{P_H+q_H+|H|-1}{P_H+q_H} \rceil = \lceil \frac{m+2+n-1}{m+2} \rceil = \lceil \frac{m+n+1}{m+2} \rceil$. Thus, $tHs(\mathrm{Shack}(H, v, n)) \ge \lceil \frac{m+n+1}{m+2} \rceil$.

Now we will show that $tHs(\mathrm{Shack}(H, v, n)) \le \lceil \frac{m+n+1}{m+2} \rceil$. Define f as a vertex and edge labeling of graph G, $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$ by the following function.

$$\begin{aligned} f(x_k) &= \lceil \tfrac{k}{m+2} \rceil \\ f(v_{ij}) \cup f(e_{lj}) &= \begin{cases} \lceil \frac{j}{m-(t+1)} \rceil; 1 \le j \le m - t + 1, 1 \le t \le m \\ \lceil \frac{j+t-(m+1)}{m+2} \rceil + 1; m - t + 2 \le j \le n, 1 \le t \le m \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{m+2, m+3, \ldots, m+n+1\}$ forms a consecutive sequence. It implies the set of $H-$weights are distinct. By considering the above label f, the minimum $tHs(\mathrm{Shack}(H, v, n))$ can be achieved by the following:

$$\begin{aligned} tHs(\mathrm{Shack}(H, v, n)) &\le \lceil \tfrac{j+t-(m+1)}{m+2} \rceil + 1, \text{ for } j = n, t = m \\ &= \lceil \tfrac{n+m-m-1}{m+2} + \tfrac{m+2}{m+2} \rceil \\ &= \lceil \tfrac{n-1+m+2}{m+2} \rceil \\ &= \lceil \tfrac{m+n+1}{m+2} \rceil \end{aligned}$$

Thus, $tHs(\mathrm{Shack}(H, v, n)) \le \lceil \frac{m+n+1}{m+2} \rceil$. It concludes that $tHs(\mathrm{Shack}(H, v, n)) = \lceil \frac{m+n+1}{m+2} \rceil$. □

Theorem 2.2 *Let $G = c\mathrm{Shack}(H, v, n)$ be disjoint union of multiple copies c of shackle of graph H. Then*

$$tHs(c\mathrm{Shack}(H, v, n)) = \lceil \frac{m + cn + 1}{m + 2} \rceil$$

where $m = p_H + q_H - 2$, p_H and q_H are the number of vertices and edges in H respectively, $n = |H|$ and c is number of copies of G.

Proof. The graph $G = c\mathrm{Shack}(H, v, n)$ is a diconnected graph with vertex set $V(c\mathrm{Shack}(H, v, n)) = \{v^u_{ij}; 1 \le i \le p_H - 2, 1 \le j \le n, 1 \le u \le c\} \cup \{x^u_k; 1 \le k \le n+1, 1 \le u \le c\}$ and edge set $E(c\mathrm{Shack}(H, v, n)) = \{e^u_{lj}; 1 \le l \le q_H, 1 \le j \le n, 1 \le u \le c\}$. Thus, the graph $c\mathrm{Shack}(H, v, n)$ has $|V(c\mathrm{Shack}(H, v, n))| = c((n - 1)p_H + 1)$ and $|E(c\mathrm{Shack}(H, v, n))| = cnq_H$. Since $m = p_H + q_H - 2$, then by Lemma 2.1

$$\begin{aligned} tHs(c\mathrm{Shack}(H, v, n)) &\ge \lceil \tfrac{p_H+q_H+|H|-1}{p_H+q_H} \rceil \\ &= \lceil \tfrac{m+2+cn-1}{m+2} \rceil \\ &= \lceil \tfrac{m+cn+1}{m+2} \rceil \end{aligned}$$

Now we will show that $tHs(c\text{Shack}(H,v,n)) \le \lceil \frac{m+cn+1}{m+2} \rceil$. The vertex and edge labeling f is a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$. Let $w = ju$; $1 \le j \le n$, $1 \le u \le c$ such that $1 \le w \le cn$.

$$\begin{aligned} f(x_k^u) &= \lceil \tfrac{u}{m+2} \rceil, 1 \le u \le c \\ f(v_{ij}^u) \cup f(e_{lj}^u) &= \begin{cases} \lceil \frac{w}{m-(t+1)} \rceil; 1 \le w \le m-t+1, 1 \le t \le m \\ \lceil \frac{w+t-(m+1)}{m+2} \rceil + 1; m-t+2 \le w \le cn, 1 \le t \le m \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{m+2, m+3, \ldots, m+cn+1\}$ which form a consecutive sequence. It implies the set of $H-$weights are distinct. Now considering the above label of f, the minimum $tHs(c\text{Shack}(H,v,n))$ can be achieved by the following:

$$\begin{aligned} tHs(c\text{Shack}(H,v,n)) &\le \lceil \tfrac{cn+m-(m+1)}{m+2} \rceil + 1 \\ &= \lceil \tfrac{cn+1}{m+2} + \tfrac{m+2}{m+2} \rceil \\ &= \lceil \tfrac{cn-1}{m+2} + \tfrac{m+2}{m+2} \rceil \\ &= \lceil \tfrac{m+cn-1}{m+2} \rceil \end{aligned}$$

Thus, $tHs(c\text{Shack}(H,v,n)) \le \lceil \frac{m+cn-1}{m+2} \rceil$. It implies that $tHs(c\text{Shack}(H,v,n)) = \lceil \frac{m+cn-1}{m+2} \rceil$. □

Theorem 2.3 *Let G be an amalgamation of any connected graph H, denoted by $G = \text{Amal}(H,v,n)$. Then the following holds*

$$tHs(\text{Amal}(H,v,n)) = \lceil \frac{r+n-1}{r} \rceil$$

where $r = p_H + q_H - 1$, p_H and q_H is the number of vertices and edges in H respectively and $n = |H|$.

Proof. The vertex set and edge set of the graph $\text{Amal}(H,v,n)$ can be split into following sets: $V(\text{Amal}(H,v,n)) = \{A\} \cup \{x_{ij}; 1 \le i \le p_H - 1, 1 \le j \le n\}$ and $E(\text{Amal}(H,v,n)) = \{e_{lj}; 1 \le l \le q_H, 1 \le j \le n\}$. Thus, the graph $\text{Amal}(H,v,n)$ has $|V(\text{Amal}(H,v,n))| = p_G$, and $|E(\text{Amal}(H,v,n))| = q_G$. Let n, m be positive integers with $n \ge 2$ and $m \ge 3$. Thus $|V(\text{Amal}(H,v,n))| = p_G = n(p_H - 1) + 1$ and $|E(\text{Amal}(H,v,n))| = q_G = nq_H$. Then by lemma 2.1,

$$\begin{aligned} tHs(\text{Amal}(H,v,n)) &\ge \lceil \tfrac{p_H+q_H+|H|-1}{p_H+q_H} \rceil \\ &= \lceil \tfrac{r+1+n-1}{r+1} \rceil \\ &= \lceil \tfrac{r+n}{r+1} \rceil \\ &= \lceil \tfrac{r+n-1}{r} \rceil \end{aligned}$$

Thus, the lower bound $tHs(\text{Amal}(H,v,n)) \ge \lceil \frac{r+n-1}{r} \rceil$. Now we will prove that $tHs(\text{Amal}(H,v,n)) \le \lceil \frac{r+n-1}{r} \rceil$. The vertex and edge labeling f is a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$.

$$\begin{aligned} f(A) &= 1 \\ f(x_{ij}) \cup f(e_{lj}) &= \begin{cases} \lceil \frac{j}{r-(t-1)} \rceil; 1 \le j \le r-t+1, 1 \le t \le r \\ \lceil \frac{j+t-(r+1)}{r} \rceil + 1; r-i+2 \le j \le n, 1 \le t \le r. \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{r+1, r+2, \ldots, r+n\}$ form a consecutive sequence. It implies the set of $H-$weights are distinct.

Now by considering the above label f, the minimum $tHs(\mathrm{Amal}(H,v,n))$ can be achieved by the following:

$$\begin{aligned} tHs(\mathrm{Amal}(H,v,n)) &\leq \lceil \tfrac{n+r-(r+1)}{r} \rceil + 1 \\ &= \lceil \tfrac{n-1}{r} + \tfrac{r}{r} \rceil \\ &= \lceil \tfrac{r+n-1}{r} \rceil \end{aligned}$$

It is clear to concludes that $tHs(\mathrm{Amal}(H,v,n)) = \lceil \frac{r+n-1}{r} \rceil$. □

Theorem 2.4 *Let G be a disjoint union of multiple copies c of amalgamation of graph H, denoted by $G = c\mathrm{Amal}(H,v,n)$. Then*

$$tHs(c\mathrm{Amal}(H,v,n)) = \lceil \frac{r+cn-1}{r} \rceil$$

where $r = p_H + q_H - 1$, p_H and q_H is the number of vertices and edges in H respectively, $n = |H|$ and c is number of copies of G.

Proof. The vertex set and edge set of the graph $G = c\mathrm{Amal}(H,v,n)$ can be split into following sets: $V(G) = \{A^k; 1 \leq k \leq c\} \cup \{{x_{ij}}^k; 1 \leq i \leq p_H - 1, 1 \leq j \leq n, 1 \leq k \leq c\}$ and $E(G) = \{{e_{lj}}^k; 1 \leq j \leq n, 1 \leq l \leq q_H, 1 \leq k \leq c\}$. Thus the graph $c\mathrm{Amal}(H,v,n)$ has with $|V(c\mathrm{Amal}(H,v,n))| = p_G$, and $|E(c\mathrm{Amal}(H,v,n))| = p_G$. Let n, r, and odd c be positive integers with $n \geq 2$ and $r, c \geq 3$. Thus $|V(G)| = p_G = c(n(p_H-1)+1)$ and $|E(G)| = q_G = cnq_H$. Then by lemma 2.1,

$$\begin{aligned} tHs(c\mathrm{Amal}(H,v,n)) &\geq \lceil \tfrac{p_H+q_H+|H|-1}{p_H+q_H} \rceil \\ &= \lceil \tfrac{r+1-cn-1}{r} \rceil \\ &= \lceil \tfrac{r+cn}{r+1} \rceil \\ &= \lceil \tfrac{r+cn-1}{r} \rceil \end{aligned}$$

Thus, the lower bound $tHs(c\mathrm{Amal}(H,v,n)) \geq \lceil \frac{r+cn-1}{r} \rceil$. Now we will show that $tHs(c\mathrm{Amal}(H,v,n)) \leq \lceil \frac{r+cn-1}{r} \rceil$. For any V and E, the labeling as follows. Let $w = jk$; $1 \leq j \leq n$, $1 \leq k \leq c$ such that $1 \leq w \leq cn$.

$$\begin{aligned} f(A^k) &= 1, 1 \leq k \leq c \\ f(x_{ij}^k) \cup f(e_{lj}^k) &= \begin{cases} \lceil \frac{w}{r-(t-1)} \rceil; 1 \leq w \leq r-t+1, 1 \leq t \leq r, 1 \leq k \leq c \\ \lceil \frac{w+t-(r+1)}{r} \rceil + 1; r-t+2 \leq w \leq cn, 1 \leq t \leq r, 1 \leq k \leq c. \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{r+1, r+2, \ldots, r+cn\}$ form a consecutive sequence. It implies the set of $H-$weights are distinct. Now considering the above label of f, the minimum $tHs(c\mathrm{Amal}(H,v,n))$ can be achieved by the following:

$$\begin{aligned} tHs(c\mathrm{Amal}(H,v,n)) &\leq \lceil \tfrac{w+t-(r+1)}{r} \rceil + 1 \\ &= \lceil \tfrac{cn+r-r-1}{r} \rceil + \lceil \tfrac{r}{r} \rceil \\ &= \lceil \tfrac{cn+r-1}{r} \rceil \end{aligned}$$

It concludes the proof. □

Theorem 2.5 *Let G be a shackle of connected graph C_m graph, denoted by $G = \mathrm{Shack}(C_m,v,n)$. Then*

$$tHs(\mathrm{Shack}(C_m,v,n)) = \lceil \frac{2m+n-1}{2m} \rceil$$

where m is an order of the cycle graph and n number of C_m.

Proof. The graph Shack(C_m, v, n) is a connected graph with vertex set $V(\text{Shack}(C_m, v, n)) = \{v_{ij}; 1 \leq i \leq p_{C_m}-2, 1 \leq j \leq n\} \cup \{x_k; 1 \leq k \leq n+1\}$ and edge set $E(\text{Shack}(C_m, v, n)) = \{e_{lj}; 1 \leq l \leq q_{C_m}, 1 \leq j \leq n\}$. The cardinalities of the graph Shack(C_m, v, n) are $|V(\text{Shack}(C_m, v, n))| = (n-1)p_{C_m} + 1$, and $|E(\text{Shack}(C_m, v, n))| = nq_{C_m}$, where $p_{C_m} = |V(C_m)|$, and $q_{C_m} = |E(C_m)|$. Then by Lemma 2.1,

$$\begin{aligned} tHs(\text{Shack}(C_m, v, n)) &\geq \lceil \tfrac{p_{C_m}+q_{C_m}+|C_m|-1}{p_{C_m}+q_{C_m}} \rceil \\ &= \lceil \tfrac{m+m+n-1}{m+m} \rceil \\ &= \lceil \tfrac{2m+n-1}{2m} \rceil \end{aligned}$$

Now we will show that $tHs(\text{Shack}(C_m, v, n)) \leq \lceil \frac{2m+n-1}{2m} \rceil$. Define the vertex and edge labelings $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$ as follows

$$\begin{aligned} f(x_k) &= \lceil \tfrac{k}{2m} \rceil \\ f(v_{ij}) \cup f(e_{lj}) &= \begin{cases} \lceil \frac{j}{2m-2-(t+1)} \rceil; 1 \leq j \leq 2m-t-1, 1 \leq t \leq 2m-2 \\ \lceil \frac{j+t-(2m-2+1)}{2m} \rceil + 1; 2m-t \leq j \leq n, 1 \leq t \leq 2m-2 \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{2m, 2m+1, \ldots, 2m+n-1\}$ form a consecutive sequence. It implies the set of $H-$weights are distinct. Now considering the above label of f, the minimum $tHs(\text{Shack}(C_m, v, n))$ can be achieved by the following:

$$\begin{aligned} tHs(\text{Shack}(C_m, v, n)) &\leq \lceil \tfrac{j+t-(2m-2+1)}{2m} \rceil + 1 \\ &= \lceil \tfrac{n+2m-2-(2m-2+1)}{2m} + \tfrac{2m}{2m} \rceil \\ &= \lceil \tfrac{n-1}{2m} + \tfrac{2m}{2m} \rceil \\ &= \lceil \tfrac{2m+n-1}{2m} \rceil \end{aligned}$$

Thus $tHs(\text{Shack}(C_m, v, n)) \leq \lceil \frac{2m+n-1}{2m} \rceil$, it implies that $tHs(\text{Shack}(C_m, v, n)) = \lceil \frac{2m+n-1}{2m} \rceil$. □

Theorem 2.6 *Let G be a disjoint union of multiple copies c of shackle of graph C_m, denoted by $G = c\text{Shack}(C_m, v, n)$. Then*

$$tHs(c\text{Shack}(C_m, v, n)) = \lceil \frac{2m+cn-1}{2m} \rceil$$

where m is an order of the cycle graph, n is a number of C_m, and c is number of multiple copies of G.

Proof. Suppose we denote the vertex and edge sets of the graph $G = c\text{Shack}(C_m, v, n)$ as follows: $V(c\text{Shack}(C_m, v, n)) = \{v^u_{ij}; 1 \leq i \leq p_{C_m} - 2, 1 \leq j \leq n, 1 \leq u \leq c\} \cup \{x^u_k; 1 \leq k \leq n+1, 1 \leq u \leq c\}$ and $E(c\text{Shack}(C_m, v, n)) = \{e^u_{lj}; 1 \leq l \leq q_{C_m}, 1 \leq j \leq n, 1 \leq u \leq c\}$. Thus, the graph $c\text{Shack}(C_m, v, n)$ has $|V(c\text{Shack}(C_m, v, n))| = c((n-1)p_{C_m} + 1)$, and $|E(c\text{Shack}(C_m, v, n))| = cnq_{C_m}$, where $p_{C_m} = |V(C_m)|$ and $q_{C_m} = |E(C_m)|$. Then by Lemma 2.1

$$\begin{aligned} tHs(c\text{Shack}(C_m, v, n)) &\geq \lceil \tfrac{p_{C_m}+q_{C_m}+|C_m|-1}{p_{C_m}+q_{C_m}} \rceil \\ &= \lceil \tfrac{2m+cn-1}{2m} \rceil \end{aligned}$$

Now we will show that $tHs(c\text{Shack}(C_m, v, n)) \leq \lceil \frac{2m+cn-1}{2m} \rceil$ by defining the vertex and edge labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$ by the following. Let $w = ju$; $1 \leq j \leq n$, $1 \leq u \leq c$ such that $1 \leq w \leq cn$.

$$\begin{aligned} f(x^u_k) &= \lceil \tfrac{u}{2m} \rceil, 1 \leq u \leq c \\ f(x^u_{ij}) \cup f(e^u_{lj}) &= \begin{cases} \lceil \frac{w}{(2m-2)-(t+1)} \rceil; 1 \leq w \leq 2m-t-1, 1 \leq t \leq 2m-2 \\ \lceil \frac{w+t-(2m-2+1)}{2m} \rceil + 1; 2m-t \leq w \leq cn, 1 \leq t \leq 2m-2 \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)$ is $W(H) = \{2m, 2m+1, \ldots, 2m+cn-1\}$ form a consecutive sequence. It implies the set of $H-$weights are distinct. Now considering the above label of f, the minimum $tHs(c\text{Shack}(C_m, v, n))$ can be achieved by the following.

$$\begin{aligned} tHs(\text{Shack}(C_m, v, n)) &\le \lceil \tfrac{w+t-(2m-2+1)}{2m} \rceil + 1 \\ &= \lceil \tfrac{cn+(2m-2)-(2m-2+1)}{2m} + \tfrac{2m}{2m} \rceil \\ &= \lceil \tfrac{cn-1}{2m} + \tfrac{2m}{2m} \rceil \\ &= \lceil \tfrac{2m+cn-1}{2m} \rceil \end{aligned}$$

It is clear to conclude that $tHs(\text{Shack}(C_m, v, n)) = \lceil \frac{2m+cn-1}{2m} \rceil$. □

Theorem 2.7 *Let* G *be an amalgamation of connected graph* C_3*, denoted by* $G = \text{Amal}(C_3, v, n)$*. Then the following holds*

$$tHs(\text{Amal}(C_3, v, n)) = \lceil \frac{n+4}{5} \rceil$$

where n *is a number of* C_3*.*

Proof. Let the graph $\text{Amal}(C_3, v, n)$ has with $|V(G)| = p_G$, $|E(G)| = q_G$, $|V(H)| = |V(C_3)| = p_H = p_{C_3}$, and $|E(H)| = |E(C_3)| = q_H = q_{C_3}$. Suppose we denote the vertex and edge sets of the graph $G = \text{Amal}(C_3, v, n)$ as follows: $V(G) = \{A\} \cup \{x_{ij}; 1 \le i \le 2, 1 \le j \le n\}$ and $E(G) = \{Ax_{ij}; 1 \le i \le 2, 1 \le j \le n\} \cup \{x_{1j}x_{2j}; 1 \le j \le n\}$. Thus, the graph $\text{Amal}(C_3, v, n)$ has $|V(\text{Amal}(C_3, v, n))| = 2n+1$, and $|E(\text{Amal}(C_3, v, n))| = 3n$, where $p_{C_3} = |V(C_3)|$ and $q_{C_3} = |E(C_3)|$. Then by Lemma 2.1, we have the following

$$\begin{aligned} tHs(\text{Amal}(C_3, v, n)) &\ge \lceil \tfrac{P_H+q_H+|H|-1}{P_H+q_H} \rceil \\ &= \lceil \tfrac{6+n-1}{6} \rceil \\ &= \lceil \tfrac{n+5}{6} \rceil \\ &= \lceil \tfrac{n+4}{5} \rceil \end{aligned}$$

Thus, the lower bound $tHs\text{Amal}(C_3, v, n)) \ge \lceil \frac{n+4}{5} \rceil$. Now we will show that $tHs(\text{Amal}(C_3, v, n)) \le \lceil \frac{n+4}{5} \rceil$. The vertex and edge labelings f is a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$.

$$\begin{aligned} f(A) &= 1 \\ f(x_{i,j}) \cup f(Ax_{i,j}) \cup f(x_{ij}x2j) &= \begin{cases} \lceil \frac{j}{5-(i-1)} \rceil; 1 \le j \le 6-i, 1 \le i \le 5 \\ \lceil \frac{j+i-(6)}{5} \rceil + 1; 7-i \le j \le n, 1 \le i \le 5. \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)$ is $W(H) = \{6, 7, \ldots, 6+(n-1)\}$ which form a consecutive sequence. It implies the set of $H-$weights are distinct. Now considering the above label of f, the minimum $tHs\text{Amal}(C_3, v, n))$ can be achieved by the following.

$$\begin{aligned} tHs(\text{Amal}(C_3, v, n)) &\le \lceil \tfrac{j+i-(6)}{5} \rceil + 1 \\ &= \lceil \tfrac{5+n-6}{5} + \tfrac{5}{5} \rceil \\ &= \lceil \tfrac{n+4}{5} \rceil \end{aligned}$$

It concludes the proof. □

Theorem 2.8 *Let G be a disjoint union of amalgamation of C_3 graph, denoted by* $c\text{Amal}(C_3, v, n)$. *Then*

$$tHs(c\text{Amal}(C_3, v, n)) = \lceil \frac{cn+4}{5} \rceil$$

Proof. Let the graph $c\text{Amal}(C_3, v, n)$ has with $|V(G)| = p_G$, $|E(G)| = q_G$, $|V(H)| = |V(C_3)| = p_H = p_{C_3}$, and $|E(H)| = |E(C_3)| = q_H = q_{C_3}$. The vertex set and edge set of the graph $G = c\text{Amal}(C_3, v, n)$ can be split into following sets: $V(G) = \{A^k; 1 \leq k \leq c\} \cup \{x^k_{ij}; 1 \leq i \leq 2, 1 \leq j \leq n, 1 \leq k \leq c\}$ and $E(G) = \{A^k x^k_{ij}; 1 \leq i \leq 2, 1 \leq j \leq n, 1 \leq k \leq c\} \cup \{x^k_{1j} x^k_{2j}; 1 \leq j \leq n, 1 \leq k \leq c\}$. Let n, m, and odd s be positive integers with $n \geq 2$ and $r, c \geq 3$. Thus $|V(G)| = p_G = c(2n+1)$ and $|E(G)| = q_G = 3cn$. Then by lemma 2.1,

$$\begin{aligned} tHs(c\text{Amal}(C_3, v, n)) &\geq \lceil \tfrac{P_H+q_H+|H|-1}{P_H+q_H} \rceil \\ &= \lceil \tfrac{6+cn-1}{6} \rceil \\ &= \lceil \tfrac{5+cn}{6} \rceil \\ &= \lceil \tfrac{4+cn}{5} \rceil \end{aligned}$$

Thus, the lower bound $tHs(c\text{Amal}(C_3, v, n)) \geq \lceil \frac{cn+4}{5} \rceil$. Now we will prove that $tHs(c\text{Amal}(H, v, n)) \leq \lceil \frac{cn+4}{5} \rceil$. Let $l = jk; 1 \leq j \leq n, 1 \leq k \leq c$ such that $1 \leq l \leq cn$. For any V and E, the labeling as follows.

$$\begin{aligned} f(A^k) &= 1, 1 \leq k \leq c \\ f(x^k_{i,j}) \cup f(A^k x^k_{i,j}) \cup f(x^k_{ij} x^k_{2j}) &= \begin{cases} \lceil \frac{l}{5-(i-1)} \rceil; 1 \leq l \leq 6-i, 1 \leq i \leq 5, 1 \leq k \leq c \\ \lceil \frac{l+i-(6)}{5} \rceil + 1; 7-i \leq l \leq cn, 1 \leq i \leq 5, 1 \leq k \leq c. \end{cases} \end{aligned}$$

Under the labeling f, the total H-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{6, 7, \ldots, 6 + (cn-1)\}$ which form a consecutive sequence. It implies the set of $H-$weights are distinct. Now considering the above label of f, the minimum $tHs(c\text{Amal}(C_3, v, n))$ can be achieved by the following.

$$\begin{aligned} tHs(c\text{Amal}(C_3, v, n)) &\geq \lceil \tfrac{l+i-(6)}{5} \rceil + 1 \\ &= \lceil \tfrac{5+cn-6}{5} + \tfrac{5}{5} \rceil \\ &= \lceil \tfrac{cn+4}{5} \rceil \end{aligned}$$

Thus $tHs(c(\text{Amal}(C_3, v, n))) \leq \lceil \frac{cn+4}{5} \rceil$, it implies that $tHs(c(\text{Amal}(C_3, v, n))) = \lceil \frac{cn+4}{5} \rceil$. □

Concluding Remarks

We have found the total H-irregularity strength of shackle and amalgamation of G, namely $tHs(\text{Shack}(H, v, n))$, $tHs(c(\text{Shack}(H, v, n)))$, $tHs(\text{Amal}(H, v, n))$ and $tHs(c(\text{Amal}(H, v, n)))$. Apart from those graphs, the study of the values of tHs are considered to be interesting research topic as it is a new extension of total edge irregularity strength of G. Therefore, we propose the following open problem.

Open Problem 2.1 *Let G be any connected and disconnected graph, apart from the above graphs determine the value of $tHs(G)$.*

Open Problem 2.2 *Let $tes(G)$ and $tHs(G)$ be total edge irregularity strength and total H-irregularity strength of graph G. Characterize the connection between $tes(G)$ and $tHs(G)$.*

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