

Speeds of light in Stueckelberg-Horwitz-Piron electrodynamics

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Abstract. Stueckelberg-Horwitz-Piron (SHP) electrodynamics formalizes the distinction between coordinate time (measured by laboratory clocks) and chronology (temporal ordering) by defining 4D spacetime events x^μ as functions of an external evolution parameter τ . As τ grows monotonically, the spacetime evolution of classical events $x^\mu(\tau)$ trace out particle worldlines dynamically and induce the *five* U(1) gauge potentials through which events interact.

In analogy with the constant c that associates a unit of length x^0 with intervals of time t in standard relativity, we introduce a constant c_5 associated with the external time τ . Whereas the nonrelativistic limit of special relativity can be found by taking $c \rightarrow \infty$, we show that 5D SHP goes over to an equilibrium state of Maxwell theory in the limit $c_5 \rightarrow 0$. Thus, the dimensionless ratio c_5/c parameterizes the deviation of SHP from standard electrodynamics, in particular the coupling of events. Put another way, Maxwell theory can be understood as currents and fields relaxing to an equilibrium independent of chronological time as $c_5\tau$ slows to zero. We find that taking $0 < c_5/c < 1$ enables the resolution of several longstanding difficulties in SHP theory.

1. Introduction

The description of an antiparticle as a particle propagating backward in time was given by Stueckelberg [1] in the context of a relativistic Hamiltonian classical mechanics. In his model, a pair process is represented by a single worldline, generated dynamically by the interaction of a spacetime event $x^\mu(\tau)$ with an electromagnetic field as the external parameter τ grows monotonically from $\tau = -\infty$ to $\tau = \infty$. By explicitly distinguishing [2] the coordinate time x^0 (measured by laboratory clocks) from chronological time τ (temporal ordering), the parameter τ is given a role formally similar to the Galilean time in Newtonian theory, serving Stueckelberg's broader goal of generalizing the techniques of nonrelativistic classical and quantum mechanics to covariant form. This formalism has been extended by Piron, Horwitz, *et al.* [1] - [18], so that in its present form Stueckelberg-Horwitz-Piron (SHP) electrodynamics describes a U(1) gauge theory in which the τ -dependence of gauge transformations requires *five* compensation potentials. The resulting theory can be understood as a microscopic event dynamics in which the chronological increment $\tau \rightarrow \tau' = \tau + d\tau$ generates, by way of a covariant Hamiltonian, evolution of a 4D block universe defined at τ to an infinitesimally close 4D block universe defined at τ' . Standard Maxwell theory emerges as an equilibrium limit in which the system becomes τ -independent, and the 4D block universe remains static.

In a formal approach to special relativity that takes Minkowski geometry as its starting point,



the constant c is introduced as a means of measuring time in units of spatial distance, and the notion of a speed of light emerges from the role played by c in wave equations for U(1) gauge fields. Defining the flat metric in Minkowski space as

$$c^2 ds^2 = -g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - \mathbf{dx}^2 \quad g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad (1)$$

and parameterizing by proper time s imposes a constraint on four-velocity

$$-c^2 = \frac{dx^\mu}{ds} \frac{dx_\mu}{ds} = -\left(\frac{dx^0}{ds}\right)^2 + \left(\frac{d\mathbf{x}}{ds}\right)^2 = -\left(\frac{dx^0}{ds}\right)^2 \left[1 - \left(\frac{d\mathbf{x}}{dx^0}\right)^2\right] \quad (2)$$

so that

$$\frac{dt}{ds} = \gamma = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \quad (3)$$

equivalent to the mass-shell constraint

$$c^2 m^2 = -\left(m \frac{dx^\mu}{ds}\right) \left(m \frac{dx_\mu}{ds}\right) = -p^2 = \frac{E^2}{c^2} - \mathbf{p}^2 \quad (4)$$

Given this constraint, the nonrelativistic limit can be recovered by taking $c \rightarrow \infty$, expressing instantaneous action at a distance.

Using natural units ($\hbar = c = 1$) in the development of SHP electrodynamics, no explicit constant was assigned to the external time τ , implicitly associating c with both times. In this paper we associate a new constant c_5 with the invariant time τ , identify the expressions in which it must appear and study its role in the classical electromagnetic theory. We will show that 5D SHP goes over to Maxwell theory as an equilibrium state in the limit $c_5 \rightarrow 0$, expressing the suppression of τ -dependent microscopic events as $c_5 \Delta\tau \rightarrow 0$ and restoring the static 4D block universe. Thus, the dimensionless ratio c_5/c parameterizes the deviation of SHP from standard electrodynamics, in particular the coupling of events. We find that taking $0 \leq c_5/c < 1$ enables the resolution of longstanding difficulties in SHP theory.

2. Stueckelberg-Horwitz-Piron (SHP) electrodynamics as gauge theory

2.1. Quantum foundation

The generalized Stueckelberg-Schrodinger equation

$$\left(i\hbar\partial_\tau + \frac{e}{c}\phi\right) \psi(x, \tau) = \frac{1}{2M} \left(p^\mu - \frac{e}{c}a^\mu\right) \left(p_\mu - \frac{e}{c}a_\mu\right) \psi(x, \tau) \quad (5)$$

describes the interaction of an event characterized by the wavefunction $\psi(x, \tau)$ with five gauge fields $a_\mu(x, \tau)$ and $\phi(x, \tau)$. Equation (5) is invariant under the local gauge transformations

$$\begin{aligned} \text{Wavefunction} \quad \psi(x, \tau) &\rightarrow \exp\left[\frac{ie}{\hbar c}\Lambda(x, \tau)\right] \psi(x, \tau) \\ \text{Vector potential} \quad a_\mu(x, \tau) &\rightarrow a_\mu(x, \tau) + \partial_\mu\Lambda(x, \tau) \\ \text{Scalar potential} \quad \phi(x, \tau) &\rightarrow \phi(x, \tau) + \partial_\tau\Lambda(x, \tau) \end{aligned} \quad (6)$$

whose τ -dependence is the essential departure from Stueckelberg's work, and determines the structure of the resulting theory [6]. The corresponding global gauge invariance leads to the conserved Noether current

$$\partial_\mu j^\mu + \partial_\tau \rho = 0 \quad (7)$$

where

$$j^\mu = -\frac{i\hbar}{2M} \left\{ \psi^* \left(\partial^\mu - \frac{ie}{c} a^\mu \right) \psi - \psi \left(\partial^\mu + \frac{ie}{c} a^\mu \right) \psi^* \right\} \quad \rho = |\psi(x, \tau)|^2 . \quad (8)$$

The scalar event density $\rho(x, \tau)$ describes the probability of finding the event at the spacetime point x^μ at the chronological time τ . Although this four-vector current is not divergenceless, the five-current conservation can be integrated

$$\partial_\mu \int \frac{d\tau}{\lambda} j^\mu(x, \tau) + \int \frac{d\tau}{\lambda} \partial_\tau \rho(x, \tau) = \partial_\mu J^\mu(x) = 0 \quad (9)$$

under the boundary condition

$$\rho(x, \tau) \xrightarrow{\tau \rightarrow \pm\infty} 0 \quad (10)$$

where λ is a new parameter with units of time. Thus, $J^\mu(x)$ can be identified with the Maxwell four-current. In earlier work, the parameter λ appeared as an explicit coefficient multiplying the potentials — here it is absorbed into the definitions of the potentials. We will see that λ plays the role of a correlation length, characterizing the range of the electromagnetic interaction.

In analogy with the notation $x^0 = ct$ we adopt the formal designations

$$x^5 = c_5 \tau \quad \partial_5 = \frac{1}{c_5} \partial_\tau \quad j^5 = c_5 \rho \quad a_5 = \frac{1}{c_5} \phi \quad (11)$$

and the index convention

$$\mu, \nu = 0, 1, 2, 3 \quad \alpha, \beta, \gamma = 0, 1, 2, 3, 5 \quad (12)$$

so that the gauge and current conditions (6) and (7) can be written

$$a_\alpha \rightarrow a_\alpha + \partial_\alpha \Lambda \quad \partial_\alpha j^\alpha = 0 . \quad (13)$$

We will see that the values of λ and c_5 are not entirely independent. It is convenient to choose the factor g_{55} and $g^{55} = 1/g_{55}$ to apply when raising and lowering the 5-index, so that

$$\partial_\alpha j^\alpha = g^{\mu\nu} \partial_\mu j_\nu + g^{55} \partial_5 j_5 . \quad (14)$$

These expressions suggest a higher dimensional symmetry that breaks to $O(3,1)$ in the presence of interactions — it may be $O(4,1)$ for $g_{55} = 1$ or $O(3,2)$ for $g_{55} = -1$.

2.2. Classical event dynamics

The classical mechanics of a relativistic event is found by rewriting the Stueckelberg-Schrodinger equation in the form

$$i\hbar \partial_\tau \psi(x, \tau) = \left[\frac{1}{2M} \left(p^\mu - \frac{e}{c} a^\mu \right) \left(p_\mu - \frac{e}{c} a_\mu \right) - \frac{e}{c} \phi \right] \psi(x, \tau) = K \psi(x, \tau) \quad (15)$$

and transforming the classical Hamiltonian to Lagrangian form as

$$L = \dot{x}^\mu p_\mu - K = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\alpha a_\alpha \quad (16)$$

where

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} \quad \dot{x}^5 = \frac{dx^5}{d\tau} \equiv c_5 . \quad (17)$$

The Euler-Lagrange equations

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0 \quad (18)$$

are

$$\frac{d}{d\tau} \left(M \dot{x}^\mu + \frac{e}{c} a^\mu \right) - \partial^\mu \left(\frac{e}{c} \dot{x}^a a_a \right) = 0 \quad (19)$$

leading to the Lorentz force

$$\begin{aligned} M \ddot{x}^\mu &= \frac{e}{c} [\dot{x}^\alpha \partial^\mu a_\alpha - \dot{x}^\alpha \partial_\alpha a^\mu] = \frac{e}{c} f^\mu_\alpha(x, \tau) \dot{x}^\alpha \\ &= \frac{e}{c} f^\mu_\nu(x, \tau) \dot{x}^\nu + \frac{e}{c} f^\mu_5(x, \tau) \dot{x}^5 \\ &= \frac{e}{c} f^\mu_\nu(x, \tau) \dot{x}^\nu - g_{55} \frac{ec_5}{c} f^{5\mu}(x, \tau) \end{aligned} \quad (20)$$

where

$$f^\mu_\alpha = \partial^\mu a_\alpha - \partial_\alpha a^\mu . \quad (21)$$

Because the four components of \dot{x}^μ are independent, the event evolution may be off-shell. In this context, on-shell evolution obeys the mass-shell constraint $\dot{x}^2 = -c^2$ of standard relativity. In SHP electrodynamics

$$\dot{x}^2 = \left(c \frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right)^2 = \left(c \frac{dt}{d\tau} \right)^2 \left(1, \frac{1}{c} \left(\frac{d\mathbf{x}}{dt} \right) \right)^2 = -c^2 \dot{t}^2 \left(1 - \frac{\mathbf{v}^2}{c^2} \right) \quad (22)$$

so that an event evolves on-shell when

$$\left| \frac{dt}{d\tau} \right| = |\dot{t}| = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \quad (23)$$

and is said to be off-shell when $|\dot{t}|$ takes any other value. In the SHP formalism, particles may exchange mass with fields through

$$\frac{d}{d\tau} \left(-\frac{1}{2} M \dot{x}^2 \right) = -M \dot{x}^\mu \ddot{x}_\mu = -\frac{e}{c} \dot{x}^\mu (c_5 f_{\mu 5} + f_{\mu\nu} \dot{x}^\nu) = \frac{ec_5}{c} \dot{x}^\mu f_{5\mu} = g_{55} \frac{ec_5}{c} f^{5\mu} \dot{x}_\mu \quad (24)$$

and the mass shell relation is demoted from the status of constraint to that of conservation law for interactions in which $\dot{x}^\mu f_{5\mu} = 0$ (which usually entails $f_{5\mu} = 0$). However, if the scale of the fields $f_{5\mu}$ is small compared to the Maxwell fields $f_{\mu\nu}$ then the exchange of mass will be correspondingly small. It would be convenient to find that

$$c_5 \rightarrow 0 \quad \Rightarrow \quad f_{5\mu} \rightarrow 0 \quad (25)$$

so that c_5/c can be understood as the scale of dynamic evolution in the microscopic system, approaching an equilibrium equivalent to standard Maxwell theory as $c_5 \rightarrow 0$. With this expectation in mind we examine the fields produced by the motions of charged events.

2.3. Electromagnetic interaction

To specify an electromagnetic action, we expand the interaction term as

$$\dot{X}^\alpha a_\alpha \rightarrow \int d^4x \dot{X}^\alpha(\tau) \delta^4(x - X(\tau)) a_\alpha(x, \tau) = \frac{1}{c} \int d^4x j^\alpha(x, \tau) a_\alpha(x, \tau) \quad (26)$$

and choose a kinetic term for the gauge field; this term must be both gauge and $O(3,1)$ invariant. We write the action as

$$S_{\text{em}} = \int d^4x d\tau \left\{ \frac{e}{c^2} j^\alpha(x, \tau) a_\alpha(x, \tau) - \int \frac{ds}{\lambda} \frac{1}{4c} \left[f^{\alpha\beta}(x, \tau) \Phi(\tau - s) f_{\alpha\beta}(x, s) \right] \right\} \quad (27)$$

where λ is the parameter introduced in (9). The five components of the local event current

$$j^\alpha(x, \tau) = c \dot{X}^\alpha(\tau) \delta^4(x - X(\tau)) \quad (28)$$

have support at the spacetime location $X^\mu(\tau)$ of the corresponding event, where again $\dot{X}^5 = c_5$. As in (9) the τ -integral of (28) along the worldline concatenates the instantaneous event current into the Maxwell particle current

$$J^\mu(x) = c \int d\tau \dot{X}^\mu(\tau) \delta^4(x - X(\tau)) \quad (29)$$

in the standard form. The field interaction kernel is taken to be

$$\Phi(\tau) = \delta(\tau) - (\xi\lambda)^2 \delta''(\tau) = \int \frac{d\kappa}{2\pi} \left[1 + (\xi\lambda\kappa)^2 \right] e^{-i\kappa\tau} \quad (30)$$

where

$$\xi = \frac{1}{2} \left[1 + \left(\frac{c_5}{c} \right)^2 \right] \quad (31)$$

is chosen so that the low energy Lorentz force agrees with Coulomb's law. The second term in $\Phi(\tau)$ is nonlocal in τ and breaks any higher dimensional symmetry associated with the fields to $O(3,1)$. The inverse function of the interaction kernel

$$\Phi^{-1}(\tau) = \int \frac{d\kappa}{2\pi} \frac{e^{-i\kappa\tau}}{1 + (\xi\lambda\kappa)^2} = \frac{1}{2\xi\lambda} e^{-|\tau|/\xi\lambda} \quad (32)$$

satisfies

$$\int ds \Phi^{-1}(\tau - s) \Phi(s) = \delta(\tau) \quad \int d\tau \Phi^{-1}(\tau) = 1 \quad (33)$$

and appears in the field equations as a smoothing of the particle current over a neighborhood of width λ (in time) around the exact τ -synchronization of each event. For convenience we will write

$$\varphi(\tau) = \lambda \Phi^{-1} = \frac{1}{2\xi} e^{-|\tau|/\xi\lambda} \quad (34)$$

so that

$$\int \frac{d\tau}{\lambda} \varphi(\tau) = 1. \quad (35)$$

This definition of φ differs by a factor of λ from earlier work.

Varying the action (27) with respect to the fields, and using (34) to remove the kernel Φ , leads to the field equations

$$\partial_\beta f^{\alpha\beta}(x, \tau) = \frac{e}{c} \int ds \varphi(\tau - s) j^\alpha(x, s) = \frac{e}{c} j_\varphi^\alpha(x, \tau) \quad (36)$$

$$\partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} = 0 \quad (37)$$

which are formally similar to Maxwell's equations in 5D, and are called pre-Maxwell equations. Rewriting the field equations in four-vector and scalar components, they take the form

$$\begin{aligned} \partial_\nu f^{\mu\nu} - \frac{1}{c_5} \partial_\tau f^{5\mu} &= \frac{e}{c} j_\varphi^\mu & \partial_\mu f^{5\mu} &= \frac{e}{c} j_\varphi^5 = \frac{c_5}{c} e \rho_\varphi \\ \partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} &= 0 & \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5} \partial_\tau f_{\mu\nu} &= 0 \end{aligned} \quad (38)$$

which may be compared with the 3-vector form of Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E} &= \frac{e}{c} \mathbf{J} & \nabla \cdot \mathbf{E} &= \frac{e}{c} J^0 \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} &= 0 \end{aligned} \quad (39)$$

with $f_{5\mu}$ playing the role of the vector electric field and $f^{\mu\nu}$ playing the role of the magnetic field.

2.4. Wave equations and induced fields

The pre-Maxwell equations in Lorenz gauge lead to the wave equation

$$\partial_\beta \partial^\beta a^\alpha = (\partial_\mu \partial^\mu + \partial_\tau \partial^\tau) a^\alpha = (\partial_\mu \partial^\mu + \frac{g_{55}}{c_5^2} \partial_\tau^2) a^\alpha = -\frac{e}{c} j_\varphi^\alpha(x, \tau) \quad (40)$$

whose solutions may respect 5D symmetries broken by the $O(3,1)$ symmetry of the event dynamics. A Green's function solution to

$$(\partial_\mu \partial^\mu + \frac{g_{55}}{c_5^2} \partial_\tau^2) G(x, \tau) = -\delta^4(x) \delta(\tau) \quad (41)$$

can be used to obtain potentials in the form

$$\begin{aligned} a^\alpha(x, \tau) &= -\frac{e}{c} \int d^4x' d\tau' G(x - x', \tau - \tau') j_\varphi^\alpha(x', \tau') \\ &= -e \int d^4x' d\tau' ds G(x - x', \tau - \tau') \varphi(\tau' - s) \dot{X}^\alpha(s) \delta^4(x' - X(s)) \\ &= -e \int ds \left[\int d\tau' G(x - X(s), \tau - \tau') \varphi(\tau' - s) \right] \dot{X}^\alpha(s). \end{aligned}$$

Comparing $\dot{X}^5(s) = c_5$ with the 4-vector $\dot{X}(s) = \dot{X}^0(s)(c, \mathbf{v})$, where $|\mathbf{v}| < c$, we see that the fifth potential $a^5(x, \tau)$ is in general scaled by a factor of c_5/c with respect to $a^\mu(x, \tau)$.

The principal part Green's function was found (in natural units, writing $c = c_5 = 1$) in [7] by adapting Schwinger's derivation of the Klein-Gordon propagator. Working through the derivation and replacing τ with $x^5 = c_5\tau$ the result takes the form

$$G_P(x, \tau) = -\frac{1}{2\pi} \delta(x^2) \delta(\tau) - \frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-g_{55}g_{\alpha\beta}x^\alpha x^\beta) \frac{1}{\sqrt{-g_{55}g_{\alpha\beta}x^\alpha x^\beta}} \quad (42)$$

$$= G_{Maxwell} + G_{Correlation} \quad (43)$$

so that both terms have units of distance⁻² × time⁻¹. The support of $G_{Correlation}$ is

$$-g_{55}g_{\alpha\beta}x^\alpha x^\beta = \begin{cases} -(x^2 + c_5^2\tau^2) = c^2t^2 - \mathbf{x}^2 - c_5^2\tau^2 > 0 & , \quad g_{55} = 1 \\ (x^2 - c_5^2\tau^2) = \mathbf{x}^2 - c^2t^2 - c_5^2\tau^2 > 0 & , \quad g_{55} = -1 \end{cases} \quad (44)$$

with causality properties discussed in [7]. In particular, we see that for $g_{55} = 1$, $G_{Correlation}$ has timelike support with respect to the event trajectory, opening the possibility of a self-interaction of a type not present in standard Maxwell theory. Since Schwinger's method involves singling out a direction for τ in the 5D space (x, τ) , we see that the first term $G_{Maxwell}$ breaks O(4,1) or O(3,2) symmetry to O(3,1).

To avoid singularities, we must take particular care in handling the distribution functions and the order of integration. Evaluating the derivative in (42) we find for $g_{55} = 1$,

$$\begin{aligned} G_{correlation}(x, \tau) &= -\frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-g_{55}g_{\alpha\beta}x^\alpha x^\beta) \frac{c_5}{\sqrt{-g_{55}g_{\alpha\beta}x^\alpha x^\beta}} \\ &= -\frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \frac{\theta(-x^2 - c_5^2\tau^2)}{(-x^2 - c_5^2\tau^2)^{1/2}} \\ G_{correlation}(x, \tau) &= -\frac{c_5}{2\pi^2} \left(\frac{1}{2} \frac{\theta(-x^2 - c_5^2\tau^2)}{(-x^2 - c_5^2\tau^2)^{3/2}} - \frac{\delta(-x^2 - c_5^2\tau^2)}{(-x^2 - c_5^2\tau^2)^{1/2}} \right) . \end{aligned} \quad (45)$$

Although the second term appears highly singular, we will see that when calculating potentials, singularities in the terms of $G_{correlation}$ cancel when the subtraction is performed before applying the limits of integration, as required by Schwinger's prescription.

The 'static' Coulomb potential in this framework is induced by an isolated event moving uniformly along the t axis. Writing the event as

$$X(\tau) = (c\tau, 0, 0, 0) \quad (46)$$

produces the currents

$$j^0(x, \tau) = c^2 \delta(t - \tau) \delta^3(\mathbf{x}) \quad \mathbf{j}(x, \tau) = 0 \quad j^5(x, \tau) = cc_5 \delta(t - \tau) \delta^3(\mathbf{x}) \quad (47)$$

$$j_\varphi^0(x, \tau) = c^2 \varphi(t - \tau) \delta^3(\mathbf{x}) \quad \mathbf{j}_\varphi(x, \tau) = 0 \quad j_\varphi^5(x, \tau) = cc_5 \varphi(t - \tau) \delta^3(\mathbf{x}) \quad (48)$$

and the Maxwell part of the Green's function induces

$$a^0(x, \tau) = \frac{e}{4\pi|\mathbf{x}|} \varphi \left(\tau - \left(t - \frac{|\mathbf{x}|}{c} \right) \right) \quad \mathbf{a} = 0 \quad a^5(x, \tau) = \frac{c_5}{c} a^0(x, \tau) . \quad (49)$$

In Appendix A we show that the contribution from $G_{Correlation}$ is smaller than the $G_{Maxwell}$ contribution by c_5/c and drops off as $1/|\mathbf{x}|^2$, so that it may be neglected at low energy. Using (34) for $\varphi(\tau)$, a test event evolving in parallel at $x(\tau) = (c\tau, \mathbf{x})$ will see the Yukawa-type potential

$$a^0(x, \tau) = \frac{e}{4\pi|\mathbf{x}|} \frac{1}{2\tilde{\zeta}} e^{-|\mathbf{x}|/\tilde{\zeta}\lambda c} \quad (50)$$

with photon mass spectrum $m_\gamma \sim \hbar/\tilde{\zeta}\lambda c^2$. If λ is small (so that $\frac{1}{\lambda}\varphi$ approaches a delta function and the current narrows to a small neighborhood around the event) the mass spectrum becomes wide. If λ is large, the support of the current spreads along the worldline, the potential becomes Coulomb-like and the photon mass is small.

The field strength components found from (50) are

$$\begin{aligned} f^{k0}(x, \tau) &= \partial^k \frac{e}{4\pi|\mathbf{x}|} \frac{1}{2\tilde{\zeta}} e^{-|\mathbf{x}|/\tilde{\zeta}\lambda c} & f^{k5}(x, \tau) &= \frac{c_5}{c} f^{k0}(x, \tau) \\ f^{ij}(x, \tau) &= f^{50}(x, \tau) = 0 \end{aligned} \quad (51)$$

and the test event will experience the Coulomb force through the Lorentz force (20) as

$$M\dot{x}^k = \frac{e}{c} f^k_{\nu} \dot{x}^\nu - g_{55} \frac{ec_5}{c} f^{5k} = -\frac{e}{c} f^{k0} (\dot{x}^0 - g_{55} c_5) \quad (52)$$

Since the test event velocity is $\dot{x}(\tau) = (c, \mathbf{0})$ this becomes

$$M\ddot{x} = -\frac{e^2}{2\tilde{\zeta}} \left(1 - g_{55} \frac{c_5}{c}\right) \nabla \left(\frac{e^{-|\mathbf{x}|/\tilde{\zeta}\lambda c}}{4\pi|\mathbf{x}|} \right) = -e^2 \frac{1 - g_{55} \frac{c_5}{c}}{1 + \left(\frac{c_5}{c}\right)^2} \nabla \left(\frac{e^{-|\mathbf{x}|/\tilde{\zeta}\lambda c}}{4\pi|\mathbf{x}|} \right) \quad (53)$$

where we used (31) for $\tilde{\zeta}$. For an antiparticle event, the source evolves backward in time with $\dot{x}^0 = -c$ so that the Coulomb force induced by a particle event (upper sign) or antiparticle event (lower sign) can be written

$$\mathbf{F} = \mp e^2 \frac{1 \mp g_{55} \frac{c_5}{c}}{1 + \left(\frac{c_5}{c}\right)^2} \nabla \left(\frac{e^{-|\mathbf{x}|/\tilde{\zeta}\lambda c}}{4\pi|\mathbf{x}|} \right) \quad (54)$$

This expression permits the resolution a longstanding difficulty in SHP theory. Under the assumption that $c_5/c = 1$ there is no choice for $g_{55} = \pm 1$ that can describe classical elastic scattering for both e^-/e^- and e^+/e^- . However, with $c_5 < c$ either signature for g_{55} is permitted, and (54) provides an experimental bound on c_5/c associated with the discrepancy between particle/particle and particle/antiparticle scattering at very low energy.

The form of $G_{Maxwell}$ allows us apply standard techniques associated with the Liénard-Wiechert potential. Writing

$$\begin{aligned} a^\alpha(x, \tau) &= -\frac{e}{c} \int d^4x' d\tau' G_{Maxwell}(x - x', \tau - \tau') j_\varphi^\alpha(x', \tau') \\ &= \frac{e}{2\pi} \int ds \varphi(\tau - s) \dot{X}^\alpha(s) \delta\left((x - X(s))^2\right) \theta^{ret} \end{aligned} \quad (55)$$

and using the identity

$$\int d\tau f(\tau) \delta[g(\tau)] = \frac{f(\tau_R)}{|g'(\tau_R)|} \quad (56)$$

where τ_R is the retarded time found from

$$g(\tau) = (x - X(\tau_R))^2 = 0 \quad \theta^{ret} = \theta(x^0 - X^0(\tau_R)) \quad , \quad (57)$$

provides

$$a^\alpha(x, \tau) = \frac{e}{4\pi} \varphi(\tau - \tau_R) \frac{\dot{X}^\alpha(\tau_R)}{|(x^\mu - X^\mu(\tau_R)) \dot{X}_\mu(\tau_R)|} \quad . \quad (58)$$

Using this potential, where we write the event velocity and line of observation as

$$u^\mu = \dot{X}^\mu(\tau) \quad z^\mu = x^\mu - X^\mu(\tau) \quad (59)$$

the potential takes the form

$$a^\mu(x, \tau) = \frac{e}{4\pi} \varphi(\tau - \tau_R) \frac{u^\mu}{|u \cdot z|} \quad a^5(x, \tau) = \frac{e}{4\pi} \varphi(\tau - \tau_R) \frac{c_5}{|u \cdot z|} \quad . \quad (60)$$

By a similar procedure we find the field strengths as

$$f^{\mu\nu}(x, \tau) = -\frac{e}{4\pi} \varphi(\tau - \tau_R) \left\{ \frac{(z^\mu u^\nu - z^\nu u^\mu) u^2}{(u \cdot z)^3} + \frac{\epsilon(\tau - \tau_R)}{\lambda} \frac{z^\mu u^\nu - z^\nu u^\mu}{(u \cdot z)^2} \right. \\ \left. \frac{(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu)(u \cdot z) - (z^\mu u^\nu - z^\nu u^\mu)(\dot{u} \cdot z)}{(u \cdot z)^3} \right\} \quad (61)$$

$$f^{5\mu}(x, \tau) = c_5 \frac{e}{4\pi} \varphi(\tau - \tau_R) \left\{ \frac{z^\mu u^2 - u^\mu (u \cdot z)}{(u \cdot z)^3} + \frac{\epsilon(\tau - \tau_R)}{\lambda} \frac{z^\mu - u^\mu (u \cdot z)}{(u \cdot z)^2} \right. \\ \left. - \frac{(\dot{u} \cdot z) z^\mu}{(u \cdot z)^3} \right\} \quad (62)$$

where we used $|u \cdot z| = -(u \cdot z)$ (easily seen in a co-moving frame) and

$$\frac{d}{d\tau_R} \varphi(\tau - \tau_R) = -\frac{1}{2\xi} \frac{d}{d\tau} e^{-|\tau - \tau_R|/\xi\lambda} = -\frac{\epsilon(\tau - \tau_R)}{\xi\lambda} \varphi(\tau - \tau_R) \quad (63)$$

where $\epsilon(\tau) = \text{signum}(\tau)$. Notice that the explicit τ -dependence in these expressions is limited to the smoothing function $\varphi(\tau - \tau_R)$ and its derivative.

3. Recovering Maxwell theory from SHP

The classical SHP framework, described by the pre-Maxwell equations (36) and (37) and the Lorentz force (20), is a theory of interacting events. Each event $x^\mu(\tau)$ induces an instantaneous five-component current density $j_\phi^\alpha(x, \tau)$ with support in a neighborhood $x^\mu(\tau \pm \lambda)$ along its dynamically evolving worldline. The five gauge potentials $a^\alpha(x, \tau)$ locally induced by these currents moderate interactions between individual events. The complete worldlines characterized by the Maxwell currents $J^\mu(x)$ can only be known *a posteriori* by concatenation of these instantaneous event currents. We now compare two approaches for associating SHP with the standard Maxwell theory of interacting particle worldlines, as an equilibrium state of this microscopic event dynamics.

3.1. Concatenation by τ -integration

In (29) it was seen that the divergenceless 4D Maxwell current is obtained by integrating the instantaneous event current over τ , concatenating the events into a particle worldline. The source of the pre-Maxwell fields is

$$j_\varphi^\alpha(x, \tau) = \int ds \varphi(\tau - s) j^\alpha(x, s) \quad (64)$$

formed by smoothing the delta function support of the instantaneous current defined in (28) by convolution with $\varphi(\tau) = \lambda \Phi^{-1}(\tau)$. It follows from (35) that

$$\int \frac{d\tau}{\lambda} j_\varphi^\mu(x, \tau) = \int ds j^\mu(x, s) = J^\mu(x) \quad (65)$$

so that smoothing the local event current leaves the concatenation unchanged. Following the Stueckelberg boundary condition (10), one may impose the additional requirement

$$f^{5\mu}(x, \tau) \xrightarrow{\tau \rightarrow \pm\infty} 0 \quad (66)$$

so that integration of the field equations provide

$$\left. \begin{aligned} \partial_\beta f^{\alpha\beta}(x, \tau) &= \frac{e}{c} j_\varphi^\alpha(x, \tau) \\ \partial_{[\alpha} f_{\beta\gamma]} &= 0 \\ \partial_\alpha j^\alpha &= 0 \end{aligned} \right\} \xrightarrow{\int \frac{d\tau}{\lambda}} \left\{ \begin{aligned} \partial_\nu F^{\mu\nu}(x) &= \frac{e}{c} J^\mu(x) \\ \partial_\nu F^{5\nu}(x) &= \frac{e}{c} J^5(x) \\ \partial_{[\mu} F_{\nu\rho]} &= 0 \\ \partial_\mu J^\mu(x) &= 0 \end{aligned} \right. \quad (67)$$

where we use (65) and

$$A^\alpha(x) = \int \frac{d\tau}{\lambda} a^\alpha(x, \tau) \quad F^{\alpha\nu}(x) = \int \frac{d\tau}{\lambda} f^{\alpha\nu}(x, \tau) . \quad (68)$$

Integration of (42)

$$\int d\tau G_{Maxwell} = D(x) = -\frac{1}{2\pi} \delta(x^2) \quad \int d\tau G_{Correlation} = 0 \quad (69)$$

similarly recovers the 4D Maxwell Green's function. As seen in equations (60) — (62), the τ -dependence of the potentials and fields resides in multiplicative factors of the type $\varphi(\tau - \tau_R)$, so that integration along the worldline recovers the standard Maxwell fields in their explicit form. In particular, integration of (50) yields the standard Coulomb potential.

In the resulting picture, the microscopic dynamics approach an equilibrium state because the boundary conditions (10) and (66) hold pointwise in x as $\tau \rightarrow \infty$, asymptotically eliminating interactions that cannot be described in Maxwell theory. The Maxwell-type description recovered by concatenating the microscopic dynamics may thus be understood as a self-consistent summary constructed *a posteriori* from the complete worldlines.

3.2. The limit $c_5 \rightarrow 0$

We have assumed that $0 \leq c_5 < c$ and we must check that SHP theory remains finite for $c_5 \rightarrow 0$.

First we notice that c_5 appears explicitly three times in the pre-Maxwell equations (38)

$$\begin{aligned} \partial_\nu f^{\mu\nu} - \frac{1}{c_5} \partial_\tau f^{5\mu} &= \frac{e}{c} j_\varphi^\mu & \partial_\mu f^{5\mu} &= \frac{e}{c} j_\varphi^5 = \frac{c_5}{c} e \rho_\varphi \\ \partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} &= 0 & \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5} \partial_\tau f_{\mu\nu} &= 0 \end{aligned}$$

twice in the form $\frac{1}{c_5} \partial_\tau$ and once multiplying the event density ρ_φ . The derivative term poses no problem in the homogeneous pre-Maxwell equation, which is satisfied identically for fields derived from potentials. Specifically, the fields $f_{5\mu}$ contain terms of the type $\partial_5 a_\mu = \frac{1}{c_5} \partial_\tau a_\mu$ that cancel the explicit τ -derivative of $f_{\mu\nu}$, evaluated before passing to the limit $c_5 \rightarrow 0$. However, the homogeneous equation does impose a new condition through

$$c_5 (\partial_\nu f_{5\mu} - \partial_\mu f_{5\nu}) + \partial_\tau f_{\mu\nu} = 0 \xrightarrow{c_5 \rightarrow 0} \partial_\tau f_{\mu\nu} = 0 \quad (70)$$

requiring that the field strength $f^{\mu\nu}$ become τ -independent in this limit. From equation (61) we see that this condition is violated by the multiplicative factor $\varphi(\tau - \tau_R)$ unless we simultaneously require $c_5 \rightarrow 0 \Rightarrow \lambda \rightarrow \infty$, in which case $\varphi(x, \tau) \rightarrow 1/2\xi = 1$, where we use (31) for ξ . This requirement effectively spreads the event current j_φ^α uniformly along the particle worldline, recovering the τ -independent particle current

$$\begin{aligned} j_\varphi^\mu(x, \tau) &= \int ds \varphi(\tau - s) j^\mu(x, s) \longrightarrow \int ds 1 \cdot j^\mu(x, s) = J^\mu(x) \\ j_\varphi^5(x, \tau) &= \int ds \varphi(\tau - s) j^5(x, s) \longrightarrow \int ds j^5(x, s) \\ \partial_\mu j_\varphi^\mu(x, \tau) + \frac{1}{c_5} \partial_\tau j_\varphi^5(x, \tau) &\longrightarrow \partial_\mu J^\mu(x) = 0 \end{aligned} \quad (71)$$

associated with Maxwell theory. Generally, because the τ -dependence of the potentials and fields is contained in φ , the condition $\lambda \rightarrow \infty$ eliminates all the terms in the pre-Maxwell equations containing ∂_τ . Similarly, the photon mass $m_\gamma \sim \hbar/\xi\lambda c^2$ must vanish.

Equation (62) shows that $f^{5\mu}$ is generally proportional to c_5 for fields of the Liénard-Wiechert type. Therefore we can write the inhomogeneous pre-Maxwell equations in the finite form

$$\partial_\nu f^{\mu\nu} = \frac{e}{c} j_\varphi^\mu \quad \partial_\mu \left(\frac{1}{c_5} f^{5\mu} \right) = \frac{e}{c} \rho_\varphi \quad (72)$$

where we see that $f^{5\mu}$ decouples from the field $f^{\mu\nu}$ that now satisfies Maxwell's equations.

To find the limiting form of the electromagnetic interactions, we consider an arbitrary event $X^\mu(\tau)$, which induces the current

$$j_\varphi^\alpha(x, \tau) = c \int ds \varphi(\tau - s) \dot{X}^\alpha(s) \delta^4[x - X(s)] \quad (73)$$

Using (20), (31), (32), (61) and (62) the Lorentz force on a test event moving in the field induced by this current can be written

$$\begin{aligned}
 M\ddot{x}^\mu &= \frac{e}{c} [f^\mu_\nu(x, \tau)\dot{x}^\nu + f^{5\mu}(x, \tau)\dot{x}^5] \\
 &= \frac{e}{c} \frac{e}{4\pi} \varphi(\tau - \tau_R) [\mathcal{F}^\mu_\nu(x, \tau)\dot{x}^\nu + c_5^2 \mathcal{F}^{5\mu}(x, \tau)] \\
 &= \frac{e}{c} \frac{e}{4\pi} \frac{1}{2\xi} e^{-|\tau - \tau_R|/\xi\lambda} [\mathcal{F}^\mu_\nu(x, \tau)\dot{x}^\nu + c_5^2 \mathcal{F}^{5\mu}(x, \tau)] \\
 &= \frac{e^2}{4\pi c} e^{-|\tau - \tau_R|/\xi\lambda} \frac{1}{2\xi} [\mathcal{F}^\mu_\nu(x, \tau)\dot{x}^\nu + c_5^2 \mathcal{F}^{5\mu}(x, \tau)] \\
 &= \frac{e^2}{4\pi c} e^{-|\tau - \tau_R|/\xi\lambda} \frac{\mathcal{F}^\mu_\nu(x, \tau)\dot{x}^\nu + c_5^2 \mathcal{F}^{5\mu}(x, \tau)}{1 + (c_5/c)^2}
 \end{aligned} \tag{74}$$

where

$$\begin{aligned}
 \mathcal{F}^{\mu\nu}(x, \tau) &= -\frac{(z^\mu u^\nu - z^\nu u^\mu) u^2}{(u \cdot z)^3} - \frac{(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu)(u \cdot z) - (z^\mu u^\nu - z^\nu u^\mu)(\dot{u} \cdot z)}{(u \cdot z)^3} \\
 &\quad + \frac{\epsilon(\tau - \tau_R)}{\lambda} \frac{z^\mu u^\nu - z^\nu u^\mu}{(u \cdot z)^2}
 \end{aligned} \tag{75}$$

$$\mathcal{F}^{5\mu}(x, \tau) = \frac{z^\mu u^2 - u^\mu (u \cdot z)}{(u \cdot z)^3} - \frac{(\dot{u} \cdot z) z^\mu}{(u \cdot z)^3} + \frac{\epsilon(\tau - \tau_R)}{\lambda} \frac{z^\mu - u^\mu (u \cdot z)}{(u \cdot z)^2} \tag{76}$$

are independent of c_5 . In the limit $c_5 \rightarrow 0$ and $\lambda \rightarrow \infty$, all $\mathcal{F}^{\alpha\mu}(x, \tau)$ become τ -independent and the Lorentz force interaction reduces to

$$M\ddot{x}^\mu = \frac{e^2}{4\pi c} \mathcal{F}^\mu_\nu(x) \dot{x}^\nu \tag{77}$$

decoupling from the field $\mathcal{F}^{5\mu}$. We thus see that in addition to recovering Maxwell's equations, the parameter c_5/c provides a continuous scaling of the Lorentz force to the standard form in Maxwell theory. That is, the combined limit $\lambda \rightarrow \infty$ and $c_5 \rightarrow 0$ restricts the possible dynamics in SHP to those of Maxwell theory.

The contribution associated with $G_{\text{Correlation}}$ is less straightforward, but in Appendix B we gain some insight by studying the δ -function term and assuming that the structure of the θ -function term must be sufficiently similar to permit cancellation of singularities.

4. Conclusions

We have reviewed classical Stueckelberg-Horwitz-Piron electrodynamics, inserting where appropriate a new constant c_5 associated with the chronological time τ in much the way that c is associated with the coordinate time t . Although it has been previously assumed (implicitly) that $c_5 = c$, we now consider the possibility that $0 \leq c_5/c < 1$ and find that this resolves longstanding difficulties in consistently formulating elastic particle/particle scattering and particle/antiparticle scattering in classical electrodynamics.

We showed that SHP remains finite for all values of $0 \leq c_5/c < 1$, where consistency of the pre-Maxwell equations requires that $\lambda \rightarrow \infty$ when $c_5 \rightarrow 0$. This limit reproduces standard Maxwell dynamics, as the pre-Maxwell equations and SHP Lorentz equation go over to Maxwell's equations and their associated Lorentz equation. We note that while a Maxwell-like description can be obtained from SHP as an *a posteriori* equilibrium limit by integration over the worldlines, the parameter c_5/c provides a smooth transition, so that its value can be determined from experimental deviations of SHP from standard Maxwell theory. In the resulting picture, the Maxwell approximation to pre-Maxwell theory can be understood as an event dynamics evolving so slowly in τ that the system is essentially in equilibrium, with vanishing event density and no mass exchange.

As noted in the introduction, off-shell event evolution involves mass exchange between particles and fields. Another longstanding issue in SHP is finding an adequate explanation for the fixed masses of the observed particles. In a subsequent paper [18] we discuss a mechanism by which the field produced through $G_{\text{Correlation}}$ for $g_{55} = 1$ leads to a self-interaction with timelike support whose effect is to restore on-shell evolution and stabilize the particle mass.

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Appendix A. Coulomb potential from $G_{\text{Correlation}}$

We are interested in an event moving as

$$X = (c\tau, \mathbf{0}) \quad u^2 = -c^2 \quad (\text{A.1})$$

where we approximate

$$\varphi(\tau' - s) = \lambda \delta(\tau' - s) \quad (\text{A.2})$$

so that

$$a^\alpha(x, \tau) = -\lambda e \int ds G(x - X(s), \tau - s) \quad \dot{X}^\alpha(s) = \frac{\lambda e}{2\pi^2} \dot{X}^\alpha(s) \int ds G(x - X(s), \tau - s) \quad (\text{A.3})$$

We introduce the function $g(s)$ to express terms of the type

$$- \left((x - X(s))^2 + c_5^2(\tau - s)^2 \right) = - \left((ct, \mathbf{x}) - (cs, \mathbf{0}) \right)^2 + c_5^2(\tau - s)^2 = c^2 g(s) \quad (\text{A.4})$$

where

$$g(s) = (t - s)^2 - \frac{R^2}{c^2} - \frac{c_5^2}{c^2}(\tau - s)^2 = Cs^2 + Bs + A \quad (\text{A.5})$$

and

$$\zeta^2 = \frac{c_5^2}{c^2} \quad C = (1 - \zeta^2) \quad B = -2(t - \zeta^2 \tau) \quad A = t^2 - \frac{R^2}{c^2} - \zeta^2 \tau^2 \quad (\text{A.6})$$

so that the potential can be written as

$$a(x, \tau) = \frac{\lambda e c_5}{2\pi^2 c^3} (c, \mathbf{0}, c_5) \int ds \left[\frac{1}{2} \frac{\theta(g(s))}{g^{3/2}(s)} - \frac{\delta(g(s))}{g^{1/2}(s)} \right] \theta(t - s) \quad (\text{A.7})$$

The zeros of $g(s)$ are found to be

$$s_{\pm} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C} = \frac{(t - \zeta^2 \tau) \pm \sqrt{\frac{R^2}{c^2} (1 - \zeta^2) + \zeta^2 (t - \tau)^2}}{(1 - \zeta^2)} \quad (\text{A.8})$$

and since we assume $\zeta^2 < 1$ there will be roots for any values of t and R . In addition, the condition $\theta^{ret} = \theta(t - s)$ requires $t > s$.

If $t < s_-$ then

$$t < \frac{(t - \zeta^2 \tau) - \sqrt{\frac{R^2}{c^2} (1 - \zeta^2) + \zeta^2 (t - \tau)^2}}{(1 - \zeta^2)} \Rightarrow -\zeta^2 (t - \tau)^2 > \frac{R^2}{c^2} \quad (\text{A.9})$$

and so $t \geq s_-$ becomes a condition of integration for the θ term. Similarly, if $t > s_+$ then

$$t > \frac{(t - \zeta^2 \tau) + \sqrt{\frac{R^2}{c^2} (1 - \zeta^2) + \zeta^2 (t - \tau)^2}}{(1 - \zeta^2)} \Rightarrow -\zeta^2 (t - \tau)^2 > \frac{R^2}{c^2} \quad (\text{A.10})$$

leading to the condition

$$\theta(g(s)) \theta(t - s) \neq 0 \Rightarrow s_- \leq s \leq t \leq s_+ \quad (\text{A.11})$$

from which

$$a(x, \tau) = \frac{\lambda e c_5}{2\pi^2 c^3} \left(1, \mathbf{0}, \frac{c_5}{c}\right) \left(\frac{1}{2} \int_{-\infty}^{s_-} ds \frac{1}{g^{3/2}(s)} - \int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t - s) \right). \quad (\text{A.12})$$

Using the well-known form [19]

$$\int \frac{dx}{(Cx^2 + Bx + A)^{3/2}} = \frac{2(2Cs + B)}{q(Cx^2 + Bx + A)^{1/2}} \quad (\text{A.13})$$

where

$$q = 4AC - B^2 \quad (\text{A.14})$$

we notice from (A.8) that

$$s_- = \frac{-B - \sqrt{B^2 - 4AC}}{2C} = \frac{-B - \sqrt{-q}}{2C} \Rightarrow -\sqrt{-q} = 2Cs_- + B \quad (\text{A.15})$$

and so

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{s_-} ds \frac{1}{g^{3/2}(s)} &= \frac{2Cs_- + B}{qg^{1/2}(s_-)} - \frac{2Cs + B}{qg^{1/2}(s)} \Big|_{-\infty} \\ &= \frac{-\sqrt{-q}}{qg^{1/2}(s_-)} + \frac{2\sqrt{C}}{(2Cs_- + B)^2} \\ &= \frac{1}{\sqrt{-q}g^{1/2}(s_-)} + \frac{1}{2} \frac{\sqrt{1 - \zeta^2}}{\frac{R^2}{c^2} (1 - \zeta^2) + \zeta^2 (t - \tau)^2} \end{aligned} \quad (\text{A.16})$$

The second term is

$$\int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) \quad (\text{A.17})$$

and using the identity

$$\int ds f(s) \delta(g(s)) = \frac{f(s^-)}{|g'(s^-)|} \Big|_{s^-=g^{-1}(0)} \quad (\text{A.18})$$

we can evaluate

$$\int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) = \frac{\theta(t-s_-)}{|g'(s_-)| g^{1/2}(s_-)} = \frac{1}{|g'(s_-)| g^{1/2}(s_-)} . \quad (\text{A.19})$$

Since

$$g'(s_-) = (Cs_-^2 + Bs_- + A)' = 2Cs_- + B = -\sqrt{-q} \quad (\text{A.20})$$

we see that this term cancels the singularity in the first term, leaving

$$\frac{1}{2} \int_{-\infty}^{s_-} ds \frac{1}{g^{3/2}(s)} - \int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) = \frac{1}{2} \frac{\sqrt{1-\zeta^2}}{\frac{R^2}{c^2} (1-\zeta^2) + \zeta^2 (t-\tau)^2} \quad (\text{A.21})$$

and

$$a(x, \tau) = \frac{\lambda e}{4\pi^2} (c, \mathbf{0}, c_5) \frac{c_5}{c} \frac{\sqrt{1-\frac{c_5}{c}}}{R^2 \left(1-\frac{c_5}{c}\right) + \frac{c_5}{c} c^2 (t-\tau)^2} . \quad (\text{A.22})$$

We notice that the potential has units of $\lambda c/\text{distance}^2 = 1/\text{distance}$, as does the potential associated with $G_{Maxwell}$. This contribution to the potential is smaller by a factor of c_5/c than the Yukawa potential found in (50), and drops off faster with distance.

Appendix B. c_5 -dependence of general potential from $\mathbf{G}_{\text{Correlation}}$

We are interested in an arbitrary event moving as

$$X(\tau) = (ct(\tau), \mathbf{x}(\tau)) \quad X^5 = c_5 \tau \quad (\text{B.1})$$

and the induced field

$$a^\alpha(x, \tau) = -e \int ds G_\varphi(x - X(s), \tau - s) \dot{X}^\alpha(s) . \quad (\text{B.2})$$

Making the approximation

$$\varphi(\tau' - s) = \lambda \delta(\tau' - s) \quad (\text{B.3})$$

leads to

$$a^\alpha(x, \tau) = \frac{\lambda e c_5}{2\pi^2} \int ds \dot{X}^\alpha(s) \left(\frac{1}{2} \frac{\theta\left(-(x-X(s))^2 - c_5^2(\tau-s)^2\right)}{\left[-(x-X(s))^2 - c_5^2(\tau-s)^2\right]^{3/2}} - \frac{\delta\left(-(x-X(s))^2 - c_5^2(\tau-s)^2\right)}{\left(-(x-X(s))^2 - c_5^2(\tau-s)^2\right)^{1/2}} \right) \theta^{ret} \quad (\text{B.4})$$

We designate

$$g(s) = -(x - X(s))^2 - c_5^2(\tau - s)^2 \quad s^\pm = g^{-1}(0) \quad (\text{B.5})$$

and assume that the observation point x is in the timelike future of $X(-\infty)$ so that

$$a^\alpha(x, \tau) = \frac{\lambda e}{2\pi^2} \int_{-\infty}^{s^\pm} ds \frac{1}{2} \dot{X}^\alpha(s) \frac{\theta\left(-(x - X(s))^2 - c_5^2(\tau - s)^2\right)}{\left[-(x - X(s))^2 - c_5^2(\tau - s)^2\right]^{3/2}} \theta(ct - X^0(s)) \\ - \frac{\lambda e}{2\pi^2} \int_{-\infty}^{\infty} ds \dot{X}^\alpha(s) \frac{\delta\left(-(x - X(s))^2 - c_5^2(\tau - s)^2\right)}{\left(-(x - X(s))^2 - c_5^2(\tau - s)^2\right)^{1/2}} \theta(ct - X^0(s)) . \quad (\text{B.6})$$

Using the identity

$$\int ds f(s) \delta[g(s)] = \sum_{s^\pm = g^{-1}(0)} \frac{f(s)}{|g'(s)|} \quad (\text{B.7})$$

the second term in the integral becomes

$$\int_{-\infty}^{\infty} ds \dot{X}(s) \frac{\delta(g(s))}{(g(s))^{1/2}} \theta(ct - X^0(s)) = \frac{\dot{X}(s^\pm) \theta(ct - X^0(s^\pm))}{(g(s^\pm))^{1/2} |g'(s^\pm)|} . \quad (\text{B.8})$$

At the observation point $(x, x^5) = (x, c_5\tau)$ we define a 5D line of observation as

$$Z = (z, z^5) = (x, x^5) - (X(s), X^5) = (x - X(s), c_5\tau - c_5s) \quad (\text{B.9})$$

and a 5-velocity

$$U = (u, u^5) \quad u^\mu = \dot{X}^\mu(s) \quad u^5 = \dot{X}^5 \quad (\text{B.10})$$

leading to a generalization of the denominator of (60) in the form

$$g'(s) = 2U \cdot Z \quad (\text{B.11})$$

so that (B.8) becomes

$$\left. \frac{\dot{X}(s) \theta(ct - X^0(s))}{(g(s))^{1/2} |2U \cdot Z|} \right|_{s \rightarrow s^\pm} \quad (\text{B.12})$$

which is singular as $s \rightarrow s^\pm$. As seen in Appendix A, we expect that this singularity is canceled by a corresponding singularity in the θ -term of (B.6). Nevertheless, for $s \neq s^\pm$ this expression remains finite if we take $c_5 \rightarrow 0$. Since we expect the δ -term to have a similar structure to the θ -term, it seems that the contribution of $G_{\text{correlation}}$ to the field induced by a general event will split as

$$\frac{\mathcal{F}_\nu^\mu(x, \tau) \dot{x}^\nu + c_5^2 \mathcal{F}^{5\mu}(x, \tau)}{1 + (c_5/c)^2} \quad (\text{B.13})$$

where $\mathcal{F}_\nu^\mu(x, \tau)$ and $\mathcal{F}^{5\mu}(x, \tau)$ remain finite as $c_5 \rightarrow 0$.

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