

# The Feynman-Dyson view

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**Abstract.** This paper is a survey of our work on the mathematical foundations for the Feynman-Dyson program in quantum electrodynamics (QED). After a brief discussion of the history, we provide a representation theory for the Feynman operator calculus. This allows us to solve the general initial-value problem and construct the Dyson series. We show that the series is asymptotic, thus proving Dyson's second conjecture for quantum electrodynamics. In addition, we show that the expansion may be considered exact to any finite order by producing the remainder term. This implies that every nonperturbative solution has a perturbative expansion. Using a physical analysis of information from experiment versus that implied by our models, we reformulate our theory as a sum over paths. This allows us to relate our theory to Feynman's path integral, and to prove Dyson's first conjecture that the divergences are in part due to a violation of Heisenberg's uncertainly relations. As a by-product, we also prove Feynman's conjecture about the relationship between the operator calculus and has path integral. Thus, providing the first rigorous justification for the Feynman formulation of quantum mechanics.

## Introduction

Following Dirac's quantization of the electromagnetic field in 1927, and his relativistic electron theory in 1928, the equations for quantum electrodynamics (QED) were developed by Heisenberg and Pauli in the years 1929-30 (see Miller [1] and Schweber [2]). From the beginning, when researchers attempted to use the straightforward and physically intuitive time-dependent perturbation expansion to compute physical observables, a number of divergent expressions appeared. Although it was known that the same problems also existed in classical electrodynamics, it was noted by Oppenheimer that there was a fundamental difference in the quantum problem when compared to the classical one. Dirac [3] had shown that, in the classical case, one could account for the problem of radiation reaction without directly dealing with the self-energy divergence by using both advanced and retarded fields and a particular limiting procedure.

Early attempts to develop subtraction procedures for the divergent expressions were very discouraging because they depended on both the gauge and the Lorentz frame, making them appear ambiguous. Although the equations of QED were both Lorentz and gauge covariant, it was generally believed that, in a strict sense, they had no solutions expandable in powers of the charge. The thinking of the times was clearly expressed by Oppenheimer in his 1948 report to the Solvay Conference, "If one wishes to explore these solutions, bearing in mind that certain infinite terms will, in a later theory, no longer be infinite, one needs a covariant way of identifying these terms; and for that, not merely the field equations themselves, but the whole method of approximation and solution must at all stages preserve covariance."



The solution to the problem posed by Oppenheimer was made independently by Tomonaga [4], Schwinger [5] and Feynman [6],[7], [8]. These papers may be found in Schwinger [5]. Tomonaga introduced what is now known as the interaction representation and showed how the approximation process could be carried out in a covariant manner. Schwinger developed the general theory and applied it to many of the important problems. Feynman took a holistic view of physical reality in his development. He suggested that we view a physical event as occurring on a film which exposes more and more of the outcome as the film unfolds. His idea was to deal directly with the solutions to the equations describing the physical system, rather than the equations themselves. In addition to solving the problem posed by Oppenheimer, Feynman's approach led to a new perturbation series, which provided an easy, intuitive, and computationally simple method to study interacting particles while giving physical meaning to each term in his expansion.

Since Feynman's method and approach was so different, it was not clear how it related to that of Schwinger and Tomonaga. Dyson [9], [10] made a major contribution. Dyson realized that Feynman and Schwinger were both dealing with different versions of Heisenberg's S-matrix. He then formally introduced time-ordering and provided a unified approach by demonstrating the equivalence of the Feynman and Schwinger-Tomonaga theories. This approach also allowed him to show how the Schwinger theory could be greatly simplified and extended to all orders of the perturbation expansion. Dyson's time-ordering idea was actually obtained from discussions with Feynman, who later explored and fully developed it into his time-ordered operator calculus.

After the problem proposed by Oppenheimer was resolved, attitudes toward the renormalization program and quantum field theory could be classified into three basic groups. The first group consisted of those who were totally dissatisfied with the renormalization program. The second group considered the renormalization program an interim step and believed that the divergences were an indication of additional physics, which could not be reached by present formulations. The first two groups will not be extensively discussed here. However, we can associate the names of Dirac and Landau with the first group, and Sakata and Schwinger with the second. (See Dirac,[11] Sakata,[12] Schwinger, [13] and also Schweber.[14].

The third group was more positive, and directed its attention toward investigating the mathematical foundations of quantum field theory with the hope of providing a more orderly approach to the renormalization program (assuming that the theory proved consistent). This direction was clearly justified since part of the problem had been consistently blamed on a mathematical issue, the perturbation expansion. Indeed, the whole renormalization program critically depended on the expansion of the S-matrix in powers of the coupling constant. This concern was further supported since attempts to use the expansion when the coupling constant was large led to meaningless results. Additional unease could be attributed to the fact that, at that time, not much was actually known about the physically important cases where one was dealing with unbounded operator-valued distributions.

Researchers working on the mathematical foundations of quantum electrodynamics and quantum field theory adopted the name axiomatic field theory starting in the 1950s. These researchers focused on trying to find out what could be learned about the existence of local relativistic quantum field theories based on certain natural assumptions which included the postulates of quantum mechanics, locality, Poincare invariance, and a reasonable spectrum. This approach was initiated by the work of Wightman [15], and Lehmann, Symanzik, and Zimmermann [16]. In this approach, the quantized field is interpreted mathematically as an operator-valued Schwartz distribution. Explicit use of the theory of distributions was a major step, which helped to partially make the theory mathematically sound by smoothing out the fields locally.

The axiomatic approach proved very fruitful, providing the first rigorous proofs of a number of important general results, and attracted many able researchers. The favored name 1960-1970

era was algebraic quantum field theory. The books by Jost [17], Streater and Wightman [18], and Bogolubov and Shirkov [19] are the classics, while more recent work can be found in Haag [20]. (See also the book by Bogolubov, Logunov, and Todorov, [21].)

Although a great deal of work was done in constructive field theory between 1950 and 1990, many difficult problems still remain. For example, the appearance of difficulties with the constructive approach to polynomial types of field theories is discussed in the paper by Sokal [22]. He identified some mathematical difficulties and conjectured that the preliminary successes in two and three space-time dimensions might not work as expected in four space-time. His conjecture was independently verified by Aizenman and Graham [23] and Fröhlich [24]. Since then, research on this and related problems has almost vanished.

### *Purpose*

During his Gibbs Lecture [25] Dyson, suggested that providing the mathematical foundations for Feynman's sum over histories, or path integral approach would also provide the foundations for quantum field theory within the framework of the  $C^*$ -algebra approach due to Haag and Kastler [26].

At the end of his book on path integrals with Hibbs [27], Feynman states: "Nevertheless, many of the results and formulations of path integrals can be reexpressed by another mathematical system, a kind of ordered operator calculus. In this form many of the results of the preceding chapters find an analogous but more general representation ... involving noncommuting variables." Thus, Feynman conjectured that mathematical meaning for the time-ordered operator calculus would automatically solve the path integral problem and lead to wider applications.

A number of mathematicians have formally investigated the time-ordering problem: Miranker and Weiss [28], Nelson [29], Araki [30] and Maslov [31]. The recent comprehensive work by Johnson and Lapidus contains references to almost all contributions to the field (see [32]). However, none were able to provide any insight into the physical problems associated with quantum electrodynamics and quantum field theory in general, nor were they able to connect time-ordering to Feynman's path integral.

In this paper, we discuss a rigorous implementation of Feynman's program (see also [33]). First, von Neumann's infinite tensor product Hilbert space [34] is used to construct the mathematical version of Feynman's film. Then, the basic methods of analysis are extended to this setting, making it possible to prove Feynman's conjecture and the remaining two conjectures of Dyson. (It is of historical note that many of the tools required for this work were developed by von Neumann for other purposes in the 1930's [34].)

This paper is part of a new investigation into the physical and mathematical foundations of relativistic quantum theory. Our overall goal is to construct a self-consistent relativistic quantum theory of particles and fields. For this paper, we have two specific objectives. Our first and major objective is to construct a physically simple and computationally useful representation theory for the Feynman time-ordered operator calculus. A correct formulation and representation theory for the Feynman time-ordered operator calculus should at least have the following desirable features:

- (i) It should provide a transparent generalization of current analytic methods without sacrificing the physically intuitive and computationally useful ideas of Feynman.
- (ii) It should provide a clear approach to some of the mathematical problems of relativistic quantum theory.
- (iii) It should explain the connection with path integrals.

In the course of his analysis, unification, and simplification of the Feynman, Schwinger Tomonaga theory, Dyson made two important suggestions (conjectures). The first conjecture concerned the

divergences in QED, while the second was concerned with the convergence of the renormalized perturbation series. In addressing the problem of divergences, Dyson conjectured that they may be due to an idealized conception of measurability resulting from the infinitely precise knowledge of the space-time positions of particles implied by our Hamiltonian formulation which leads to a violation of the Heisenberg uncertainty principle. This point of view can be traced directly to the Bohr-Rosenfeld theory of measurability for field operators. In addressing the renormalized S-matrix, Dyson suggested that it might be more reasonable to expect the expansion to be asymptotic rather than convergent and gave physical arguments to support his claim. The lack of a clear mathematical framework made it impossible to formulate and investigate his suggestions. Schweber [2] notes that Dyson made two other well-known conjectures. The overlapping divergences conjecture was proved by Salam. Dyson's conjecture that a certain Feynman integral converges, necessary for showing that the ultraviolet divergences cancel to all orders, was proved by Weinberg. Our second objective is to provide proofs of the above two conjectures under general conditions that should apply to any formulation of quantum field theory which does not abandon Hamiltonian generators for unitary solution operators. The proof of the first conjecture is, to some extent expected, and is a partial vindication of our belief in the consistency of quantum electrodynamics in the sense that the ultraviolet problem is caused by an effect that is basically simple. Such a result is partly anticipated since the effect can be made to disappear via appropriate cutoffs. We also identify special conditions under which the renormalized perturbation series may actually converge.

## 1. Time-Ordering

In the Feynman view, particles live and die on a photograph of their motion, in which time acts as a director. If the time interval (for convenience) is  $I = [-T, T]$ , physical analysis shows that the actual model can be represented as a curve on  $\mathbf{X} = \prod_{-T}^T \mathbb{R}_p^3(t)$ , where  $\mathbb{R}_p^3(t)$  is a physical coordinate system, envisioned as  $\mathbb{R}^3$  attached to an observer, including measuring devices and any background effects (either local or distant), which affect the observer's ability to obtain precise information about the physical evolution at time  $t$ .

Thus, true physical events occur on  $\mathbf{X}$ , where actual experimental information is modified by fluctuations in  $\mathbb{R}_p^3(t)$  and by the interaction of the micro-system with the measuring equipment. However, in terms of theoretical representations, mathematical models of the evolution are represented by wave functions, amplitudes, and/or operator-valued distributions, etc, on a Hilbert space. It follows that there are three spaces, the physical space of evolution for the micro-system, the observer's space of obtainable information concerning this evolution and the mathematical modeling space.

### 1.1. The Film

The modeling space of the obtainable information or film, is a Hilbert bundle over the time interval  $I$ . Fix a separable space  $\mathcal{H}$  and let  $\hat{\otimes}_{t \in I} \mathcal{H}(t) =: \mathcal{H}_{\otimes}^2$  be the infinite tensor product Hilbert space of von Neumann [34]. In order to gain a feel for this space, we need to defined uncountable products.

In order to avoid trivialities, we always assume that, in any product, all terms are nonzero.

**Definition 1.1.** If  $\{z_t\}$  is a sequence of complex numbers indexed by  $t \in I$ ,

- (i) We say that the product  $\prod_{t \in I} z_t$  is convergent with limit  $z$  if, for every  $\varepsilon > 0$ , there is a finite set  $J(\varepsilon)$  such that, for all finite sets  $J \subset I$ , with  $J(\varepsilon) \subset J$ , we have  $|\prod_{t \in J} z_t - z| < \varepsilon$ .
- (ii) We say that the product  $\prod_{t \in I} z_t$  is quasi-convergent if  $\prod_{t \in I} |z_t|$  is convergent. (If the product is quasi-convergent, but not convergent, we assign it the value zero.)

To see why the second definition is natural, let  $z_t = e^{i\theta(t)}$ , where  $\theta(t)$  is an arbitrary real-valued function. Since  $I$  is not countable, we note that

$$0 < \left| \prod_{t \in I} z_t \right| < \infty \text{ if and only if } \sum_{t \in I} |1 - z_t| < \infty. \quad (1.1)$$

Thus, it follows that convergence implies that at most a countable number of the  $z_t \neq 1$ .

**Definition 1.2.** Let  $\phi = \bigotimes_{t \in I} \phi_t$  and  $\psi = \bigotimes_{t \in I} \psi_t$  be in  $\mathcal{H}_{\otimes}^2$ .

- (i) We say that  $\phi$  is strongly equivalent to  $\psi$  ( $\phi \equiv^s \psi$ ), if and only if  $\sum_{t \in I} |1 - \langle \phi_t, \psi_t \rangle_t| < \infty$ .
- (ii) We say that  $\phi$  is weakly equivalent to  $\psi$  ( $\phi \equiv^w \psi$ ), if and only if  $\sum_{t \in I} |1 - |\langle \phi_t, \psi_t \rangle_t|| < \infty$ .

We note the slight difference between strong and weak,  $\langle \phi_t, \psi_t \rangle_t$  verses  $|\langle \phi_t, \psi_t \rangle_t|$ .

The proofs of results in this section are due to von Neumann [34].

**Theorem 1.3.** The relations defined above are equivalence relations on  $\mathcal{H}_{\otimes}^2$ , which decomposes  $\mathcal{H}_{\otimes}^2$  into disjoint equivalence classes.

**Definition 1.4.** For  $\varphi = \bigotimes_{t \in I} \varphi_t \in \mathcal{H}_{\otimes}^2$ , we define  $\mathcal{H}_{\otimes}^2(\varphi)$  to be the closed subspace generated by the span of all  $\psi \equiv^s \varphi$  and we call it the strong partial tensor product space generated by the vector  $\varphi$ . (von Neumann called them incomplete tensor product spaces.)

**Theorem 1.5.** For the partial tensor product spaces, we have the following:

- (i) If  $\psi_t \neq \varphi_t$  occurs for at most a finite number of  $t$ , then  $\psi = \bigotimes_{t \in I} \psi_t \equiv^s \varphi = \bigotimes_{t \in I} \varphi_t$ .
- (ii) The space  $\mathcal{H}_{\otimes}^2(\varphi)$  is the closure of the linear span of  $\psi = \bigotimes_{t \in I} \psi_t$  such that  $\psi_t \neq \varphi_t$  occurs for at most a finite number of  $t$ .
- (iii) If  $\Phi = \bigotimes_{t \in I} \varphi_t$  and  $\Psi = \bigotimes_{t \in I} \psi_t$  are in different equivalence classes of  $\mathcal{H}_{\otimes}^2$ , then  $(\Phi, \Psi)_{\otimes} = \prod_{t \in I} \langle \varphi_t, \psi_t \rangle_t = 0$ .
- (iv)  $\mathcal{H}_{\otimes}^2(\varphi)^w = \bigoplus_{\psi \equiv^w \varphi} [\mathcal{H}_{\otimes}^2(\psi)^s]$ .

If  $L[\mathcal{H}_{\otimes}^2]$  is the set of bounded linear operators on  $\mathcal{H}_{\otimes}^2$ , for each  $t \in I$ , define  $L[\mathcal{H}(t)] \subset L[\mathcal{H}_{\otimes}^2]$  by:

$$L[\mathcal{H}(t)] = \left\{ \mathbb{H}(t) = \left( \bigotimes_{T \geq s > t} \mathbf{I}_s \right) \otimes H(t) \otimes \left( \bigotimes_{t > s \geq -T} \mathbf{I}_s \right), \forall H(t) \in L[\mathcal{H}] \right\}, \quad (1.2)$$

where  $\mathbf{I}_s$  is the identity operator. Let  $L^{\#}[\mathcal{H}_{\otimes}^2]$  be the uniform closure of the algebra generated by  $\{L[\mathcal{H}(t)], t \in I\}$ . If the family  $\{H(t), t \in I\}$  is in  $L[\mathcal{H}]$ , then the operators  $\{\mathbb{H}(t), t \in I\} \in L^{\#}[\mathcal{H}_{\otimes}^2]$  commute when acting at different times:

$$\mathbb{H}(t)\mathbb{H}(\tau) = \mathbb{H}(\tau)\mathbb{H}(t) \text{ for } t \neq \tau.$$

Let  $E = \bigotimes_{t \in I} e_t$  be a unit vector, let  $\mathcal{H}_{\otimes}^2(E)$  be the partial tensor product space generated by  $E$  and let  $\mathbf{P}_E$  denote the projection from  $\mathcal{H}_{\otimes}^2$  onto  $\mathcal{H}_{\otimes}^2(E)$ . If  $\mathbf{T}_{\theta}^t$  is used to represent the mapping from  $L(\mathcal{H}) \rightarrow L(\mathcal{H}(t))$  in (2), von Neumann [34] proved that:

**Theorem 1.6.** The set  $L(\mathcal{H}(t))$  is a uniformly closed algebra and  $\mathbf{T}_{\theta}^t : L(\mathcal{H}) \rightarrow L(\mathcal{H}(t))$  is an isometric isomorphism of algebras. (We call  $\mathbf{T}_{\theta}^t$  the time-ordering morphism.)

**Theorem 1.7.** If  $\mathbf{T} \in L^{\#}[\mathcal{H}_{\otimes}^2]$ , then  $\mathbf{P}_E \mathbf{T} = \mathbf{T} \mathbf{P}_E$ .

Thus, every operator in  $L^\#[\mathcal{H}_\otimes^2]$  is the uniform limit of operators from the algebra generated by the family  $\{L[\mathcal{H}(t)], t \in I\}$  and is invariant on  $\mathcal{H}_\otimes^2(E)$  (for any  $E$ ).

**Definition 1.8.** An exchange operator  $\mathbf{E}[t, t']$  is a linear map defined for pairs  $t, t'$  such that:

- (i)  $\mathbf{E}[t, t'] : L[\mathcal{H}(t)] \rightarrow L[\mathcal{H}(t')]$ , (isometric isomorphism),
- (ii)  $\mathbf{E}[s, t']\mathbf{E}[t, s] = \mathbf{E}[t, t']$ ,
- (iii)  $\mathbf{E}[t, t']\mathbf{E}[t', t] = I$ ,
- (iv) for  $s \neq t, t'$ ,  $\mathbf{E}[t, t']\mathcal{H}(s) = \mathcal{H}(s)$ , for all  $\mathcal{H}(s) \in L[\mathcal{H}(s)]$ .

The exchange operator acts to interchange the time positions of a pair of operators in an expression. We can now construct our film. Let  $\{e^i\}$  be an orthonormal basis for  $\mathcal{H}$  and set  $E^i = \otimes e_t^i$ , where  $e_t^i = e^i$  for each  $i$ . Let  $\mathcal{FD}_2^i = \mathcal{H}_\otimes^2(E^i)$  be the partial tensor product space generated by the vector  $E^i$  and set  $\mathcal{FD}_\otimes^2 = \bigoplus_{i=1}^\infty \mathcal{FD}_2^i$ . It is not hard to see that the fiber at each time-slice is isomorphic to  $\mathcal{H}$ .

**Definition 1.9.** We call  $\mathcal{FD}_\otimes^2$  the Feynman-Dyson space (FD-space) on  $\mathcal{H}$  over  $I$ .

In order to understand why  $\mathcal{FD}_\otimes^2$  is the correct structure for our film, we need an explicit basis for each  $\mathcal{FD}_2^i$ . Let  $\mathbf{F}$  be the set of all functions  $f(\cdot) : I \rightarrow \mathbb{N} \cup \{0\}$  such that  $f(t)$  is zero for all but a finite number of  $t$ , and let  $F(f)$  denote the image of the function  $f(\cdot)$ . Set  $E_{F(f)}^i = \otimes_{t \in I} e_{t, f(t)}^i$  with  $e_{t, 0}^i = e^i$ , and  $f(t) = k$  implies  $e_{t, k}^i = e^k$ .

It is shown by von Neumann that the set of vectors  $\{E_{F(f)}^i | F(f) \in \mathbf{F}\}$  is an orthonormal basis for  $\mathcal{FD}_2^i$  (so that  $\langle E_{F(f)}^i, E_{F(g)}^i \rangle = \prod_{t \in I} \langle e_{t, f(t)}^i, e_{t, g(t)}^i \rangle = 0$  unless  $f(t) = g(t)$  for all  $t \in I$ .)

## 1.2. Integrals and Evolutions

Recall that a unitary group  $U(t)$  is said to be a  $C_0$ -unitary group if it is strongly continuous,  $U(0) = I$ , and  $\lim_{|t| \rightarrow 0} U(t)\phi = \phi$  for all  $\phi \in \mathcal{H}$ .

We now assume that  $H(t)$  is strongly continuous in  $t$  and is the generator of a  $C_0$ -unitary group on  $\mathcal{H}$ , with domain  $D(H(t))$ , independent of  $t \in I$ . It follows that  $Q[T, -T] = \int_{-T}^T H(t)dt \in C(\mathcal{H})$  (the closed densely defined linear operators on  $\mathcal{H}$ ) as a strong Riemann integral.

If  $\Phi \in \mathcal{FD}_\otimes^2$ , with  $\Phi = \sum_i \sum_{F(f)} a_{F(f)}^i E_{F(f)}^i$  and if  $\sum_{F(f)}^K |a_{F(f)}^i|^2 = |b^i|^2$ , we have: (using the basis)

**Theorem 1.10.** (Fundamental Theorem for Time-Ordered Integrals)

Let  $\mathbb{H}(t)$  be the corresponding time-ordered version of the family  $\{H(t), t \in I\}$ . If  $\Phi \in D(H(t))$ ,  $-T < t \leq T$ , then

- (i) The operator  $\mathbf{Q}[t, -T] = \int_{-T}^t \mathbb{H}(s)ds \in C(\mathcal{FD}_\otimes^2)$  is a strong Riemann integral, which generates the  $C_0$ -unitary group  $\exp\{i\mathbf{Q}[t, -T]\}$ , on  $\mathcal{FD}_\otimes^2$  and, for each  $t \leq T$ .
- (ii)  $\mathbf{Q}[t, s]\Phi + \mathbf{Q}[s, -T]\Phi = \mathbf{Q}[t, -T]\Phi$  (a.s.).
- (iii)  $\|\mathbf{Q}[t, -T]\Phi\|_\otimes^2 = \sum_i^J |b^i|^2 |\langle \mathbf{Q}[t, -T]e^i, e^i \rangle|^2$  (a.s.).
- (iv) If  $\Phi(t) = \exp\{i\mathbf{Q}[t, -T]\}\Phi$ , then

$$i \frac{\partial \Phi(t)}{\partial t} = \mathbb{H}(t)\Phi(t), \quad \Phi(-T) = \Phi$$

The proof can be found in [33]. In (2) and (3) of the above theorem, the exceptional set is at most countable. This theorem shows that using time to order operators offers much more than just an index. It is shown in [33] that we need only assume weak continuity of  $H(t)$ , along with a growth condition, in order to obtain the same result. This theorem implies that all of analysis can be lifted to this setting under less conditions with no losses.

In physics, the interesting models represent interacting particles with  $\mathbb{H}(t) = \mathbb{H}_0(t) + w\mathbb{H}_1(t)$ , where  $w\mathbb{H}_1(t)$  is small in some sense. (For example,  $\|\mathbb{H}_1\Phi\|/\|\mathbb{H}_0\Phi\| \ll w$  with a few mathematical conditions will suffice.) In the most useful cases, everything is calculated using perturbation theory. The relevant computational algorithm starts with:

$$\mathbf{U}_0[t, -T] = \exp\left\{-\frac{i}{\hbar}\mathbf{Q}_0[t, -T]\right\}, \quad \mathbf{Q}_0[t, -T] = \int_{-T}^t \mathbb{H}_0(\tau)d\tau.$$

The interaction representation is then defined by  $\mathbb{H}_I(t) = \mathbf{U}_0[-T, t]\mathbb{H}_1(t)\mathbf{U}_0[t, -T]$  and ( $v = -\frac{iw}{\hbar}$ )

$$\mathbf{U}_I^w[t, -T] = \exp\{v\mathbf{Q}_I[t, -T]\}, \quad \mathbf{Q}_I[t, -T] = \int_{-T}^t \mathbb{H}_I(\tau)d\tau.$$

One then writes

$$\Phi(t) = \Phi + \sum_{k=1}^{\infty} v^k \int_{-T}^t ds_1 \int_{-T}^{s_1} ds_2 \cdots \int_{-T}^{s_{k-1}} ds_k \mathbb{H}_I(s_1)\mathbb{H}_I(s_2) \cdots \mathbb{H}_I(s_k)\Phi,$$

where  $\Phi(t) = \mathbf{U}_I^w[t, -T]\Phi$  and  $w$  is a small constant (charge).

An important theorem of Haag [35] shows that the interaction representation in sharp time (as above) does not exist in a rigorous sense (see Streater and Wightman [36], pg. 161). Haag shows that the equal time commutation relations for the canonical variables of an interacting field are equivalent to those of a free field. In trying to explain this unfortunate result, Streater and Wightman point out that (see [36], p. 168) "... What is even more likely in physically interesting quantum field theories is that equal-time commutation relations will make no sense at all; the field might not be an operator unless smeared in time as well as space." It has now been experimentally confirmed that there is quantum interference in time (see [37]). Thus, Hagg's assumption of sharp time is not physically valid.

The mathematical problem is to provide some justification for the amazingly accurate results obtained by Feynman and others. To address this problem, we begin with the notion of an asymptotic expansion. We follow Hille and Phillips [38]. Let  $\mathbf{U}_I^w[t, -T] = \exp\{v\mathbf{Q}_I[t, a]\}$ .

**Definition 1.11.** We say that  $\mathbf{U}_I^w[t, -T]$  is asymptotic in the sense of Poincaré if, for each  $n$  and each  $\Phi \in D[(\mathbf{Q}_I[t, -T])^{n+1}]$ , we have

$$\lim_{w \rightarrow 0} v^{-(n+1)} \left\{ \mathbf{U}_I^w[t, -T] - \sum_{k=1}^n \frac{(v\mathbf{Q}_I[t, -T])^k}{k!} \right\} \Phi = \frac{\mathbf{Q}_I[t, -T]^{n+1}}{(n+1)!} \Phi.$$

This is the operator version of an asymptotic expansion in the classical sense due to Poincaré (see Coddington and Levinson [39]). In order to get time smearing, we use the exchange operator  $\mathbf{E}[t, s]$  of Definition 3, and let  $\bar{\mathbf{U}}_0[t, -T]$  be defined by:

$$\bar{\mathbf{U}}_0[t, -T] = \exp\left\{(-i/\hbar) \int_{-T}^t \mathbf{E}[t, s]\mathbb{H}_0(s)ds\right\}. \quad (1.3)$$

In this case the interaction representation for  $\mathbb{H}_{\mathbf{I}}(t)$  is given by:

$$\mathbb{H}_{\mathbf{I}}(t) = \bar{\mathbf{U}}_0[-T, t] \mathbb{H}_1(t) \bar{\mathbf{U}}_0[t, -T].$$

**Theorem 1.12.** *The time-ordered integral  $\mathbf{Q}_{\mathbf{I}}[t, -T]$  generates the  $C_0$  unitary group  $\mathbf{U}_{\mathbf{I}}^w[t, -T]$  on  $\mathcal{FD}_{\otimes}^2$  and, for  $\Phi(t) = \mathbf{U}_{\mathbf{I}}^w[t, -T]\Phi$ , we have:*

(i)

$$i\hbar \frac{\partial}{\partial t} \Phi(t) = w \mathbb{H}_{\mathbf{I}}(t) \Phi(t), \quad \Phi(-T) = \Phi. \quad (1.4)$$

(ii) *The operator  $\mathbf{U}_{\mathbf{I}}^w[t, a] = \exp\{v \mathbf{Q}_{\mathbf{I}}[t, a]\}$  is asymptotic in the sense of Poincaré.*

(iii) *For each  $n$  and each  $\Phi \in D\left[(\mathbf{Q}_{\mathbf{I}}[t, -T])^{n+1}\right]$ , we have*

$$\begin{aligned} \Phi(t) = & \Phi + \sum_{k=1}^n \nu^k \int_{-T}^t ds_1 \int_{-T}^{s_1} ds_2 \cdots \int_{-T}^{s_{k-1}} ds_k \mathbb{H}_{\mathbf{I}}(s_1) \mathbb{H}_{\mathbf{I}}(s_2) \cdots \mathbb{H}_{\mathbf{I}}(s_k) \Phi \\ & + \int_0^w (\nu - \xi)^n d\xi \int_{-T}^t ds_1 \int_{-T}^{s_1} ds_2 \cdots \int_{-T}^{s_n} ds_{n+1} \mathbb{H}_{\mathbf{I}}(s_1) \mathbb{H}_{\mathbf{I}}(s_2) \cdots \mathbb{H}_{\mathbf{I}}(s_{n+1}) \mathbf{U}_{\mathbf{I}}^{\xi}[s_{n+1}, -T] \Phi. \end{aligned} \quad (1.5)$$

where  $\Phi(t) = \mathbf{U}_{\mathbf{I}}^w[t, -T]\Phi$ .

**Remark 1.13.** *The above theorem includes all generators of  $C_0$ -unitary groups and provides a precise formulation and proof of Dyson's second conjecture for quantum electrodynamics that, in general, we can only expect the expansion to be asymptotic. Actually, more is proved, in that the remainder term above makes the perturbation expansion exact for all finite  $n$ . The theory is more general and applies to all of applied analysis.*

*The existence of a remainder should not lead one to conclude that it is useful for all problems. In the case of bound states, all important information resides in the remainder term, independent of  $n$ . It was this problem that led to work on nonperturbative methods. An early approach was developed by Salpeter and Bethe [40].*

## 2. Sum Over Paths

As noted earlier, Feynman stated in his book with Hibbs [27] that the operator calculus is more general than the path integral, and includes it. We now show that his expectation was indeed warranted. First we construct what we call the experimental evolution operator. This allows us to rewrite our theory as a sum over paths. We use a general argument so that the ideas apply to all cases.

Assume that the family  $\{\tau_1, \tau_2, \dots, \tau_n\}$  represents the time positions of  $n$  possible measurements of a micro system trajectory, as it appears on a film of its history. We assume that information is available beginning at time  $-T$  and ends at time  $t$ . Define  $\mathbf{Q}_{ex}[\tau_1, \tau_2, \dots, \tau_n]$  by

$$\mathbf{Q}_{ex}[\tau_1, \tau_2, \dots, \tau_n] = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E[\tau_j, s] \mathbb{H}(s) ds. \quad (2.1)$$

Here,  $t_0 = \tau_0 = -T$ ,  $t_j = (1/2)[\tau_j + \tau_{j+1}]$  (for  $1 \leq j \leq n$ ), and  $E[\tau_j, s]$  is the exchange operator. The effect of  $E[\tau_j, s]$  is to concentrate all information contained in  $[t_{j-1}, t_j]$  at  $\tau_j$ , the midpoint of the time interval around  $\tau_j$  relative to  $\tau_{j-1}$  and  $\tau_{j+1}$ . We can rewrite  $\mathbf{Q}_{ex}[\tau_1, \tau_2, \dots, \tau_n]$  as

$$\mathbf{Q}_{ex}[\tau_1, \tau_2, \dots, \tau_n] = \sum_{j=1}^n \Delta t_j \left[ \frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} E[\tau_j, s] \mathbb{H}(s) ds \right]. \quad (2.2)$$

Thus, we have an average over each adjacent interval, with information concentrated at the midpoint. The evolution operator is given by

$$U[\tau_1, \tau_2, \dots, \tau_n] = \exp \left\{ -\frac{i}{\hbar} \sum_{j=1}^n \Delta t_j \left[ \frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} E[\tau_j, s] \mathbb{H}(s) ds \right] \right\}.$$

For  $\Phi \in \mathcal{FD}_{\otimes}^2$ , we define the function  $\mathbf{U}[N(t), -T]\Phi$  by:

$$\mathbf{U}[N(t), -T]\Phi = U[\tau_1, \tau_2, \dots, \tau_n]\Phi. \quad (2.3)$$

$\mathbf{U}[N(t), -T]\Phi$  is a  $\mathcal{FD}_{\otimes}^2$ -valued random variable, which represents the distribution of the number of measurements,  $N(t)$ , that are possible up to time  $t$ . In order to relate  $\mathbf{U}[N(t), -T]\Phi$  to actual experimental results, we must compute its expected value. Let  $\lambda^{-1}$  denote the smallest time interval in which a measurement can be made, and define  $\bar{\mathbf{U}}_{\lambda}[t, -T]\Phi$  by:

$$\bar{\mathbf{U}}_{\lambda}[t, -T]\Phi = \mathcal{E}[\mathbf{U}[N(t), -T]\Phi] = \sum_{n=0}^{\infty} \mathcal{E}\{\mathbf{U}[N(t), -T]\Phi | N(t) = n\} \Pr[N(t) = n],$$

where

$$\begin{aligned} & \mathcal{E}\{\mathbf{U}[N(t), -T]\Phi | N(t) = n\} \\ &= \int_0^t \frac{d\tau_1}{t} \int_0^t \frac{d\tau_2}{t - \tau_1} \dots \int_0^t \frac{d\tau_n}{t - \tau_{n-1}} \mathbf{U}[\tau_n, \dots, \tau_1]\Phi = \mathbf{U}_n[t, 0]\Phi. \end{aligned}$$

We make the natural assumption that: (see Gill and Zachary [33])

$$\Pr[N(t) = n] = (n!)^{-1} (\lambda t)^n \exp\{-\lambda t\}.$$

Since we are only interested in what happens when  $\lambda \rightarrow \infty$ , and, as the mean number of possible measurements up to time  $t$  is  $\lambda t$ , we can take  $\tau_j = (jt/n)$ ,  $1 \leq j \leq n$ , ( $\Delta t_j = t/n$  for each  $n$ ). We now replace  $\bar{\mathbf{U}}_n[t, -T]\Phi$  by  $\mathbf{U}_n[t, -T]\Phi$  and, with this understanding, we continue to use  $\tau_j$ , so that

$$\mathbf{U}_n[t, -T]\Phi = \exp \left\{ -\frac{i}{\hbar} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E[\tau_j, s] \mathbb{H}(s) ds \right\} \Phi. \quad (2.4)$$

We define the experimental evolution operator  $\mathbf{U}_{\lambda}[t, -T]\Phi$  by

$$\mathbf{U}_{\lambda}[t, 0]\Phi = \sum_{n=0}^{[\lambda t]} \frac{(\lambda t)^n}{n!} \exp\{-\lambda t\} \mathbf{U}_n[t, -T]\Phi. \quad (2.5)$$

The following result is a consequence of the fact that Borel summability is regular.

**Theorem 2.1.** *Assume that  $\mathbb{H}(t)$  is strongly continuous in  $t$  and the generator of a  $C_0$ -unitary group. Then*

$$\lim_{\lambda \rightarrow \infty} \bar{\mathbf{U}}_{\lambda}[t, -T]\Phi = \lim_{\lambda \rightarrow \infty} \mathbf{U}_{\lambda}[t, -T]\Phi = \mathbf{U}[t, -T]\Phi. \quad (2.6)$$

Since  $\lambda \rightarrow \infty$  implies  $\lambda^{-1} \rightarrow 0$ , this means that the average time between measurements is zero (in the limit) so that we obtain a continuous path. It should be observed that this continuous path arises from averaging the sum over an infinite number of (discrete) paths. The first term in (3.5) corresponds to the path of a system that created no information (i.e., the film is blank). This event has probability proportional to  $\exp\{-\lambda t\}$  (which approaches zero as  $\lambda \rightarrow \infty$ ). The  $n$ -th term corresponds to the path that creates  $n$  possible measurements, (with probability  $[(\lambda t)^n/n!] \exp\{-\lambda t\}$ ) etc.

### 3. The S-matrix

We are now in a position to investigate the sense in which we can believe Dyson's first conjecture. At the end of his second paper on the relationship between the Feynman and Schwinger-Tomonaga theories, he explored the difference between the divergent Hamiltonian formalism that one must begin with and the finite S-matrix that results from renormalization. He took the view that it is a contrast between a real observer and a fictitious (ideal) observer. The real observer can only determine particle positions with limited accuracy and always gets finite results from his measurements. Dyson then suggests that "... The ideal observer, however, using non-atomic apparatus whose location in space and time is known with infinite precision, is imagined to be able to disentangle a single field from its interactions with others, and to measure the interaction. In conformity with the Heisenberg uncertainty principle, it can perhaps be considered a physical consequence of the infinitely precise knowledge of particle location allowed to the ideal observer, that the value obtained when he measures the interaction is infinite." He goes on to remark that, if his analysis is correct, the problem of divergences is attributable to an idealized concept of measurability.

In order to explore this idea, we work in the interaction representation with an obvious change in notation, replace  $\mathcal{H}(t)$  by  $-\frac{i}{\hbar}\mathcal{H}(t)$  and  $\mathbf{U}_\lambda[t, -T]$  by

$$\mathbf{S}_\lambda[T, -T]\Phi = \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \exp[-2\lambda T] \mathbf{S}_n[T, -T]\Phi,$$

$$\mathbf{S}_n[T, -T]\Phi = \exp \left\{ -\frac{i}{\hbar} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E[\tau_j, s] \mathbb{H}_\mathbf{I}(s) ds \right\} \Phi,$$

and  $\mathbb{H}_\mathbf{I}(t) = \int_{\mathbf{R}^3} \mathbb{H}_\mathbf{I}(\mathbf{x}(t), t) d\mathbf{x}(t)$  is the interaction energy. We now give a physical interpretation of our formalism. If we now note that the term  $\exp\{-2\lambda T\}$  may be expressed as

$$\exp\{-2\lambda T\} = \exp \left\{ -\frac{i}{\hbar} \left[ \int_{-T}^T (-i\lambda \hbar) ds \right] \right\},$$

we can rewrite  $\mathbf{S}_\lambda[T, -T]\Phi$  as:

$$\mathbf{S}_\lambda[T, -T]\Phi = \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \exp \left\{ -\frac{i}{\hbar} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} [w E[\tau_j, s] \mathbb{H}_\mathbf{I}(s) - i\lambda \hbar I_\otimes] ds \right\} \Phi.$$

In this form, the term  $-i\lambda \hbar I_\otimes$  has the physical interpretation as the absorption of photon energy of amount  $\lambda \hbar$  in each subinterval  $[t_j, t_{j-1}]$  (see Mott and Massey [41]). When we compute the  $\lim_{\lambda \rightarrow \infty}$ , we get the standard S-matrix on  $[-T, T]$ . It follows that we must add an infinite amount of photon energy to the mathematical description of the experimental picture at each point in time. This is the ultraviolet divergence and shows explicitly that the transition from the experimental to the ideal scattering operator requires that we illuminate the particle throughout its entire path. Thus, it appears that we have violated the uncertainty relation. This is further supported if we look at the form of the standard S-matrix:

$$\mathbf{S}[T, -T]\Phi = \exp \left\{ -\frac{i}{\hbar} \int_{-T}^T \mathbb{H}_\mathbf{I}(s) ds \right\} \Phi$$

and note that the differential  $ds$  in the exponent implies perfect infinitesimal time knowledge at each point, so that the energy should be totally undetermined. If violation of the Heisenberg

uncertainty relation is the cause for the ultraviolet divergence then, since it is a variance relation, it will not appear in any first order perturbation calculation. However, it should show up in all higher-order terms (as it does). On the other hand, if the divergent terms are eliminated in second order, we would expect the method to prevent them from appearing in any higher order term of the expansion (as it does). The fact that this is precisely the case in quantum electrodynamics is a clear verification of Dyson's conjecture. (If we allow  $T$  to become infinite, we once again introduce an infinite amount of energy into the mathematical description of the experimental picture, as this is also equivalent to precise time knowledge at infinity. Of course, this is the well-known infrared divergence and can be eliminated by keeping  $T$  large but finite.)

### 3.1. Path Integrals

We now investigate the relationship of the operator calculus and the path integral. Suppose that the unitary group  $U[t, -T]$  for the generator  $H(t)$  (defined on  $\mathcal{H}$ ) has an associated kernel  $\mathbf{K}[\mathbf{x}(t), t; \mathbf{x}(s), s]$  such that:

$$\begin{aligned}\mathbf{K}[\mathbf{x}(t), t; \mathbf{x}(s), s] &= \int_{\mathbf{R}^3} \mathbf{K}[\mathbf{x}(t), t; d\mathbf{x}(\tau), \tau] \mathbf{K}[\mathbf{x}(\tau), \tau; \mathbf{x}(s), s], \\ U[t, s]\varphi(s) &= \int_{\mathbf{R}^3} \mathbf{K}[\mathbf{x}(t), t; d\mathbf{x}(s), s] \varphi(s).\end{aligned}$$

Let  $\mathbf{U}[t, s]$  be the corresponding time-ordered version, with kernel  $\mathbb{K}_{\mathbf{f}}[\mathbf{x}(t), t; \mathbf{x}(s), s]$ , on  $\mathcal{FD}_{\otimes}^2$ . Then, since  $\mathbf{U}[t, \tau]\mathbf{U}[\tau, s] = \mathbf{U}[t, s]$ , we have:

$$\mathbb{K}_{\mathbf{f}}[\mathbf{x}(t), t; \mathbf{x}(s), s] = \int_{\mathbf{R}^3} \mathbb{K}_{\mathbf{f}}[\mathbf{x}(t), t; d\mathbf{x}(\tau), \tau] \mathbb{K}_{\mathbf{f}}[\mathbf{x}(\tau), \tau; \mathbf{x}(s), s].$$

From our sum over paths representation for  $\mathbf{U}[t, s]$ , we see that:

$$\mathbf{U}_k[t, s]\Phi(s) = \exp \left\{ (-i/\hbar) \sum_{j=1}^k \int_{t_{j-1}}^{t_j} \mathbf{E}[(j/\lambda), \tau] \mathbb{H}(\tau) d\tau \right\} \Phi(s)$$

corresponds to

$$\prod_{j=1}^k \int_{\mathbf{R}^3} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(t_j); d\mathbf{x}(t_{j-1}), t_{j-1}] \Big|^{(j/\lambda)} \Phi(s),$$

where  $\Big|^{(j/\lambda)}$  denotes the fact that the integration is performed in time slot  $(j/\lambda)$ . We define  $\mathbb{K}_{\mathbf{f}}[\mathcal{D}_{\lambda}\mathbf{x}(\tau); \mathbf{x}(s)]$  by:

$$\begin{aligned}& \int_{\mathbf{R}^{3[t, s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_{\lambda}\mathbf{x}(\tau); \mathbf{x}(s)] \\ &= e^{-\lambda(t-s)} \sum_{k=0}^{\llbracket \lambda(t-s) \rrbracket} \frac{[\lambda(t-s)]^k}{k!} \left\{ \prod_{j=1}^k \int_{\mathbf{R}^3} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(t_j); d\mathbf{x}(t_{j-1}), t_{j-1}] \Big|^{(j/\lambda)} \right\},\end{aligned}$$

where  $\llbracket \lambda(t-s) \rrbracket$  is the greatest integer in  $\lambda(t-s)$ .

**Definition 3.1.** The Feynman path integral associated with  $\mathbf{U}[t, s]$  is defined by:

$$\mathbf{U}[t, s] = \int_{\mathbf{R}^{3[t, s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(s)] = \lim_{\lambda \rightarrow \infty} \int_{\mathbf{R}^{3[t, s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_{\lambda}\mathbf{x}(\tau); \mathbf{x}(s)].$$

**Theorem 3.2.** *For the time-ordered theory, whenever a kernel exists, we have that:*

$$\lim_{\lambda \rightarrow \infty} \mathbf{U}_\lambda[t, s]\Phi(s) = \mathbf{U}[t, s]\Phi(s) = \int_{\mathbf{R}^{3[t, s]}} \mathbb{K}_f[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(s)]\Phi[\mathbf{x}(s)].$$

*Our approach is independent of the space of continuous functions and makes it clear that the need for a measure is more of a natural expectation based on past experience than a death blow to the foundations for the Feynman path integral.*

For an intrinsic approach to the path integral in the manner intended by Feynman, see [42].

### 3.2. Feynman-Kac

We now assume that  $H = H_0(t) + H_1(t)$ , where  $H_0(t)$  and  $H_1(t)$  are strongly continuous in  $t$  and generators of  $C_0$ -unitary groups on  $I$ , and let  $\mathbb{H}_{1,\rho}(t) = \rho\mathbb{H}_1(t)\mathbf{R}(\rho, \mathbb{H}_1(t))$  be the Yosida approximator for the time-ordered version of  $H_1(t)$ , where  $\mathbf{R}(\rho, \mathbb{H}_1(t))$  is the corresponding resolvent operator. (It is a standard result that  $\mathbb{H}_{1,\rho}(t)$  is bounded and  $\lim_{\rho \nearrow \infty} \mathbb{H}_{1,\rho}(t)\Phi = \mathbb{H}_1(t)\Phi$ , for  $\Phi \in D(\mathbb{H}_1(t))$ .) Define  $\mathbf{U}_\rho[t, -T]$  and  $\mathbf{U}_0[t, -T]$  by:

$$\mathbf{U}_\rho[t, -T] = \exp\left\{(-i/\hbar) \int_{-T}^t [\mathbb{H}_0(s) + \mathbb{H}_{1,\rho}(s)] ds\right\},$$

$$\mathbf{U}_0[t, -T] = \exp\left\{(-i/\hbar) \int_{-T}^t \mathbb{H}_0(s) ds\right\}.$$

If  $\mathbf{U}_0[t, -T]$  has an associated kernel, so that  $\mathbf{U}_0[t, s] = \int_{\mathbf{R}^{3[t, s]}} \mathbb{K}_f[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(-T)]$ , we have the following generalization of the famous Feynman-Kac Theorem, which is independent of the space of continuous functions or the concept of measure.

**Theorem 3.3.** *(Feynman-Kac)\* If  $\mathbb{H}(s) = \mathbb{H}_0(s) + \mathbb{H}_1(s)$  is the generator of a  $C_0$ -unitary group, then*

$$\begin{aligned} & \lim_{\rho \rightarrow \infty} \mathbf{U}^\rho[t, -T]\Phi = \mathbf{U}[t, -T]\Phi \\ &= \int_{\mathbf{R}^{3[t, -T]}} \mathbb{K}_f[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(-T)] \exp\left\{(-i/\hbar) \int_{-T}^t \mathbb{H}_1(s) ds\right\} \Phi[\mathbf{x}(-T)]. \end{aligned}$$

This result can be extended to all  $C_0$ -semigroup evolution operators that have a kernel. Thus, this result is a very strong generalization and shows that Feynman's conjecture was indeed correct.

## 4. Conclusion

In this paper we have provided a mathematically rigorous development of Feynman's operator calculus, in which the time index determines when operators operate (rather than their position on paper). This allows operators acting at different times to commute. This approach also allows time to assume its natural role in the representation of physical reality. We have constructed the mathematical film that allows us to view physical reality as a three-dimensional motion picture, on which one becomes more and more aware of the future as more and more of the film unfolds. The actual physical path traveled between any two space-time points

is seen as an average over all possible paths with the same beginning and end point. This leads to the first mathematically justified theory for Feynman's path integral formulation of quantum mechanics. As an application, we proved the last two remaining conjectures of Dyson's concerning the foundations of quantum electrodynamics (QED). First, we showed that the perturbation expansion is asymptotic in a well-defined sense. Secondly, we proved explicitly that the ultraviolet divergence is caused by a violation of the uncertainty relation as conjectured by Dyson.

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