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Mixed Estimator of Kernel and Fourier Series in Semiparametric Regression

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Abstract. Given paired observation $\left(x_i, v_{1i}, v_{2i}, ..., v_{pi}, t_{1i}, t_{2i}, ..., t_{qi}, y_i\right), i = 1, 2, ..., n,$ follow the additive semiparametric regression model $y_i = \mu\left(x_i, \mathbf{v}_i, \mathbf{t}_i\right) + \varepsilon_i$, where

$$\mu\left(x_i, \mathbf{v}_i, \mathbf{t}_i\right) = f\left(x_i\right) + \sum_{j=1}^{p} g_j\left(v_{ji}\right) + \sum_{s=1}^{q} h_s\left(t_{si}\right)$$

$\mathbf{v}_i = \left(v_{1i}, v_{2i}, ..., v_{pi}\right)'$, and $\mathbf{t}_i = \left(t_{1i}, t_{2i}, ..., t_{qi}\right)'$. Random errors ε_i is a normal distribution with mean 0 and variance σ^2. To obtain a mixed estimator $\mu\left(x_i, \mathbf{v}_i, \mathbf{t}_i\right)$, the regression curve $f\left(x_i\right)$ is approached by linier parametric, $g_j\left(v_{ji}\right)$ is kernel with bandwidths $\mathbf{\Phi} = \left(\phi_1, \phi_2, \ldots, \phi_p\right)'$ and the regression curve component fourier series $h_s\left(t_{si}\right)$ is approached by $H_s\left(t_{si}\right) = b_s t_{si} + \frac{1}{2} a_{0s} + \sum_{k=1}^{N} a_{ks} \cos kt_{si}$ with oscillation paremeter N. The estimator $\sum_{j=1}^{p} g_j\left(v_{ji}\right)$ is $\sum_{j=1}^{p} \hat{\mathbf{g}}_{j\phi_j}\left(v_{ji}\right)$ where $\sum_{j=1}^{p} \hat{\mathbf{g}}_{j\phi_j}\left(v_{ji}\right) = \mathbf{V}\left(\mathbf{\Phi}\right)\mathbf{y}$. Penalized Least Squares (PLS) method give

$$\underset{\mathbf{c},\boldsymbol{\beta}}{Min}\left\{L\left(\mathbf{c}\right) + L\left(\boldsymbol{\beta}\right) + \sum_{s=1}^{q} \theta_s S\left(H_s\left(t_{si}\right)\right)\right\}$$

with smoothing parameter $\boldsymbol{\theta} = \left(\theta_1, \theta_2, \ldots, \theta_q\right)'$, the estimator $\mathbf{f}\left(x\right)$ is $\hat{\mathbf{f}}\left(x\right)$ and $\sum_{s=1}^{q} h_s\left(t_{si}\right)$ is $\sum_{s=1}^{q} \hat{\mathbf{h}}_{\theta_s}\left(t_{si}\right)$, where $\hat{\mathbf{f}}\left(x\right) = \mathbf{X}\hat{\boldsymbol{\beta}}$ and $\sum_{s=1}^{q} \hat{\mathbf{h}}_{\theta_s}\left(t_{si}\right) = \mathbf{X}\hat{\mathbf{c}}\left(\theta\right)$. So that,

$$\hat{\boldsymbol{\mu}}_{\mathbf{\Phi},\theta,N}\left(\mathbf{v}_i, \mathbf{t}_i\right) = \mathbf{Z}\left(\mathbf{\Phi}, \theta, N\right)\mathbf{y}$$

is the mixed estimator of $\boldsymbol{\mu}\left(\mathbf{v}_i, \mathbf{t}_i\right)$ where $\mathbf{Z}\left(\mathbf{\Phi}, \theta, N\right) = \mathbf{C}\left(\mathbf{\Phi}, \theta, N\right) + \mathbf{V}\left(\mathbf{\Phi}\right) + \mathbf{E}\left(\mathbf{\Phi}, \theta, N\right)$ Matrix $\mathbf{C}\left(\mathbf{\Phi}, \theta, N\right), \mathbf{V}\left(\mathbf{\Phi}\right)$ and $\mathbf{E}\left(\mathbf{\Phi}, \theta, N\right)$ are depended on $\mathbf{\Phi}, \theta$ and N. Optimal $\mathbf{\Phi}, \theta$ and N can be obtained by the smallest Generalized Cross Validation (GCV).

1. Introduction

One of the statistical methods that is often used by researchers in various research fields is regression analysis. It is used to investigate the pattern of the functional relationship between two or more variables. There are three approaches to estimate the regression curve, namely the approach of parametric regression, nonparametric regression and semiparametric regression [1]. If the regression curve shape is known, it is called parametric regression. If there is no shape information, it is called nonparametric regression [2]. Combination of parametric and nonparametric components is called semiparametric regression [3].

The semiparametric model with spline function has been applied by [4], [5] [6] and [2], kernel function is developed by [7], and [8], fourier series is developed by [9], and [10], local polynomial is developed by [11] and [12]. Kernel estimator is a linier estimator in nonparametric regression estimator that is depended on bandwidth. Kernel estimator is able to model data that have no particular pattern [1]. Fourier series estimators are used when there is a trend repeated data patterns [13]. Fourier series is used to describe the curve that show sine or cosine wave [9].

For nonparametric component, it has two heavy and basic assumptions. The first is each predictor pattern in the multivariable nonparametric regression model of the predictor is considered to have the same pattern. The second is researchers insist on using only one shape of model estimator for every predictor variable. In some cases, the two assumptions causing regression model estimation produced are not correct and tend to produce larger errors [1]. To overcome these problems, some researchers have developed the mixed estimator in which each data pattern is approached with appropriate estimator. They are [1] who developed the combination of spline and kernel estimator for nonparametric regression and [14] who developed the combined estimator fourier series and spline truncated in multivariable nonparametric regression. Meanwhile, there has been no research involving mixed estimator in semiparametric regression.

This study will review mixed estimator of kernel and fourier series in semiparametric regression. The mixed estimator assumed a predictor variable follow linier parametric, other variables follow kernel function and the others follow fourier series. It obtained through Penalized Least Squares (PLS) method.

2. Main Results

Given paired observation $\left(x_i, v_{1i}, v_{2i}, ..., v_{pi}, t_{1i}, t_{2i}, ..., t_{qi}, y_i\right)$, follow the additive semiparametric regression model

$$y_i = \mu\left(x_i, \mathbf{v}_i, \mathbf{t}_i\right) + \varepsilon_i, \tag{1}$$

where $i = 1, 2, ..., n,\ \mathbf{v}_i = \left(v_{1i}, v_{2i}, ..., v_{pi}\right)'$, and $\mathbf{t}_i = \left(t_{1i}, t_{2i}, ..., t_{qi}\right)'$. Random errors ε_i has normal distribution with $E\left(\varepsilon_i\right) = 0$ dan $Var\left(\varepsilon_i\right) = \sigma^2$. The regression curve $\mu\left(x_i, \mathbf{v}_i, \mathbf{t}_i\right)$ is assumed to be additive, it can be written as:

$$\mu\left(x_i, \mathbf{v}_i, \mathbf{t}_i\right) = f\left(x_i\right) + \sum_{j=1}^{p} g_j\left(v_{ji}\right) + \sum_{s=1}^{q} h_s\left(t_{si}\right) \tag{2}$$

with $f\left(x_i\right)$, $g_j\left(v_{ji}\right)$ and $h_s\left(t_{si}\right)$ being smooth function.

In equation (2), curve $f\left(x_i\right)$ is approached by linier parametric regression. Each regression curve $g_j\left(v_{ji}\right)$ is approached by kernel function. Each regression curve component $h_s\left(t_{si}\right)$ is approached by fourier series. Linier parametric regression is

$$f\left(x_i\right) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_p x_{ip}. \tag{3}$$

Lemma 2.1. *If linier parametric curve component in the equation (2) is estimated by estimator component in the equation (3), then*

$$\mathbf{f}(x) = \mathbf{X}\boldsymbol{\beta} \tag{4}$$

where

$$\mathbf{f}(x) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix}, \text{ and } \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix},$$

vector $\mathbf{f}(x)$ is $n \times 1$, *vector* $\boldsymbol{\beta}$ is $(p+1)\times 1$ *and matrix* $\mathbf{X}$ is $n \times (1+p)$.

Proof. In equation (3), for *i* = 1 to *i* = n then $f(x_1), f(x_2), \ldots, f(x_n)$. So that,

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_1 x_{11} + \ldots + \beta_p x_{1p} \\ \beta_0 + \beta_1 x_{21} + \ldots + \beta_p x_{2p} \\ \vdots \\ \beta_0 + \beta_1 x_{n1} + \ldots + \beta_p x_{np} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$$

$$= \mathbf{X}\boldsymbol{\beta}$$

Each kernel curve component is

$$\hat{g}_{j\phi_j}(v_j) = n^{-1} \sum_{i=1}^{n} W_{\phi_j i}(v_j) y_i \tag{5}$$

where

$$W_{\phi_j i}(v_j) = \frac{K_{\phi_j}(v_j - v_{ji})}{n^{-1} \sum_{i=1}^{n} K_{\phi_j}(v_j - v_{ji})} \text{ and } K_{\phi_j}(v_j - v_{ji}) = \frac{1}{\phi_j} K\left(\frac{v_j - v_{ji}}{\phi_j}\right).$$

Regression curve estimator (5) is depended on kernel function *K* and bandwidth parameter ϕ_j. Kernel function *K* can be a epanechnikov kernel, uniform kernel, triangle kernel, squares kernel, triweigth kernel, cosine kernel and Gaussian kernel [15].

Lemma 2.2. *If kernel curve component in the equation (2) is estimated by estimator component in the equation (5), then*

$$\sum_{j=1}^{p} \hat{\mathbf{g}}_{j\phi_j}(v_j) = \mathbf{V}(\boldsymbol{\Phi})\mathbf{y} \tag{6}$$

where

$$\hat{\mathbf{g}}_{j\phi_j}(v_j)=\begin{bmatrix}\hat{g}_{j\phi_j}(v_{j1})\\ \hat{g}_{j\phi_j}(v_{j2})\\ \vdots\\ \hat{g}_{j\phi_j}(v_{jn})\end{bmatrix},\ \mathbf{y}=\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix},\text{ and }\mathbf{V}(\boldsymbol{\Phi})=\begin{bmatrix} n^{-1}\sum_{j=1}^{p}W_{\phi_j1}(v_{j1}) & n^{-1}\sum_{j=1}^{p}W_{\phi_j2}(v_{j1}) & \cdots & n^{-1}\sum_{j=1}^{p}W_{\phi_jn}(v_{j1})\\ n^{-1}\sum_{j=1}^{p}W_{\phi_j1}(v_{j2}) & n^{-1}\sum_{j=1}^{p}W_{\phi_j2}(v_{j2}) & \cdots & n^{-1}\sum_{j=1}^{p}W_{\phi_jn}(v_{j2})\\ \vdots & \vdots & \ddots & \vdots\\ n^{-1}\sum_{j=1}^{p}W_{\phi_j1}(v_{jn}) & n^{-1}\sum_{j=1}^{p}W_{\phi_j2}(v_{jn}) & \cdots & n^{-1}\sum_{j=1}^{p}W_{\phi_jn}(v_{jn})\end{bmatrix} \tag{7}$$

Vector $\hat{\mathbf{g}}_{j\phi_j}(v_j)$ *is* $n\times 1$, *vector* $\mathbf{y}$ *is* $n\times 1$, *and matrix* $\mathbf{V}(\boldsymbol{\Phi})$ *is* $n\times n$

Proof. Each component $\hat{g}_{j\phi_j}(v_j)$ can be written by (5). So that, for each component $v_j=v_{j1}$ to $v_j=v_{jn}$, then

$$\begin{bmatrix}\hat{g}_{j\phi_j}(v_{j1})\\ \hat{g}_{j\phi_j}(v_{j2})\\ \vdots\\ \hat{g}_{j\phi_j}(v_{jn})\end{bmatrix}=\begin{bmatrix} n^{-1}W_{\phi_j1}(v_{j1}) & n^{-1}W_{\phi_j2}(v_{j1}) & \cdots & n^{-1}W_{\phi_jn}(v_{j1})\\ n^{-1}W_{\phi_j1}(v_{j2}) & n^{-1}W_{\phi_j2}(v_{j2}) & \cdots & n^{-1}W_{\phi_jn}(v_{j2})\\ \vdots & \vdots & \ddots & \vdots\\ n^{-1}W_{\phi_j1}(v_{jn}) & n^{-1}W_{\phi_j2}(v_{jn}) & \cdots & n^{-1}W_{\phi_jn}(v_{jn})\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}.$$

In matrix notation can be written as:

$$\hat{\mathbf{g}}_{j\phi_j}(v_j)=\mathbf{V_j}(\boldsymbol{\phi}_\mathbf{j})\mathbf{y}, \tag{8}$$

where

$$\mathbf{V_j}(\boldsymbol{\phi}_\mathbf{j})=\begin{bmatrix} n^{-1}W_{\phi_j1}(v_{j1}) & n^{-1}W_{\phi_j2}(v_{j1}) & \cdots & n^{-1}W_{\phi_jn}(v_{j1})\\ n^{-1}W_{\phi_j1}(v_{j2}) & n^{-1}W_{\phi_j2}(v_{j2}) & \cdots & n^{-1}W_{\phi_jn}(v_{j2})\\ \vdots & \vdots & \ddots & \vdots\\ n^{-1}W_{\phi_j1}(v_{jn}) & n^{-1}W_{\phi_j2}(v_{jn}) & \cdots & n^{-1}W_{\phi_jn}(v_{jn})\end{bmatrix}$$

matrix $\mathbf{V_j}(\boldsymbol{\phi}_\mathbf{j})$ is $n\times n$. The sum of component (8) from j = 1 to j = p is

$$\begin{aligned}\sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}(v_j)&=\sum_{j=1}^{p}\left\{\begin{bmatrix} n^{-1}W_{\phi_j1}(v_{j1}) & n^{-1}W_{\phi_j2}(v_{j1}) & \cdots & n^{-1}W_{\phi_jn}(v_{j1})\\ n^{-1}W_{\phi_j1}(v_{j2}) & n^{-1}W_{\phi_j2}(v_{j2}) & \cdots & n^{-1}W_{\phi_jn}(v_{j2})\\ \vdots & \vdots & \ddots & \vdots\\ n^{-1}W_{\phi_j1}(v_{jn}) & n^{-1}W_{\phi_j2}(v_{jn}) & \cdots & n^{-1}W_{\phi_jn}(v_{jn})\end{bmatrix}\mathbf{y}\right\}\\
&=\begin{bmatrix} n^{-1}\sum_{j=1}^{p}W_{\phi_j1}(v_{j1}) & n^{-1}\sum_{j=1}^{p}W_{\phi_j2}(v_{j1}) & \cdots & n^{-1}\sum_{j=1}^{p}W_{\phi_jn}(v_{j1})\\ n^{-1}\sum_{j=1}^{p}W_{\phi_j1}(v_{j2}) & n^{-1}\sum_{j=1}^{p}W_{\phi_j2}(v_{j2}) & \cdots & n^{-1}\sum_{j=1}^{p}W_{\phi_jn}(v_{j2})\\ \vdots & \vdots & \ddots & \vdots\\ n^{-1}\sum_{j=1}^{p}W_{\phi_j1}(v_{jn}) & n^{-1}\sum_{j=1}^{p}W_{\phi_j2}(v_{jn}) & \cdots & n^{-1}\sum_{j=1}^{p}W_{\phi_jn}(v_{jn})\end{bmatrix}\mathbf{y}\\
&=\mathbf{V}(\boldsymbol{\Phi})\mathbf{y}.\end{aligned}$$

Thus proved that $\sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}\left(v_j\right)=\mathbf{V}\left(\mathbf{\Phi}\right)\mathbf{y}$.

Each fourier series curve component is approached by function $H_s\left(t_{si}\right)$, where

$$H_s\left(t_{si}\right)=b_s t_{si}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{si}. \tag{9}$$

Lemma 2.3 *If fourier series curve component in the equation (2) is estimated by function in the equation (9), then*

$$\sum_{s=1}^{q}\mathbf{H}_s\left(t_s\right)=\mathbf{U}\mathbf{c} \tag{10}$$

where

$$\mathbf{U}=\left[\mathbf{U}_1\quad \mathbf{U}_2\quad \cdots\quad \mathbf{U}_p\right],\ \mathbf{c}=\left[\mathbf{c}_1'\quad \mathbf{c}_2'\quad \cdots\quad \mathbf{c}_p'\right]',\ \mathbf{H}_s\left(t_s\right)=\left[H_s\left(t_{s1}\right)\quad H_s\left(t_{s2}\right)\quad \cdots\quad H_s\left(t_{sn}\right)\right]'. \tag{11}$$

Vector $\mathbf{H}_s\left(t_s\right)$ *is* $n\times 1$, *vector* $\mathbf{c}$ *is* $p\left(N+2\right)\times 1$ *and matrix* $\mathbf{U}$ *is* $n\times\left(p\left(N+2\right)\right)$.

Proof. In equation (9), for i = 1 to i = n then $H_s\left(t_{s1}\right), H_s\left(t_{s2}\right),\ldots, H_s\left(t_{sn}\right)$. So that

$$\begin{bmatrix} H_s\left(t_{s1}\right)\\ H_s\left(t_{s2}\right)\\ \vdots\\ H_s\left(t_{sn}\right)\end{bmatrix}=\begin{bmatrix} b_s t_{s1}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{s1}\\ b_s t_{s2}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{s2}\\ \vdots\\ b_s t_{sn}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{sn}\end{bmatrix}$$

$$=\begin{bmatrix} t_{s1} & 1 & \cos t_{s1} & \cos 2t_{s1} & \ldots & \cos Nt_{s1}\\ t_{s2} & 1 & \cos t_{s2} & \cos 2t_{s2} & \ldots & \cos Nt_{s2}\\ t_{s3} & 1 & \cos t_{s3} & \cos 2t_{s3} & \ldots & \cos Nt_{s3}\\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots\\ t_{sn} & 1 & \cos t_{sn} & \cos 2t_{sn} & \ldots & \cos Nt_{sn}\end{bmatrix}\begin{pmatrix} b_s\\ \frac{1}{2}a_{0s}\\ a_{1s}\\ \vdots\\ a_{Ns}\end{pmatrix}$$

In matrix notation can be written as:

$$\mathbf{H}_s\left(t_s\right)=\mathbf{U}_s\mathbf{c}_s \tag{12}$$

where

$$\mathbf{U}_s=\begin{bmatrix} t_{s1} & 1 & \cos t_{s1} & \cos 2t_{s1} & \ldots & \cos Nt_{s1}\\ t_{s2} & 1 & \cos t_{s2} & \cos 2t_{s2} & \ldots & \cos Nt_{s2}\\ t_{s3} & 1 & \cos t_{s3} & \cos 2t_{s3} & \ldots & \cos Nt_{s3}\\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots\\ t_{sn} & 1 & \cos t_{sn} & \cos 2t_{sn} & \ldots & \cos Nt_{sn}\end{bmatrix},\ \text{and}\ \mathbf{c}_s=\begin{pmatrix} b_s\\ \frac{1}{2}a_{0s}\\ a_{1s}\\ \vdots\\ a_{Ns}\end{pmatrix}$$

Matrix $\mathbf{U}_s$ is $n\times\left(N+2\right)$ and vector $\mathbf{c}_s$ is $\left(N+2\right)\times 1$.

The sum of component (12) from $s=1$ to $s=q$ is

$$\sum_{s=1}^{q}\mathbf{H}_s\left(t_s\right)=\sum_{s=1}^{q}\begin{bmatrix} b_s t_{s1}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{s1} \\ b_s t_{s2}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{s2} \\ \vdots \\ b_s t_{sn}+\frac{1}{2}a_{0s}+\sum_{k=1}^{N}a_{ks}\cos kt_{sn} \end{bmatrix}=\begin{bmatrix} \sum_{s=1}^{q}b_s t_{s1}+\sum_{s=1}^{q}\frac{1}{2}a_{0s}+\sum_{s=1}^{q}\sum_{k=1}^{N}a_{ks}\cos kt_{s1} \\ \sum_{s=1}^{q}b_s t_{s2}+\sum_{s=1}^{q}\frac{1}{2}a_{0s}+\sum_{s=1}^{q}\sum_{k=1}^{N}a_{ks}\cos kt_{s2} \\ \vdots \\ \sum_{s=1}^{q}b_s t_{sn}+\sum_{s=1}^{q}\frac{1}{2}a_{0s}+\sum_{s=1}^{q}\sum_{k=1}^{N}a_{ks}\cos kt_{sn} \end{bmatrix}=\mathbf{Uc}$$

where matrix $\sum_{s=1}^{q}\mathbf{H}_s\left(t_s\right)=\mathbf{Uc}$ is given by equation (10).

PLS is optimization methods in statistics. It combines the goodness of fit with the smoothness of curves, which between them controlled by a smoothing parameter [16] dan [17]. PLS method can be defined as

$$\underset{\mathbf{c},\boldsymbol{\beta}}{Min}\left\{\left[L(\mathbf{c})+L(\boldsymbol{\beta})\right]+\sum_{s=1}^{q}\theta_s S\left(H_s\left(t_{si}\right)\right)\right\} \tag{13}$$

where $\left[L(\mathbf{c})+L(\boldsymbol{\beta})\right]$ is the goodness of fit, $S\left(H_s\left(t_{si}\right)\right)$ is penalty function and θ_s is a smoothing parameter.

Lemma 2.4. *If regression model is given by equation* (1), *linier parametric curve is given by Lemma 2.2, kernel curve is given by Lemma 2.2 and fourier series is given by Lemma 2.3, then goodness of fit function is*

$$L(\mathbf{c})+L(\boldsymbol{\beta})=n^{-1}\left\|\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}-\mathbf{P}\right\|^2 \tag{14}$$

where matrix $\mathbf{I}$ *is* $n\times n$ *identity matrix and vector* $\mathbf{P}=\mathbf{X}\boldsymbol{\beta}+\mathbf{Uc}$ *is* $n\times 1$.

Proof. The equation (1), (2), (4), (6) and (10) give

$$\begin{aligned} L(\mathbf{c})+L(\boldsymbol{\beta}) &= n^{-1}\left\|\mathbf{y}-\mathbf{f}(x)-\sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}\left(v_j\right)-\sum_{s=1}^{q}\mathbf{H}_s\left(t_s\right)\right\|^2 \\ &= n^{-1}\left\|\mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{V}(\boldsymbol{\Phi})\mathbf{y}-\mathbf{Uc}\right\|^2 \\ &= n^{-1}\left\|\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}-\left(\mathbf{X}\boldsymbol{\beta}+\mathbf{Uc}\right)\right\|^2 \\ &= n^{-1}\left\|\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}-\mathbf{P}\right\|^2 \end{aligned}$$

Lemma 2.5. *If fourier series is given by Lemma 2.3, then penalty function and smoothing parameter for PLS method is*

$$\sum_{s=1}^{q}\theta_s S\left(H_s\left(t_{si}\right)\right)=\mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c} \tag{15}$$

where $\mathbf{D}(\boldsymbol{\theta})=diag\left(\mathbf{D}_1(\boldsymbol{\theta})\quad \mathbf{D}_2(\boldsymbol{\theta})\quad \cdots\quad \mathbf{D}_p(\boldsymbol{\theta})\right)$ *and* $\mathbf{D}_s(\boldsymbol{\theta})=diag\left(0\quad 0\quad \theta_s 1^4\quad \theta_s 2^4\quad \cdots\quad \theta_s N^4\right)$. *Matrix* $\mathbf{D}(\boldsymbol{\theta})$ *is* $\left(p\times(N+2)\right)\times\left(p\times(N+2)\right)$ *and* $\mathbf{D}_s(\boldsymbol{\theta})$ *is* $(N+2)\times(N+2)$.

Proof. According Bilodeau [7]

$$\int_0^{\pi} \frac{2}{\pi}\left(H_s^{(2)}\left(t_{si}\right)\right)^2 dt_{si} = \left(\sum_{k=1}^{N} k^4 a^2{}_{ks}\right) \tag{16}$$

then

$$\sum_{s=1}^{q} \theta_s S\left(H_s\left(t_{si}\right)\right) = \sum_{s=1}^{q} \theta_s \int_0^{\pi} \frac{2}{\pi}\left(H_s^{(2)}\left(t_{si}\right)\right)^2 dt_s = \sum_{s=1}^{q} \theta_s \left(\sum_{k=1}^{N} k^4 a^2{}_{ks}\right) = \mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c}$$

Theorem 2.1. *If the goodness of fit function is given by Lemma 2.4 while penalty function and smoothing parameter is given by Lemma 2.5, then the PLS estimator for* $\mathbf{c}$ *parameter is obtained from optimization:*

$$\underset{\mathbf{c},\boldsymbol{\beta}}{Min}\left\{\left[L(\mathbf{c})+L(\boldsymbol{\beta})\right]+\sum_{s=1}^{q}\theta_s S\left(H_s\left(t_{si}\right)\right)\right\} = \underset{\mathbf{c},\boldsymbol{\beta}}{Min}\left\{\mathbf{Q}(\mathbf{c},\boldsymbol{\beta})\right\} \tag{17}$$

where $\mathbf{Q}(\mathbf{c},\boldsymbol{\beta}) = n^{-1}\left\|\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}-\mathbf{P}\right\|^2 + \mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c}$

Proof.

$$\underset{\mathbf{c},\boldsymbol{\beta}}{Min}\left\{\left[L(\mathbf{c})+L(\boldsymbol{\beta})\right]+\sum_{s=1}^{q}\theta_s S\left(H_s\left(t_{si}\right)\right)\right\} = \underset{\mathbf{a},\boldsymbol{\beta}}{Min}\left\{n^{-1}\left\|\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}-\mathbf{P}\right\|^2 + \mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c}\right\} = \underset{\mathbf{c},\boldsymbol{\beta}}{Min}\left\{\mathbf{Q}(\mathbf{c},\boldsymbol{\beta})\right\}$$

Theorem 2.2. *If the PLS estimator for parameter* $\mathbf{c}$ *and* $\boldsymbol{\beta}$ *is given by Theorem 2.1, then PLS estimator for the mixed regression curve* $\boldsymbol{\mu}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right)$ *is given by*

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{\Phi},\theta}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right) = \hat{\mathbf{f}}(x) + \sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}\left(v_j\right) + \sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}\left(t_s\right) \tag{18}$$

where $\hat{\mathbf{f}}(x) = \mathbf{X}\hat{\boldsymbol{\beta}}$, $\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}\left(t_s\right) = \mathbf{U}\hat{\mathbf{c}}(\boldsymbol{\theta})$, $\hat{\boldsymbol{\beta}} = \mathbf{A}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y}$ *and* $\hat{\mathbf{c}}(\boldsymbol{\theta}) = \mathbf{B}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y}$.

Proof. The PLS estimator for parameter $\mathbf{c}$ and $\boldsymbol{\beta}$ is obtained from optimization Theorem 2.1, so that

$$\begin{aligned}
\mathbf{Q}(\mathbf{c},\boldsymbol{\beta}) &= n^{-1}\left\|\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}-\mathbf{P}\right\|^2 + \mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c} \\
&= n^{-1}\mathbf{y}'\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)'\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y} - 2n^{-1}\mathbf{P}'\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y} + n^{-1}\mathbf{P}'\mathbf{P} + \mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c} \\
&= n^{-1}\mathbf{y}'\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)'\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y} - 2n^{-1}\left(\boldsymbol{\beta}'\mathbf{X}+\mathbf{c}'\mathbf{U}'\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y} + n^{-1}\left(\boldsymbol{\beta}'\mathbf{X}'+\mathbf{c}'\mathbf{U}'\right)\left(\mathbf{X}\boldsymbol{\beta}+\mathbf{U}\mathbf{c}\right) + \mathbf{c}'\mathbf{D}(\boldsymbol{\theta})\mathbf{c}
\end{aligned}$$

Furthermore, the partial derivatives on $\mathbf{c}$ is $\frac{\partial \mathbf{Q}(\mathbf{c},\boldsymbol{\beta})}{\partial \mathbf{c}} = 0$. So that, the estimator for $\mathbf{c}$ is given by:

$$\hat{\mathbf{c}}(\boldsymbol{\theta}) = \left(\mathbf{U}'\mathbf{U} + n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\left(\mathbf{U}'\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y} - \mathbf{U}'\mathbf{X}\hat{\boldsymbol{\beta}}\right) \tag{19}$$

The partial derivatives on $\boldsymbol{\beta}$ is $\frac{\partial \mathbf{Q}(\mathbf{c},\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = 0$. So that, the estimator for $\boldsymbol{\beta}$ is given by:

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y} - \mathbf{X}'\mathbf{U}\hat{\mathbf{c}}(\boldsymbol{\theta})\right) \tag{20}$$

The equation (19) and (20) give

$$\hat{\boldsymbol{\beta}} = \mathbf{A}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y} \tag{21}$$

where

$$\mathbf{A}(\boldsymbol{\Phi},\boldsymbol{\theta},N)=\left[\left((\mathbf{X}'\mathbf{X})-\mathbf{X}'\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\mathbf{X}\right)\right]^{-1}\left(\mathbf{X}-\mathbf{X}'\mathbf{U}\left(\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\right)\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)$$

The equation (19) and (21) give

$$\hat{\mathbf{c}}(\boldsymbol{\theta})=\mathbf{B}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y} \tag{22}$$

where

$$\mathbf{B}(\boldsymbol{\Phi},\boldsymbol{\theta},N)=(\mathbf{U}'\mathbf{U}+n\mathbf{D})^{-1}\mathbf{U}'\left(\mathbf{I}-\mathbf{X}\left[\left((\mathbf{X}'\mathbf{X})-\mathbf{X}'\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\mathbf{X}\right)\right]^{-1}\left(\mathbf{X}-\mathbf{X}'\mathbf{U}\left(\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\right)\right)\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)$$

Considering the equation (21) then estimator for linier parametric curve is given by:

$$\hat{\mathbf{f}}(x)=\mathbf{X}\hat{\boldsymbol{\beta}} \tag{23}$$

and considering the equation (22) then estimator for fourier series curve is given by:

$$\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}(t_s)=\mathbf{U}\hat{\mathbf{c}}(\boldsymbol{\theta}). \tag{24}$$

So that, based on the equation (23), (24) and Lemma 2.2, then the mixed estimators for semiparametric regression curve (2) is

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{\Phi},\boldsymbol{\theta},N}(x_i,\mathbf{v}_i,\mathbf{t}_i)=\hat{\mathbf{f}}(x)+\sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}(v_j)+\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}(t_s)$$

Corollary. *If the estimator* $\hat{\boldsymbol{\mu}}_{\boldsymbol{\Phi},\boldsymbol{\theta},N}(x_i,\mathbf{v}_i,\mathbf{t}_i),\hat{\mathbf{f}}(x)$, $\sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}(v_j)$, $\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}(t_s)$ $\hat{\boldsymbol{\beta}}$, *and* $\hat{\mathbf{c}}(\boldsymbol{\theta})$ *are given by Theorem 2.2, then* $\hat{\mathbf{f}}(x)=\mathbf{C}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y}$, $\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}(t_{si})=\mathbf{E}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y}$ *and* $\hat{\boldsymbol{\mu}}_{\boldsymbol{\Phi},\boldsymbol{\theta},N}(x_i,\mathbf{v}_i,\mathbf{t}_i)=\mathbf{Z}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y}$
where

$$\mathbf{C}(\boldsymbol{\Phi},\boldsymbol{\theta},N)=\mathbf{X}\left[\left((\mathbf{X}'\mathbf{X})-\mathbf{X}'\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\mathbf{X}\right)\right]^{-1}\left(\mathbf{X}-\mathbf{X}'\mathbf{U}\left(\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\right)\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right),$$

$$\mathbf{E}(\boldsymbol{\Phi},\boldsymbol{\theta},N)=\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\left(\mathbf{I}-\mathbf{X}\left[\left((\mathbf{X}'\mathbf{X})-\mathbf{X}'\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\mathbf{X}\right)\right]^{-1}\left(\mathbf{X}-\mathbf{X}'\mathbf{U}\left(\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\right)\right)\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)$$

and $\mathbf{Z}(\boldsymbol{\Phi},\boldsymbol{\theta},N)=\mathbf{C}(\boldsymbol{\Phi},\boldsymbol{\theta},N)+\mathbf{V}(\boldsymbol{\Phi})+\mathbf{E}(\boldsymbol{\Phi},\boldsymbol{\theta},N)$.

Proof. The equation (21) and (23) given an equation

$$\hat{\mathbf{f}}(x)=\mathbf{X}\left[\left((\mathbf{X}'\mathbf{X})-\mathbf{X}'\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\mathbf{X}\right)\right]^{-1}\left(\mathbf{X}-\mathbf{X}'\mathbf{U}\left(\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\right)\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}$$

or it can written as:

$$\hat{\mathbf{f}}(x)=\mathbf{C}(\boldsymbol{\Phi},\boldsymbol{\theta},N)\mathbf{y} \tag{25}$$

The equation (22) and (24) given an equation

$$\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}(t_{si})=\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\left(\mathbf{I}-\mathbf{X}\left[\left((\mathbf{X}'\mathbf{X})-\mathbf{X}'\mathbf{U}\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\mathbf{X}\right)\right]^{-1}\left(\mathbf{X}-\mathbf{X}'\mathbf{U}\left(\left(\mathbf{U}'\mathbf{U}+n\mathbf{D}(\boldsymbol{\theta})\right)^{-1}\mathbf{U}'\right)\right)\right)\left(\mathbf{I}-\mathbf{V}(\boldsymbol{\Phi})\right)\mathbf{y}$$

or it can be written as:

$$\sum_{s=1}^{q} \hat{\mathbf{h}}_{\theta_s,N}\left(t_{si}\right) = \mathbf{E}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y} \tag{26}$$

Furthermore, the Theorem 2.2, equation (25) and (26) produce:

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_{\mathbf{\Phi},\boldsymbol{\theta},N}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right) &= \hat{\mathbf{f}}\left(x\right) + \sum_{j=1}^{p} \hat{\mathbf{g}}_{j\phi_j}\left(v_j\right) + \sum_{s=1}^{q} \hat{\mathbf{h}}_{\theta_s,N}\left(t_s\right) \\
&= \mathbf{C}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y} + \mathbf{V}\left(\mathbf{\Phi}\right)\mathbf{y} + \mathbf{E}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y} \\
&= \left(\mathbf{C}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right) + \mathbf{V}\left(\mathbf{\Phi}\right) + \mathbf{E}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\right)\mathbf{y} \\
&= \mathbf{Z}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y}
\end{aligned}$$

where

$$\mathbf{Z}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right) = \mathbf{C}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right) + \mathbf{V}\left(\mathbf{\Phi}\right) + \mathbf{E}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right) \tag{27}$$

Estimator $\hat{\boldsymbol{\mu}}_{\mathbf{\Phi},\boldsymbol{\theta},N}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right)$ are depended on the bandwidth $\mathbf{\Phi} = \left(\phi_1,\phi_2,\ldots,\phi_p\right)'$, smoothing parameter $\boldsymbol{\theta} = \left(\theta_1,\theta_2,\ldots,\theta_q\right)'$ and oscillation parameter *N*. Generalized Cross Validation (GCV) method can give the best bandwidth, smoothing parameter and oscillation parameter.

$$GCV\left(\mathbf{\Phi},\boldsymbol{\theta},N\right) = \frac{MSE\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)}{\left(n^{-1}\left(\mathbf{I}-\mathbf{Z}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\right)\right)^2} \tag{28}$$

where $MSE\left(\mathbf{\Phi},\boldsymbol{\theta},N\right) = n^{-1}\mathbf{y}'\left(\mathbf{I}-\mathbf{Z}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\right)'\left(\mathbf{I}-\mathbf{Z}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\right)\mathbf{y}$. Optimal bandwidth $\mathbf{\Phi}_{(\mathrm{opt})} = \left(\phi_{1(\mathrm{opt})},\phi_{2(\mathrm{opt})},\ldots,\phi_{p(\mathrm{opt})}\right)'$, smoothing parameter $\boldsymbol{\theta}_{(opt)} = \left(\theta_{1(opt)},\theta_{2(opt)},\ldots,\theta_{q(opt)}\right)'$ and oscillation parameter $N_{(opt)}$ are obtained by optimation

$$GCV\left(\mathbf{\Phi}_{(opt)},\boldsymbol{\theta}_{(opt)},N_{(opt)}\right) = \underset{\mathbf{\Phi},\boldsymbol{\theta},N}{\mathrm{Min}}\left\{GCV\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\right\}.$$

3. Conclusions

If given additive regression semiparametric model:

$$\mathbf{y} = \boldsymbol{\mu}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right) + \boldsymbol{\varepsilon} = \mathbf{f}\left(x_i\right) + \sum_{j=1}^{p}\mathbf{g}_j\left(v_{ji}\right) + \sum_{s=1}^{q}\mathbf{h}_s\left(t_{si}\right) + \boldsymbol{\varepsilon}$$

then

1. Mixed estimator of kernel and fourier series in semiparametric regression is given by

$$\hat{\boldsymbol{\mu}}_{\mathbf{\Phi},\theta,N}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right) = \hat{\mathbf{f}}\left(x\right) + \sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}\left(v_j\right) + \sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}\left(t_s\right)$$

where $\hat{\mathbf{f}}\left(x\right) = \mathbf{C}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y}$, $\sum_{j=1}^{p}\hat{\mathbf{g}}_{j\phi_j}\left(v_j\right) = \mathbf{V}\left(\mathbf{\Phi}\right)\mathbf{y}$, $\sum_{s=1}^{q}\hat{\mathbf{h}}_{\theta_s,N}\left(t_s\right) = \mathbf{E}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y}$ and $\hat{\boldsymbol{\mu}}_{\mathbf{\Phi},\boldsymbol{\theta},N}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right) = \mathbf{Z}\left(\mathbf{\Phi},\boldsymbol{\theta},N\right)\mathbf{y}$.

2. Mixed estimator kernel and fourier series $\hat{\boldsymbol{\mu}}_{\mathbf{\Phi},\boldsymbol{\theta},N}\left(x_i,\mathbf{v}_i,\mathbf{t}_i\right)$ is depended on $\mathbf{\Phi}_{(opt)},\boldsymbol{\theta}_{(opt)}$, and $N_{(opt)}$ that can be obtained by the smallest GCV value.

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