On Rainbow k-Connection Number of Special Graphs and It's Sharp Lower Bound

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Abstract. Let $G = (V, E)$ be a simple, nontrivial, finite, connected and undirected graph. Let c be a coloring $c : E(G) \to \{1, 2, \dots, s\}, s \in$ N. A path of edge colored graph is said to be a rainbow path if no two edges on the path have the same color. An edge colored graph G is said to be a rainbow connected graph if there exists a rainbow $u - v$ path for every two vertices u and v of G. The rainbow connection number of a graph G, denoted by $rc(G)$, is the smallest number of k colors required to edge color the graph such that the graph is rainbow connected. Furthermore, for an l-connected graph G and an integer k with $1 \leq k \leq l$, the rainbow k-connection number $rc_k(G)$ of G is defined to be the minimum number of colors required to color the edges of G such that every two distinct vertices of G are connected by at least k internally disjoint rainbow paths. In this paper, we determine the exact values of rainbow connection number of some special graphs and obtain a sharp lower bound.

Keywords: Rainbow k-Connection Number, Special Graphs, Sharp Lower Bound

1. Introduction

Suppose G is a simple connected graph with a set of vertices $V(G)$ and edges $E(G)$. For a further reference please see Gross, *et. al.* [6]. Let G be a nontrivial connected graph on which it is defined a coloring $c : E(G) \to \{1, 2, \dots, s\}, s \in N$, of the edges of G, where adjacent edges may be colored the same. A $u - v$ path P in G is a rainbow path if no two edges of P are colored the same. The graph G is rainbow-connected (with respect to c) if G contains a rainbow $u - v$ path for every two vertices u and v of G. In this case, the coloring c is called a rainbow coloring of G. If k colors are used, then c is a rainbow k-coloring. The minimum k for which there exists a *rainbow k-coloring* of the edges of G is the rainbow connection number $rc(G)$. The completes concept can be found in Chartrand in [4].

A simple observation can be proposed that if G has n vertices then $rc(G) \leq n-1$ but is not sharp. Since a given spanning tree can be assigned with distinct colors, and color the remaining edges with one of the already used colors then the upper bound of $rc(G) \leq n-1$, see Caro [1] for detail. It is also easy to understand that $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of G, Caro in [1]. Thus, it gives the following

$$diam(G) \leq rc(G) \leq n-1$$

There have been some results regarded to rainbow connection numbers. Chandran, *et.al.* in [2] determined rainbow connection number and connected dominating sets, Chakraborty, *et.al.* in [3] considered hardness and algorithms for rainbow connectivity. Furthermore, Li *et.al.* in [7] stated Rainbow connections of graphs - A survey. Also Li *et.al.* in [8] characterized graphs with rainbow connection number and rainbow connection numbers of some graph operations. Schiermeyer in [10] studied rainbow connection in graphs with minimum degree three.

A well-known result shows that in every $l-$connected graph G with $l \geq 1$, there are k internally disjoint $u-v$ paths connecting any two distinct vertices u and v for every integer k with $1 \leq k \leq l$ [9]. Chartrand et al. [5] defined the rainbow k-connectivity $rc_k(G)$ of G to be the minimum integer j for which there exists a j-edge-coloring of G such that for every two distinct vertices u and v of G, there exist at least k internally disjoint $u-v$ rainbow paths.

By the definition of rainbow k-connectivity $rc_k(G)$, we realize that it is almost impossible to derive the exact value or a nice bound of the rainbow k-connectivity for a general graph G [9]. To answer the problem: given that any connected graph G, determine the rainbow connection number $rc_k(G)$ of any graph G? It tends to be NP-hard problem. Thus, the study of rainbow k-connectivity of some classes of special graphs is still needed. In this paper we will study the rainbow connection number $rc_k(G)$ of Triangular Ladder, Wheel graphs, and edge comb of graph $G = C_n \trianglerighteq TL_m$ and $G = C_n \trianglerighteq K_m$. The edge comb between L and H, denoted by $L \trianglerighteq H$, is a graph obtained by taking one copy of L and $|E(L)|$ copies of H and grafting the i-th copy of H at the i-th edges of L. The result show that all the rainbow k-connection number $rc_k(G)$ of the graph studied in this paper achieve the minimum value.

2. The Results

Before presenting the main results we need to establish the lower bound of $rc_k(G)$ of any graph G such that the graph G is considered to be a k-connected graph. Note that the length of the shortest graph cycle (if any) in a given graph is known as a *girth*, and the length of a longest cycle is known as the *graph circumference*.

Theorem 1. *Let $d(u,v)$ be a distance between u and v, $C(u,v)$ is a shortest cycle that contains the vertices u and v. If G is 2-connected graph then $rc_2(G) \geq max$ $\{|C(u,v)| - d(u,v), \forall u,v \in V(G)\}$, where $C(u,v)$ and $d(u,v)$ are in one cycle.*

Proof. Let G be a connected cyclical graph. Thus, the length of second alternative internally disjoint rainbow path for any two vertices u and v is $|C(u,v)| - d(u,v)$ where $C(u,v)$ is a girth that contain vertices u and v. The greatest lower bound of

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$rc_2(G) \geq \max \{|C(u,v)| - d(u,v)\}$. By contradiction, if we color the edges of G by any value less than $\max \{|C(u,v)| - d(u,v)\}$ then there exist two vertices u and v that do not present two internally disjoint paths. □

We can extend the theorem for l-connected graph.

Lemma 1. *If G is l-connected graph, $l \geq 2$, then for every two vertices $u, v \in V(G)$, there exist at least $l-1$ cycles of G containing the vertices u and v.*

Proof. We can prove this theorem by contradiction. Suppose that there exist two vertices $u, v \in V(G)$ that contain one less than $l-1$ cycles of G. Suppose that the number of cycles that contain $u, v \in V(G)$ is $l-k$ where $k \geq 2$. The set $\{C_i | 1 \leq i \leq l-k\}$ is $l-k$ cycles that contain any two vertices in $V(G)$. One cycle is used to make two internally disjoint paths between u and v. Two cycles are used to make three internally disjoint paths between u and v. Since u and v are on $l-k$ cycles then the number of disjoint paths between u and v is $l-k+1$. Since $k \geq 2$ and we have two vertices with $l-k+1$ disjoint paths connecting u and v, then G is $(l-k+1 < l)$-connected graph. It is a contradiction. □

Theorem 2. *Let $d(u,v)$ be a distance between u and v, $C_i(u,v)$ be a shortest cycles that contain vertices u and v. Let C_i be cycles whose their common edge is uv. If G is l-connected graph then $rc_l(G) \geq max\{max\{|C_i(u,v)| - d(u,v), 1 \leq i \leq l-1\}, \forall u, v \in V(G)\}$, where $C(u,v)$ and $d(u,v)$ are in one cycle.*

Proof. If G is l-connected graph, then by Lemma 1 every vertex in $V(G)$ lays on at least $l-1$ cycles. Suppose the element of $\{C_i(u,v) | 1 \leq i \leq l-1\, u, v \in V(G)\}$ have $l-1$ cycles containing $u, v \in V(G)$, the $l-1$ cycles that contain u and v has to be minimum of size $|C_i(u,v)|$. The number of $rc_k(G)$ is at least $\max \{|C_i(u,v)| - d(u,v), 1 \leq i \leq l-1\}$. Otherwise there exist two vertices u, v that do not give k internally disjoint rainbow path. □

Now we will present some classes of graphs which can be determined their rainbow k-connection number.

Theorem 3. *Let G be a triangular ladder graph, the rainbow 2-connection number of G is $rc_2(G) = n$.*

Proof. Suppose $G = TL_n$. The graph G has vertex set $V(G) = \{x_i, y_i; 1 \leq i \leq n\}$ and edge set $E(G) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i y_i; 1 \leq i \leq n\}$. Define a color c of the edges $c : E(G) \to \{1, 2, \ldots, s\}, s \in N$:

$$c(e) = \begin{cases} n-i & , e \in \{x_i x_{i+1}; 1 \leq i \leq n-1\} \\ i & , e \in \{y_i y_{i+1}; 1 \leq i \leq n-1\} \\ 1 & , e \in \{x_i y_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 y_1\} \\ n & , e \in \{x_i y_i; 2 \leq i \leq n\} \end{cases}$$

It is easy to see that the color $c(e)$ reach a maximum value when $e = x_i y_i$ and $c(e) = n$. Thus, $rc_2(G) \leq n$. Now we will show that $rc_2(G) \geq n$. Consider the vertex $u = y_1$ and $v = x_n$. The vertex u and v lay on the cycle of size $2n$. Since distance, $d(u,v) = n$, then by Theorem 1, we have $rc_2(G) \geq 2n - n = n$. It concludes that $rc_2(G) = n$. □

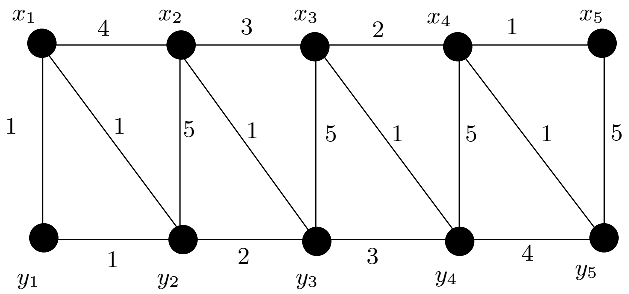

Figure 1. Graph $G = TL_5$ with $rc_2(G) = 5$

Theorem 4. *Let G be a wheel graph of order $n+1$, the rainbow 3-connection number G is $rc_3(W_n) = n$.*

Proof. Given that $G = W_n$. The graph G has vertex set $V(G) = \{x_i; 1 \le i \le n\} \cup \{A\}$ and edge set $E(G) = \{Ax_i; 1 \le i \le n\} \cup \{x_ix_{i+1}; 1 \le i \le n-1\} \cup \{x_1x_n\}$. Define a color c of the edges $c : E(G) \to \{1, 2, \ldots, s\}, s \in N$:

$$c(e) = \begin{cases} i, & e \in \{x_ix_{i+1}; 1 \le i \le n-1\} \cup \{Ax_i; 1 \le i \le n\} \\ n, & e \in \{x_1x_n\} \end{cases}$$

It is easy to see that the color $c(e)$ reach a maximum value when $e = x_1x_n$, thus $rc_3(W_n) \le n$. No we will show that $rc_3(W_n) \ge n$. We will use a contradiction. Suppose that $rc_3(W_n) \le n-1$, take $rc_3(W_n) = n-1$. Consider edge set $E' = \{x_ix_{i+1}|1 \le i \le n-1\} \cup \{x_1x_n\}$ and $|E'| = n+1$. If we color $n+1$ edges of E' by $n-1$ colors, then there exist $e_1, e_2 \in E'$ such that $c(e_1) = c(e_2)$, without loss of generality we can choose $e_1 = x_1x_2$ and $e_2 = x_ix_{i+1}$. Since W_n is 3-connected graph and $rc_3(W_n) = n-1$ then there must exist three disjoint paths between any two vertices. Consider vertex x_1 and vertex x_{i+1} which give three disjoint paths between x_1 and x_{i+1}. The first possible rainbow path is x_1Ax_{i+1}, the second is $x_1x_nx_{n-1}\ldots x_j$, however the third path $x_1x_2\ldots x_ix_{i+1}$, for x_1 and x_{i+1} is not rainbow path as $c(x_1x_2) = c(x_ix_{i+1})$. It is a contradiction, thus $rc_3(W_n) \ge n$. It concludes $rc_3(W_n) = n$. $\square$

Theorem 5. *If $G = C_n \trianglerighteq TL_m$ then $rc(G) = \frac{n}{2} + 2m - 2$ for n even and $rc_2(G) = 2m+1$ for $n = 4$.*

Proof. The graph $G = Cn \trianglerighteq Lt_m$ is a connected graph with vertex set $V(G) = \{x_i|1 \le i \le n\} \cup \{y_{i,j}|1 \le i \le n, 1 \le j \le m-1\} \cup \{z_{i,j}|1 \le i \le n, 1 \le j \le m-1\}$ and edge set $E(G) = \{x_ix_{i+1}|i \le i \le n-1\} \cup \{x_nx_1\} \cup \{x_iy_{i,1}|1 \le i \le n\} \cup \{x_{i+1}z_{i,1}|1 \le i \le n-1\} \cup \{x_1z_{n,1}\} \cup \{y_{i,j}y_{i,j+1}|1 \le i \le n, 1 \le j \le m-2\} \cup \{z_{i,j}z_{i,j+1}|1 \le i \le n, 1 \le j \le m-2\} \cup \{y_{i,j}z_{i,j}|1 \le i \le n, 1 \le j \le m-1\} \cup \{x_iz_{i,1}|1 \le i \le n\} \cup \{y_{i,j}z_{i,j+1}|1 \le i \le n, 1 \le j \le m-2\}$. The value $|V(G)| = n(2m-1)$ and $|E(G)| = 3n + 2n(m-2) + 2n(m-1)$. The diameter of G, $diam(G) = \frac{n}{2} + 2(m-1)$. The number $rc(G)$ is given by the following

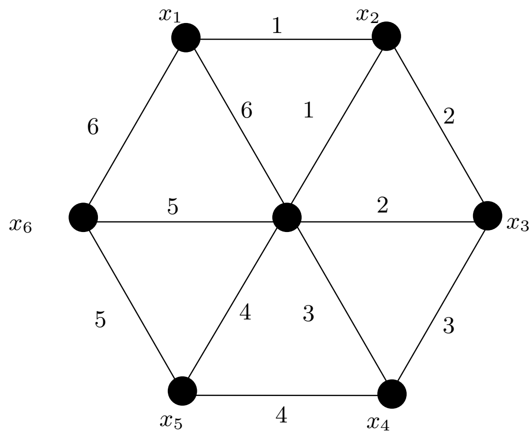

Figure 2. Graph $G = W_6$ with $rc_3(W_6) = 6$

coloring function:

$$c(e) = \begin{cases} i \bmod \frac{n}{2} & , e \in \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \\ & \{y_{i,j} z_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m-1\} \\ n \bmod \frac{n}{2}, & e \in \{x_n x_1\} \\ \frac{n}{2} + 1 & , e \in \{x_i y_{i,1} | 1 \leq i \leq n\} \cup \{x_i z_{i,1} | 1 \leq i \leq n\} \\ \frac{n}{2} + 1 + j & , e \in \{y_{i,j} y_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \\ & \cup \{y_{i,j} z_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \\ \frac{n}{2} + m & , e \in \{x_{i+1} z_{i,1} | 1 \leq i \leq n-1\} \cup \{x_1 z_{n,1}\} \\ \frac{n}{2} + m + j & , e \in \{z_{i,j} z_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \end{cases}$$

The maximum value of function is $c(e) = \frac{n}{2} + 2m - 2$ so $rc(G) \leq \frac{n}{2} + 2m - 2$. By applying Innequality 1 $rc(G) \geq \frac{n}{2} + 2m - 2$, it implies that $rc(G) = \frac{n}{2} + 2m - 2$.

The number $rc_2(G) \geq 2m + 1$ for $n = 4$ and any m, is obtained by coloring mapping:

$$c(e) = \begin{cases} i \bmod 2 & , e \in \{x_i x_{i+1} | 1 \leq i \leq 3\} \\ 2 & , e \in \{x_4 x_1\} \\ 3 & , e \in \{x_i y_{i,1} | 1 \leq i \leq n\} \cup \{x_i z_{i,1} | 1 \leq i \leq n\} \\ 3 + j & , e \in \{y_{i,j} y_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \\ & \cup \{y_{i,j} z_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \\ m + 2 & , e \in \{x_{i+1} z_{i,1} | 1 \leq i \leq n-1\} \cup \{x_1 z_{n,1}\} \\ m + 2 + j & , e \in \{z_{i,j} z_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \\ 2m + 1 & , e \in \{y_{i,j} z_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m-1\} \end{cases}$$

To prove $rc_2(G) \leq 2m + 1$, consider vertex $y_{2,m-1}$ and vertex $z_{1,m-1}$, the vertex $y_{2,m-1}$ and vertex $z_{1,m-1}$ lay on cycle of size of at least $4m - 1$. The distance between $y_{2,m-1}$ and $z_{1,m-1}$ is $2(m - 1)$ so the lengt of remaining shortest path between $y_{2,m-1}$ and $z_{1,m-1}$ is $2m + 1$. This path is the shortest alternative path from $y_{2,m-1}$ to $z_{1,m-1}$ to get the second internally disjoint rainbow path. □

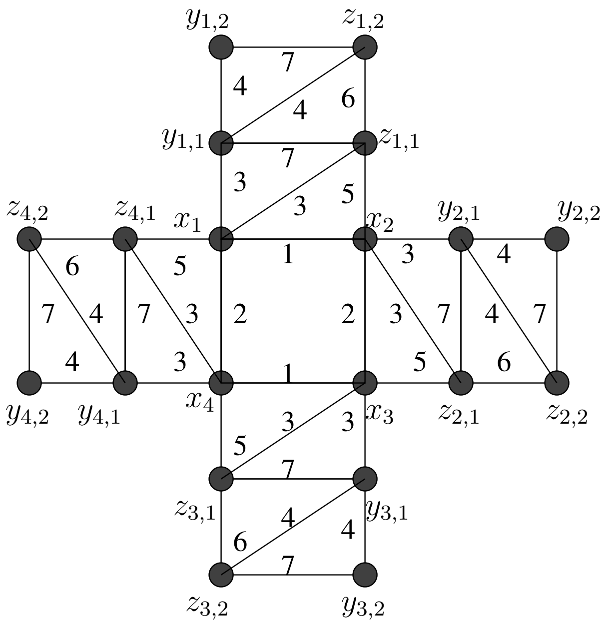

Figure 3. Graph edge comb $G = C_4 \trianglerighteq TL_3$ with $rc_2(G) = 7$.

Theorem 6. *If $G = C_n \trianglerighteq K_m$, then the number $rc(G) = \frac{n}{2} + 1$ for n even and $rc_2(G) = 4$, for $n = 4$.*

Proof. The graph $G = C_n \trianglerighteq K_m$ is a connected graph with vertex set $V(G) = \{x_i | 1 \leq i \leq n\} \cup \{y_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m-2\}$ and edge set $E(G) = \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{x_i y_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \{x_{i+1} y_{i,j} | 1 \leq i \leq n-1, 1 \leq j \leq m-2\} \cup \{x_1 y_{n,j} | 1 \leq j \leq m-2\} \cup (\bigcup_{l=1}^{m-3} (\{y_{i,l} y_{i,j+l} | 1 \leq i \leq n, 1 \leq j \leq m-2-l\})$. The number of vertices and edges of G is $|V(G)| = n + n(m-2)$ and $|E(G)| = n(1 + 2(m-2) + \frac{(m-2)(m-3)}{2})$. The Diameter of G, $diam(G) = \frac{n}{2} + 1$

The value $rc(G) = \frac{n}{2} + 1$ obtained by the following edge mapping function:

$$c(e) = \begin{cases} i \bmod \frac{n}{2} & , e \in \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \\ & \{x_i y_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \\ & (\bigcup_{l=1}^{m-3} (\{y_{i,l} y_{i,j+l} | 1 \leq i \leq n, \\ & 1 \leq j \leq m-2-l\}) \\ n \bmod \frac{n}{2} & , e \in \{x_n x_1\} \\ \frac{n}{2} + 1 & , e \in \{x_1 y_{n,j} | 1 \leq j \leq m-2\} \cup \\ & \{x_{i+1} y_{i,j} | 1 \leq i \leq n-1, 1 \leq j \leq m-2\} \end{cases}$$

The maximum value of $c(e)$ is $\frac{n}{2} + 1$ so $rc(G) \leq \frac{n}{2} + 1$, by applying Innequality 1 $rc(G) \geq \frac{n}{2} + 1$ and finally we get $rc(G) = \frac{n}{2} + 1$.

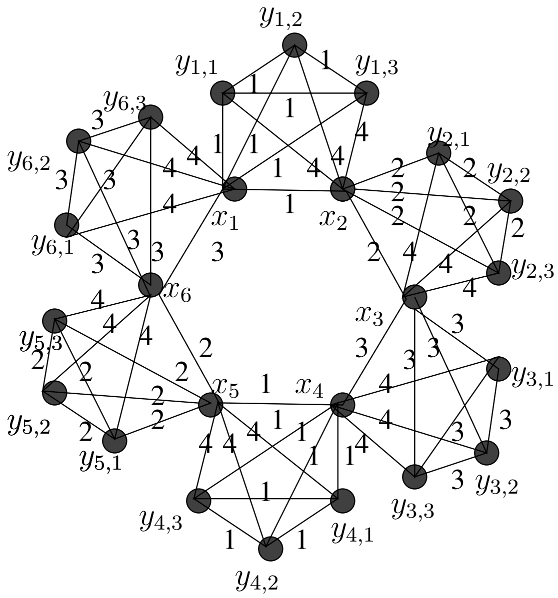

Figure 4. Graph edge comb $C_6 \trianglerighteq K_5$ with $rc(G) = 4$.

The value $rc_2(G) \geq 4$ for $n = 4$ and any m, is obtained by the following

$$c(e) = \begin{cases} i \bmod 2 & , e \in \{x_i x_{i+1} | 1 \leq i \leq 3\} \cup \{x_i y_{i,j} | 1 \leq i \leq 4, \\ & 1 \leq j \leq m-2\} \cup \{y_{i,j} y_{i,j+1} | 1 \leq i \leq 4, \\ & 1 \leq j \leq m-3\} \\ 4 \bmod 2 & , e \in \{x_4 x_1\} \\ 3 & , e \in \{x_1 y_{4,j} | 1 \leq j \leq m-2\} \cup \\ & \{x_{i+1} y_{i,j} | 1 \leq i \leq 3, 1 \leq j \leq m-2\} \\ 4 & , e \in \{x_i y_{i,j} | 1 \leq i \leq 4, 1 \leq j \leq m-2\} \\ & \cup (\bigcup_{l=1}^{m-3} (\{y_{i,l} y_{i,j+l} | 1 \leq i \leq 4, \\ & 1 \leq j \leq m-2-l\} - \{y_{i,j} y_{i,j+1} | 1 \leq i \leq 4, \\ & 1 \leq j \leq m-3\}) \end{cases}$$

To prove $rc_2(G) \leq 4$ consider vertex $y_{1,j}$ and $y_{2,k}$ for $1 \leq j, k \leq m-2$. This vertices is contained on cycle with size at least 6. The distance between $y_{1,j}$ and $y_{2,k}$ is 2 so the lengt of remaining shortest path between $y_{1,j}$ and $y_{2,k}$ is 4. This path is the shortest alternative path from $y_{1,j}$ to $y_{2,k}$ to make second internally disjoint rainbow path. □

Concluding Remarks

We have studied the rainbow k-connection number of G. The result show that all the rainbow k-connection number $rc_k(G)$ of the graph studied in this paper achieve the minimum value. We have also characterized any graph to have a minimum k-connection number, through the following theorem: If G is l-connected graph then

$rc_l(G) \geq \max\ \{|C_i(u,v)| - d(u,v), 1 \leq i \leq l-1\}$, where $|C_i(u,v)|$ is a girth that contains the vertices u and v. However, it is just lower bound, we have not found the sharper upper bound of $rc_k(G)$ of any graph. Thus we propose the following open problem.

Open Problem 1. *Given that any connected graph G, determine a sharp upper bound of the rainbow k-connection number $rc_k(G)$ of G.*

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