IOP Conf. Series: Journal of Physics: Conf. Series **845** (2017) 012013 doi:10.1088/1742-6596/845/1/012013

The electromagnetic field equations for moving media

T Ivezić

Ruđer Bošković Institute, P.O.B. 180, 10002 Zagreb, Croatia

E-mail: `ivezic@irb.hr`

Abstract.
In this paper a formulation of the field equation for moving media is developed by the generalization of an axiomatic geometric formulation of the electromagnetism in vacuum (Ivezić T 2005 *Found. Phys. Lett.* **18** 401). First, the field equations with bivectors $F(x)$ and $\mathcal{M}(x)$ are presented and then these equations are written with the 4D vectors $E(x)$, $B(x)$, $P(x)$ and $M(x)$. The latter contain both the 4D velocity vector u of a moving medium and the 4D velocity vector v of the observers who measure E and B fields. They do not appear in previous literature. All these equations are also written in the standard basis and compared with Maxwell's equations with 3D vectors. In this approach the Ampère-Maxwell law and Gauss's law are inseparably connected in one law and the same happens with Faraday's law and the law that expresses the absence of magnetic charge. It is shown that Maxwell's equations with 3D vectors and the field equations with 4D geometric quantities are not equivalent in 4D spacetime

PACS numbers: 03.30.+p, 03.50.De

1. Introduction

The field equations for moving media in a relativistically covariant formulation were first presented by Minkowski [1]. An axiomatic geometric formulation of electromagnetism in vacuum is presented in [2]. That formulation is with only one axiom: the field equation for the bivector field F. In this paper the formulation from [2] is generalized to moving media. The geometric approach to special relativity (SR) that is used in [2], and in this paper as well, exclusively deals either with the abstract, coordinate-free, four-dimensional (4D) geometric quantities, e.g., vectors (4-vectors in the usual notation) $E(x)$, $B(x)$, .. (x is the position vector), or with their representations in some basis, the 4D coordinate-based geometric quantities (CBGQs) comprising *both* components and *a basis*, e.g., $E = E^{\nu}\gamma_{\nu}$. In that approach, which is called "invariant special relativity" (ISR), an independent physical reality is attributed to 4D geometric quantities and not, as usual, to 3D quantities. Every 4D CBGQ is *invariant* under the passive Lorentz transformations (LT); the components transform by the LT and the basis by the inverse LT leaving the whole CBGQ unchanged. The invariance of a 4D CBGQ under the passive LT reflects the fact that such 4D geometric quantity represents *the same physical quantity* for relatively moving inertial observers. This invariance is the reason for the name ISR. The principle of relativity is naturally satisfied, if the physical laws are written with the abstract 4D geometric quantities or with 4D CBGQs. There is no need to postulate it outside the mathematical formulation of the theory as in Einstein's formulation of SR [3]. The paper is organized as follows.

In Sec. 2 the basic field equation for moving media (Eq. (7)) is expressed in terms of the bivector $F = F(x)$ that represents the electromagnetic field and the generalized magnetization-polarization bivector $\mathcal{M} = \mathcal{M}(x)$. Then that equation with abstract quantities (AQs) is written with CBGQs in the standard basis. The resulting equation is separated into two equations, the equation with sources and the equation without sources. The field equation with sources is also written in the "source representation" both with AQs and with CBGQs in the $\{\gamma_\mu\}$ basis, according to which the sources of F field are the true currents $j^{(C)}$ and the magnetization-polarization current density $\partial \cdot \mathcal{M}$.

In Sec. 3 the decomposition of F is presented. F is decomposed into the electric field vector E, the magnetic field vector B and the velocity vector v of the observers who measure E and B fields (in the usual notation E, B, v, ... are called 4-vectors). Similarly, $\mathcal{M}$ is decomposed into the polarization vector $P(x)$, the magnetization vector $M(x)$ and the bulk velocity vector u of the medium. Inserting these decompositions into the basic field equation with F and $\mathcal{M}$ (Eq. (7)) we find the general form of the field equations for a magnetized and polarized moving medium expressed in terms of $E(x)$, $B(x)$, $P(x)$ and $M(x)$, which are named the field equations in the Ampèrian form. In equation with the geometric product (Eq. (26)), i.e., in its vector part (Eq. (27)), there are two different velocities u and v and, as I am aware, these field equations do not appear in previous literature. They are important results that are obtained in this paper. All AQs E, B, P and M are represented in the standard basis $\{\gamma_\mu\}$ in order to compare these basic field equations with usual formulations that deal with 3D vectors. The equation (27) is also written in the "source representation" both with AQs and with CBGQs. It is visible from that representation that the sources of E and B fields are the true current density $j^{(C)}$ and the P and M vectors. *The field equations with AQs* (Eq. (7)) *and* (Eq. (26)) *or the corresponding equations with CBGQs comprise and generalize all usual Maxwell's equations (with 3D vectors) for moving media.*

In Sec. 4 we first present a brief review of the existence of the fundamental difference between the usual transformations (UT) of the electric and magnetic fields as 3D vectors and the mathematically correct LT of 4D geometric quantities that represent the electric and magnetic fields in 4D spacetime. Then, the basic field equations with CBGQs are compared with usual Maxwell's equations with 3D vectors for the case when the observers are at rest in a stationary medium. It is shown that these two formulations are not equivalent, because the UT are not the LT.

In Sec. 5 the similar comparison is presented for the case when the observers are at rest in the laboratory frame, but material medium is moving. Again the same result is obtained as in Sec. 4 that these two formulations are not equivalent since 3D quantities do not properly transform under the LT.

In Sec. 6 the comparison with Galilean Electromagnetism is presented. It is shown that both Galilean limits are ill-defined in 4D spacetime and that they are not a quasi-static approximation of the relativistically correct field equations.

In Secs. 7, 7.1 and 7.2 the motional emf ε is calculated using 3D quantities and their UT, Sec. 7.1, and 4D geometric quantities and their mathematically correct LT, Sec. 7.2. In Sec. 7.1 it is shown that if ε is defined by the usual definition with 3D quantities then ε is different in relatively moving inertial frames, $\varepsilon = UBl$ in S and $\varepsilon' = \gamma UBl$ in S', which means that the principle of relativity is not satisfied in the usual formulation of electromagnetism with 3D quantities and their UT of **E** and **B** (and the UT of **P** and **M**). In Sec. 7.2 it is shown that if ε is defined as an invariant 4D quantity, the Lorentz scalar, then always the same value for ε is obtained, $\varepsilon = \gamma UBl$. This result unambiguously shows that the principle of relativity is naturally satisfied in the approach with 4D geometric quantities and their LT.

In Sec. 8 the discussion of the results and the conclusions are presented.

2. The basic field equation for moving media in terms of F and $\mathcal{M}$

In this paper, the geometric algebra formalism [4] will be used. In order to compare Eq. (26), i.e., Eqs. (27) and (28), with the usual formulations of electromagnetism with 3D vectors we shall represent all abstract quantities in (26), i.e., (27) and (28), with 4D CBGQs in the standard basis $\{\gamma_\mu\}$. Thus, for the reader's convenience, all equations will be written not only with the abstract multivectors but also with CBGQs in the standard basis. Therefore, *the knowledge of the geometric algebra is not required for the understanding of this presentation.*

However, a very brief summary of the geometric algebra will be provided here. The geometric (Clifford) product is written by simply juxtaposing multivectors AB. The geometric product of a grade-r multivector A_r with a grade-s multivector B_s decomposes into $A_r B_s = \langle AB\rangle_{\ r+s} + \langle AB\rangle_{\ r+s-2} \ldots + \langle AB\rangle_{\ |r-s|}$. The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series; $A_r \cdot B_s \equiv \langle AB\rangle_{\ |r-s|}$ and $A_r \wedge B_s \equiv \langle AB\rangle_{\ r+s}$. For vectors a and b we have: $ab = a\cdot b + a\wedge b$, where $a\cdot b \equiv (1/2)(ab+ba)$, $a\wedge b \equiv (1/2)(ab-ba)$. Usually the above mentioned standard basis is introduced. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors $\{\gamma_\mu\}, \mu = 0, ..., 3$, satisfying $\gamma_\mu\cdot\gamma_\nu = \eta_{\mu\nu} = diag(+---)$. The basis vectors γ_μ generate by multiplication a complete basis for the spacetime algebra: 1, γ_μ, $\gamma_\mu\wedge\gamma_\nu$, $\gamma_\mu\gamma_5$, γ_5 ($2^4 = 16$ independent elements). γ_5 is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra.

This basis, the standard basis $\{\gamma_\mu\}$, is a right-handed orthonormal frame of vectors in the Minkowski spacetime M^4 with γ_0 in the forward light cone, $\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ ($k = 1, 2, 3$). The $\{\gamma_\mu\}$ basis corresponds to Einstein's system of coordinates in which the Einstein synchronization of distant clocks [3] and Cartesian space coordinates x^i are used in the chosen inertial frame of reference.

The field equation in vacuum in the geometric algebra formalism is:

$$\partial F = j/\varepsilon_0 c, \quad \partial\cdot F + \partial\wedge F = j/\varepsilon_0 c, \tag{1}$$

i.e., with the CBGQs in the $\{\gamma_\mu\}$ basis, that equation becomes

$$\partial_\alpha F^{\alpha\beta}\gamma_\beta - \partial_\alpha\ {}^*F^{\alpha\beta}\gamma_5\gamma_\beta = (1/\varepsilon_0 c) j^\beta\gamma_\beta, \tag{2}$$

where ${}^*F^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ is the usual dual tensor, $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally skew-symmetric Levi-Civita pseudotensor. The usual covariant form of Eq. (2), i.e., *only the basis components in the* $\{\gamma_\mu\}$ *basis,* are two equations, the equation with sources $\partial_\alpha F^{\alpha\beta} = j^\beta/\varepsilon_0 c$, and that one without sources $\partial_\alpha\ {}^*F^{\alpha\beta} = 0$. It is shown in [2] that the bivector $F = F(x)$, which represent the electromagnetic field, can be taken as the primary quantity for electromagnetism and the field equation for F, Eq. (1), is the basic equation. As shown in [2], the bivector field F yields a complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. For the given sources the Clifford algebra formalism enables one to find in a simple way the electromagnetic field F, see Eqs. (7) and (8) in [2].

If j is the total current density then (1), i.e., (2), *holds unchanged in moving medium as well.* The equation (1) can be separated into the field equation with sources and that one without sources as

$$\partial \cdot F = j/\varepsilon_0 c, \quad \partial \wedge F = 0, \tag{3}$$

i.e., with CBGQs in the $\{\gamma_\mu\}$ basis they are

$$\partial_\alpha F^{\alpha\beta}\gamma_\beta = (1/\varepsilon_0 c) j^\beta \gamma_\beta, \quad \partial_\alpha \, {}^*F^{\alpha\beta}\gamma_5\gamma_\beta = 0. \tag{4}$$

Since j is a vector the trivector part is identically zero (in the absence of a magnetic charge).

The total current density vector j can be decomposed as

$$j = j^{(C)} + j^{(\mathcal{M})}, \tag{5}$$

where $j^{(C)}$ is the conduction current density of the *free* charges and $j^{(M)}$ is the magnetization-polarization current density of the *bound* charges

$$j^{(\mathcal{M})} = -c\partial\mathcal{M} = -c\partial \cdot \mathcal{M} \tag{6}$$

($\partial \wedge \mathcal{M} = 0$, since $j^{(\mathcal{M})}$ is a vector). $\mathcal{M}$ is the generalized magnetization-polarization bivector $\mathcal{M} = \mathcal{M}(x)$.

Then (1) (i.e., (3)) can be written as

$$\partial(\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c; \quad \partial \cdot (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c, \ \partial \wedge F = 0. \tag{7}$$

The trivector part, i.e., the field equation without sources, remained unchanged, because it is not affected by the separation of the current density vector j into free and bound parts; that part does not contain j . The equations (7) are the primary equations for electromagnetism in moving media. In most materials $\mathcal{M}$ is a function of the field F and this dependence is determined by the constitutive relations. Recently, they are discussed in detail in [5]. In that case (7) are well-defined equations for F. The constitutive relations from [5] with 4D geometric quantities, which correctly transform under the LT, are compared with Minkowski's constitutive relations with 3D vectors and several essential differences are pointed out. They are caused by the fact that, as in the case with the field equations that are investigated here, the UT of 3D vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$, $\mathbf{M}$, etc. are not the LT. Furthermore, in [5], the physical explanation is presented for the existence of the magnetoelectric effect in moving media that essentially differs from the traditional explanation.

If (7) is written with CBGQs in the standard basis it becomes

$$\partial_\alpha(\varepsilon_0 F^{\alpha\beta} + \mathcal{M}^{\alpha\beta})\gamma_\beta - \partial_\alpha(\varepsilon_0 \, {}^*F^{\alpha\beta})\gamma_5\gamma_\beta = c^{-1} j^{(C)\beta}\gamma_\beta, \tag{8}$$

which can be separated into two equations, the equation with sources

$$\partial_\alpha(\varepsilon_0 F^{\alpha\beta} + \mathcal{M}^{\alpha\beta})\gamma_\beta = c^{-1} j^{(C)\beta}\gamma_\beta \tag{9}$$

and the equation without sources, which is the same as in the vacuum

$$\partial_\alpha\, {}^*F^{\alpha\beta}\gamma_5\gamma_\beta = 0. \tag{10}$$

Instead of dealing with the axiomatic formulation of electromagnetism for moving media that uses only the *local* form of the field equation (7) one can construct the equivalent integral form simply replacing F by $F + \mathcal{M}/\varepsilon_0$ in Eqs. (18), (21), (22) and also j by $j^{(C)}$ in (21), (22) in [2]. However, the integral form will not be investigated here.

Proceeding in the same way as in [2] one can derive from (7) the stress-energy vector $T(n)$ for a moving medium simply replacing F by $F + \mathcal{M}/\varepsilon_0$ in Eqs. (26), (37-47) in [2]. For example, Eq. (26) in [2] becomes

$$T(n) = T(n(x), x) = -(\varepsilon_0/2)\left\langle (F + \mathcal{M}/\varepsilon_0) n (F + \mathcal{M}/\varepsilon_0) \right\rangle_1 . \tag{11}$$

$T(n)$ is a vector-valued linear function on the tangent space at each spacetime point x describing the flow of energy-momentum through a hypersurface with unit normal vector $n = n(x)$. The expression for $T(n)$, $T(n) = Un + (1/c)S$, Eq. (41) in [2], will remain unchanged, but the energy density U and the Poynting vector S will change according to the described replacement. All this with $T(n)$ will not be discussed in this paper, but in a separate paper in which the Abraham-Minkowski controversy will also be examined in a new way.

Another form of the field equation with sources (9) is the "source representation"

$$\partial \cdot \varepsilon_0 F = j^{(C)}/c - \partial \cdot \mathcal{M}, \tag{12}$$

i.e., with the CBGQs in the $\{\gamma_\mu\}$ basis

$$\partial_\alpha(\varepsilon_0 F^{\alpha\beta})\gamma_\beta = (c^{-1} j^{(C)\beta} - \partial_\alpha \mathcal{M}^{\alpha\beta})\gamma_\beta \tag{13}$$

according to which the sources of the fundamental electromagnetic field F are the true currents $j^{(C)}$ and the magnetization-polarization current density $\partial \cdot \mathcal{M}$, i.e., the space-time changes of the generalized magnetization-polarization bivector $\mathcal{M}$.

In previous formulations of electromagnetism in media (at rest, or moving), starting with Minkowski (his f_{hk}), [1], the electromagnetic excitation tensor is introduced, see, e.g., a modern textbook on classical electromagnetism, [6], or the papers [7-9] and references therein, in the recent - Annalen der Physik, Special Topic Issue 9-10/2008: The Minkowski spacetime of special relativity - 100 years after its discovery. Here, in (7), $\mathcal{H}$ can be introduced as

$$\mathcal{H} = \varepsilon_0 F + \mathcal{M}. \tag{14}$$

However, it is worth noting that (14) is in some sense unsatisfactory, since physically different kind of entities are mixed in it; an electromagnetic field F and a matter field $\mathcal{M}$, i.e., the magnetization-polarization bivector. Moreover, as will be seen in the next section, in general, two different velocity vectors, v - the velocity of the observers and u - the velocity of the moving medium, enter into the decompositions of F and $\mathcal{M}$, Eqs. (17) and (21), respectively. This fact causes that *the usual decomposition of $\mathcal{H}$ into the electric and magnetic excitations, Eq.* (25), *is not possible in the general case but only in the case if* $u = v$, or if both decompositions (17) and (21) are made with the same velocity vector, either u or v. In that case $\mathcal{H}$ can be introduced in (7) and the usual form of the field equations in moving media is obtained

$$\partial \cdot \mathcal{H} = j^{(C)}/c, \quad \partial \wedge F = 0, \tag{15}$$

i.e., with CBGQs in the $\{\gamma_\mu\}$ basis,

$$\partial_\alpha \mathcal{H}^{\alpha\beta}\gamma_\beta = c^{-1} j^{(C)\beta}\gamma_\beta, \quad \partial_\alpha\, {}^*F^{\alpha\beta}\gamma_5\gamma_\beta = 0. \tag{16}$$

3. The basic field equation for moving media in terms of E, B and P, M

In this paper instead of using (14), (15) and (16) we deal with (7), i.e., with (8), as the basic field equations. In that equation bivectors F and $\mathcal{M}$ can be decomposed. First, the decomposition of F is considered. There is *a mathematical theorem according to which any antisymmetric tensor of the second rank can be decomposed into two space-like vectors and the unit time-like vector.* For the proof of that theorem in geometric terms see, e.g., [10]. When applied to the bivector F, e.g., Eq. (13) in [2], this yields

$$F = E \wedge v/c + (IcB) \cdot v/c, \tag{17}$$

where the electric and magnetic fields are represented by vectors $E(x)$ and $B(x)$. The unit pseudoscalar I is defined algebraically without introducing any reference frame as in Sec. 1.2 in [4] (Hestenes D and Sobczyk G). We choose I in such a way that when I is represented in the $\{\gamma_\mu\}$ basis it becomes $I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_5$. With such choice for I, $\{\gamma_1, \gamma_2, \gamma_3\}$ form a right-handed orthonormal set, as usual for a 3D Cartesian frame. The LT (boosts) do not change the orientation for spacetime. Here, in the whole paper, under the name LT we shall only consider - boosts.

If (17) is written with CBGQs in the $\{\gamma_\mu\}$ basis it becomes

$$F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \; F^{\mu\nu} = (1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta} v_\alpha B_\beta, \tag{18}$$

where $\gamma_\mu \wedge \gamma_\nu$ is the bivector basis. In the same way as for any other CBGQ it holds that bivector F *is the same 4D quantity* for relatively moving inertial observers and for all bases chosen by them, $F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)F'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu = \ldots$.

Minkowski, Sec. 11.6, [1], see also [11], was the first who introduced vectors (4-vectors in the usual notation) of the electric and magnetic fields and the velocity vector, Φ, Ψ and w, respectively, in his notation, and presented the decomposition of F, his equation (55), that corresponds to (17). Note that he considered that w, Φ and Ψ are 1×4 matrices and F is a 4×4 matrix. Thus he worked with components of the geometric quantities taken in the standard basis $\{\gamma_\mu\}$.

The vector v in the decomposition (17) is interpreted as the velocity vector of the observers who measure E and B fields. Then $E(x)$ and $B(x)$ are defined with respect to v, i.e., with respect to the observer, as

$$E = F \cdot v/c, \quad B = -(1/c)I(F \wedge v/c). \tag{19}$$

It also holds that $E \cdot v = B \cdot v = 0$; both E and B are space-like vectors. It is visible from (19) that *E and B depend not only on F but on v as well.*

If (19) is written with CBGQs in the $\{\gamma_\mu\}$ basis it becomes

$$E = E^\mu\gamma_\mu = (1/c)F^{\mu\nu}v_\nu\gamma_\mu, \quad B = B^\mu\gamma_\mu = (1/2c^2)\varepsilon^{\mu\nu\alpha\beta}F_{\nu\alpha}v_\beta\gamma_\mu. \tag{20}$$

As F is antisymmetric it holds that $E^\mu v_\mu = B^\mu v_\mu = 0$. Only three components of E and B in any basis are independent. However, as E and B depend not only on F but on v as well this result does not mean that three spatial components of E, or B, are necessarily independent components. The form of v in a given inertial frame will determine which three components are independent. The relations (17) - (20) are mathematically correct definitions.

Similarly, using the same theorem, the bivector $\mathcal{M}(x)$ can be decomposed into two space-like vectors, the polarization vector $P(x)$ and the magnetization vector $M(x)$ and the unit time-like vector u/c

$$\mathcal{M} = P \wedge u/c + (MI) \cdot u/c^2. \tag{21}$$

There is the rest frame for a medium, i.e., for $\mathcal{M}$, or P and M , and therefore the vector u in the decomposition (21) may be identified with bulk velocity vector of the medium in spacetime. Integral curves of u define the averaged world-lines of identifiable constituents of the medium. If (21) is written with CBGQs in the $\{\gamma_\mu\}$ basis it becomes

$$\mathcal{M} = (1/2)\mathcal{M}^{\mu\nu}\gamma_\mu \wedge \gamma_\nu,\ \mathcal{M}^{\mu\nu} = (1/c)(P^\mu u^\nu - P^\nu u^\mu) + (1/c^2)\varepsilon^{\mu\nu\alpha\beta}M_\alpha u_\beta. \tag{22}$$

The vectors $P(x)$ and $M(x)$ are determined by $\mathcal{M}(x)$ and the unit time-like vector u/c as

$$P = \mathcal{M} \cdot u/c, \quad M = cI(\mathcal{M} \wedge u/c) \tag{23}$$

and it holds that $P \cdot u = M \cdot u = 0$. As in the case with F, it can be seen from (23) that P and M depend not only on $\mathcal{M}$ but on u as well. $P(x)$ and $M(x)$ from (23) can be written as CBGQs in the $\{\gamma_\mu\}$ basis

$$P = (1/c)\mathcal{M}^{\mu\nu}u_\nu\gamma_\mu, \quad M = (1/2)\varepsilon^{\mu\nu\alpha\beta}\mathcal{M}_{\alpha\nu}u_\beta\gamma_\mu, \tag{24}$$

with $P^\mu u_\mu = M^\mu u_\mu = 0$.

Usually, only the velocity vector u of the moving medium is taken into account, or the case $u = v$ is considered, i.e., it is supposed that the observer frame is comoving with medium, or both decompositions (17), i.e., (18), and (21), i.e., (22), are made with the same velocity vector, either u or v.

Such assumptions enable the introduction of the electromagnetic excitation bivector $\mathcal{H}$, Eq. (14), and, by using (17) and (21), one finds the decomposition of $\mathcal{H}$ into the electric and magnetic excitations (other names of which are “electric displacement” and “magnetic field intensity”)

$$\mathcal{H} = D \wedge u/c + (IH) \cdot u/c^2, \tag{25}$$

where, as usual, the electric displacement vector $D = \varepsilon_0 E + P$ and the magnetic field intensity vector $H = (1/\mu_0)B - M$ are introduced. The bivector $\mathcal{H}$ in (25) can be written as a CBGQ in the $\{\gamma_\mu\}$ basis

$$\mathcal{H} = (1/2)\mathcal{H}^{\mu\nu}\gamma_\mu \wedge \gamma_\nu,\ \mathcal{H}^{\mu\nu} = (1/c)(D^\mu u^\nu - D^\nu u^\mu) + (1/c^2)\varepsilon^{\mu\nu\alpha\beta}u_\alpha H_\beta.$$

The decomposition (25) was first introduced by Minkowski, Eq. (56) Sec. 11.6, [1]. Notice that Minkowski dealt only with bulk velocity vector of the medium u; in both his Eqs. (55) (our Eq. (17) but with $v = u$) and (56) (our Eq. (25)) the vector w (our u) appears. The same treatment with the decomposition of $\mathcal{H}$ and consequently with only one velocity, the velocity u, is used in several textbooks, e.g., [12], and papers, e.g., [13, 14]. However, in general, $u \neq v$, e.g., the observers are at rest in the laboratory frame ($v = c\gamma_0$) in which the considered medium is moving with velocity u ($u \neq c\gamma_0$). Therefore, we continue with an alternative approach which deals with two different velocity vectors v and u, i.e., with Eqs. (17) and (21).

Inserting (17) and (21) into the field equation (7) one gets the general form of the field equation for a magnetized and polarized moving medium expressed in terms of $E(x)$, $B(x)$, $P(x)$ and $M(x)$

$$\partial\{\varepsilon_0[E \wedge v/c + (IB) \cdot v] + [P \wedge u/c + (1/c^2)(MI) \cdot u]\} = j^{(C)}/c. \tag{26}$$

In the same way as in (7), Eq. (26) with the geometric product can be divided into the vector part (with sources)

$$\partial \cdot \{\varepsilon_0[E \wedge v/c + (IB) \cdot v] + [P \wedge u/c + (1/c^2)(MI) \cdot u]\} = j^{(C)}/c \tag{27}$$

and the trivector part (without sources)

$$\partial \wedge [E \wedge v/c + (IB) \cdot v] = 0. \tag{28}$$

The field equation without sources (28) remains unchanged relative to the corresponding equation for vacuum, because the same assertion holds for the trivector part of (7). We call (26), i.e., (27) and (28), the field equations in the Ampèrian form, in analogy with Maxwell's equations when they are written in terms of the 3D vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{M}$; for the latter ones and the name see, e.g., Eqs. (4.5) in [15]. Observe that in (26), i.e., (27), there are two different velocities u and v. The equation (26) is a fundamental result, which is not previously reported in the physics literature, as I am aware.

If the geometric product is used then there is *only one equation for electromagnetism in moving media, Eq.* (7), i.e., in the Ampèrian form Eq. (26). They are written with abstract 4D geometric quantities and *they comprise and generalize all usual Maxwell's equations (with 3D vectors) for moving media.*

If Eq. (27) is written in the $\{\gamma_\mu\}$ basis, it becomes

$$\partial_\alpha\{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu}E^\mu v^\nu + c\varepsilon^{\alpha\beta\mu\nu}v_\mu B_\nu] + [\delta^{\alpha\beta}{}_{\mu\nu}P^\mu u^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu}M_\mu u_\nu]\}\gamma_\beta = j^{(C)\beta}\gamma_\beta, \tag{29}$$

where $\delta^{\alpha\beta}{}_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu$. Similarly, in the $\{\gamma_\mu\}$ basis, (28) becomes

$$\partial_\alpha(c\delta^{\alpha\beta}{}_{\mu\nu}B^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu}E_\mu v_\nu)\gamma_5\gamma_\beta = 0. \tag{30}$$

Again, as for (28), Eq. (30) is the same as in vacuum. In (29), as in (26), i.e., (27), there are two different velocities u and v. The equation (29) does not appear in the entire previous literature. The equation (26), i.e., (27) and (28) and also (29) and (30) are the fundamental results that are obtained in this paper and they enable an alternative, but viable, treatment of electromagnetism of moving media.

The equation (27) can be written in another form, i.e., in the "source representation" as with F and $\mathcal{M}$, Eq. (12),

$$\partial \cdot \{\varepsilon_0[E \wedge v/c + (IB) \cdot v]\} = j^{(C)}/c - \partial \cdot [P \wedge u/c + (1/c^2)(MI) \cdot u], \tag{31}$$

according to which the sources of E and B fields are the true current density $j^{(C)}$ and the P and M vectors.

If the abstract quantities, e.g., $j^{(C)}$, ∂, E , etc. in (31) are replaced by their representations, i.e., CBGQs in the standard basis, $j^{(C)} = j^{(C)\beta}\gamma_\beta$, $\partial = \gamma^\beta\partial_\beta$, $E = E^\mu\gamma_\mu$, ..., then (31) becomes

$$\begin{aligned} &\partial_\alpha\{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu}E^\mu v^\nu + c\varepsilon^{\alpha\beta\mu\nu}v_\mu B_\nu]\}\gamma_\beta \\ = \; &\{j^{(C)\beta} - \partial_\alpha[\delta^{\alpha\beta}{}_{\mu\nu}P^\mu u^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu}M_\mu u_\nu]\}\gamma_\beta. \end{aligned} \tag{32}$$

From the equations (27) or (31), i.e., with CBGQs, (29) or (32), it is clear that *in 4D spacetime it is not possible to separate the field equation with sources for the E field from that one for the*

B field. Thus, *the usual Ampère-Maxwell law and Gauss's law are inseparably connected in one law - Eq.* (27) or *Eq.* (31), i.e., (29) or (32). Similarly, *in* (28), i.e., Eq. (30), *Faraday's law and the law that expresses the absence of magnetic charge are also inseparably connected in one law.* This is an essential difference relative to Maxwell's equations with 3D vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{M}$. Of course, the same statement holds for the original Eq. (3), i.e., for the vacuum as well.

The mathematical reason for such an inseparability is that, e.g., the gradient operator ∂ is a vector field defined on 4D spacetime. If represented in some basis then its vector character remains unchanged only when *all its components together with associated basis vectors* are taken into account in the considered equation. The same holds for other vectors E, B, j, P, etc. and multivectors like F, $\mathcal{M}$, ... For example, in general, in 4D spacetime, the current density vector j is a well-defined physical quantity, but it is not the case with the usual charge density ρ and the usual current density $\mathbf{j}$ as a 3-vector. Similarly, in general, the gradient operator ∂ cannot be divided into the usual time derivation and the spatial derivations. In 4D spacetime, an independent physical reality is attributed to the position vector x, the gradient operator ∂, the current density vector j , the vectors of the electric and magnetic fields E and B, respectively, etc., but not to the 3-vector $\mathbf{r}$ and the time t, to 3D vectors $\mathbf{j}$, $\mathbf{E}$, $\mathbf{B}$, etc. Therefore, in 4D spacetime, it is not possible to speak about the static case in electromagnetism, i.e., about the electrostatics and magnetostatics.

An important consequence stems from the above mentioned inseparability of 4D spacetime into the 3D space and the time and therefore from the inseparability of Eq. (27), i.e. (31), into two laws, and similarly for Eq. (28). It can be seen from Maxwell's equations with 3D vectors, e.g., Eqs. (43) and (44), and also (46) and (47), which all are given below, that *in the static case* the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, respectively, are completely decoupled. However, as already stated, in 4D spacetime there is no static case. The equations (27), i.e., (31), and (28) reveal that *the vectors of the electric and magnetic fields E and B*, respectively, *are never decoupled.* This statement holds for the vacuum as well. Thus if, for example, we have a magnetization M (a permanent magnet) but with a negligible permanent polarization P and without $j^{(C)}$, then, as can be seen from (31), M will induce *both B and E*. Such a result is completely understandable because E and B are derived from *one* fundamental quantity, the electromagnetic field bivector F, by the decomposition of F (17) and by (19), and similarly P and M are derived from *one* quantity, the generalized magnetization-polarization bivector $\mathcal{M}$, by the decomposition of $\mathcal{M}$ (21) and by (23). The equations (7), i.e., their "source representation" (12), are the basic field equations with the bivectors F and $\mathcal{M}$; F unites E and B and $\mathcal{M}$ unites P and M. Besides, F is independent on v and $\mathcal{M}$ is independent on u. The formulation of electromagnetism of moving media could be done exclusively in terms of F and $\mathcal{M}$ in the same way as in [2] for vacuum.

It is worth mentioning that in the integral form the equation that corresponds to the local equation (27) can be obtained from Eq. (21) in [2] replacing F by $F + \mathcal{M}/\varepsilon_0$, j by $j^{(C)}$ and inserting into it the decompositions (17) and (21), and similarly for (28) and Eq. (18) in [2].

4. Observers are at rest in a stationary medium. Comparison with the usual formulation with 3D vectors

Recently, [16 - 21], [11], it is proved that, contrary to the general belief, the UT of the 3D $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, see, e.g., Eqs. (11.148) and (11.149) in [6], or Eq. (6) in [5], i.e., Eq. (33) here, *differ* from the LT (boosts) of the 4D E and B vector fields. Note that, in previous literature, starting with Einstein [3], the UT of $\mathbf{E}$ and $\mathbf{B}$ are always considered to be the relativistically correct LT. The same holds for the UT of $\mathbf{P}$ and $\mathbf{M}$, Eq. (35) below, and the LT of vectors P and M. For a recent more detailed review see Secs. 3.1 and 3.2 in [22]. The essential point is that in the UT *the transformed* $\mathbf{E}'$ *is expressed by the mixture of the 3D vectors* $\mathbf{E}$ *and* $\mathbf{B}$, Eq. (11.149) in [6], which is

$$\mathbf{E}' = \gamma(\mathbf{E} + \beta \times c\mathbf{B}) - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{E}), \tag{33}$$

and similarly for $\mathbf{B}'$. For the components implicitly taken in the standard basis it holds that

$$E_1' = E_1, \quad E_2' = \gamma(E_2 - \beta c B_3), \quad E_3' = \gamma(E_3 + \beta c B_2), \tag{34}$$

and similarly for the components of $\mathbf{B}'$, Eq. (11.148) in [6].

The same holds for the couple of the 3D vectors $\mathbf{P}$ and $\mathbf{M}$ and their UT

$$\begin{aligned} \mathbf{P} &= \gamma(\mathbf{P}' + \beta \times \mathbf{M}'/c) - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{P}'), \\ \mathbf{M} &= \gamma(\mathbf{M}' - \beta \times c\mathbf{P}') - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{M}'). \end{aligned} \tag{35}$$

These UT of $\mathbf{P}$ and $\mathbf{M}$, Eq. (35), are given, e.g., by Eq. (4.2) in [15].

On the other hand, in ISR, as shown in [16 - 21], [11], *the correct LT always transform the 4D algebraic object representing, e.g., the electric field only to the electric field; there is no mixing with the magnetic field.* These correct LT are given by, e.g., Eq. (8) in [5], i.e., Eq. (13) in [11]. The same happens with P and M. The LT of the components E^μ (in the $\{\gamma_\mu\}$ basis) of $E = E^\mu \gamma_\mu$ are given as

$$E'^0 = \gamma(E^0 - \beta E^1), \; E'^1 = \gamma(E^1 - \beta E^0), \; E'^{2,3} = E^{2,3}, \tag{36}$$

for a boost along the x^1 axis. As already mentioned, any CBGQ is unchanged under the LT, i.e., it holds that $E = E^\nu \gamma_\nu = E'^\nu \gamma_\nu' = E_r^\nu r_\nu = E_r'^\nu r_\nu'$, where the primed quantities in both bases $\{\gamma_\mu\}$ and $\{r_\mu\}$ are the Lorentz transforms of the unprimed ones. For the $\{r_\mu\}$ basis, with the "radio," "r"synchronization, and the LT in that basis, see [23] and [24]. The same LT hold for any other vector, e.g., x, B, P, M, etc.

A short derivation of the above LT is presented in [21]. Let us introduce the γ_0-frame in which the standard basis is chosen and $v = c\gamma_0$. Therefore, in the γ_0-frame, E from (19) becomes $E = F \cdot \gamma_0$. If, in the γ_0-frame, that E is written as a CBGQ, then the components are given as $E^0 = B^0 = 0$ and only the spatial components remain, $E^i = F^{i0}$, $B^i = (1/2c)\varepsilon^{ijk0} F_{jk}$; the same components as in, e.g., Eq. (11.137) in [6]. It is proved by Minkowski [1], and reinvented and generalized in [16 - 21], [11], see also Sec. 5 in [22], that in the mathematically correct procedure for the derivation of the LT of E *both* F *and* the velocity vector v have to be transformed by the LT, e.g., for the LT from the γ_0-frame;

$$E = E^\mu \gamma_\mu = [(1/c)F^{i0} v_0]\gamma_i = [(1/c)F'^{\mu\nu} v_\nu']\gamma_\mu' = E'^\mu \gamma_\mu'. \tag{37}$$

The velocity vector v transforms as any other vector and it holds that $v = v^0 \gamma_0 = c\gamma_0 = v'^\mu \gamma_\mu'$. Hence, *the components* E^μ *transform by the LT again to the components* E'^μ *of the same electric field vector*, i.e., the above quoted LT (36) of the components E'^μ are obtained. *The main point is that the transformed components* E'^μ *are not determined only by* $F'^{\mu\nu}$, *as in all usual approaches*, e.g., Eqs. (11.147) and (11.148) in [6], *but also by* v'^μ.

As already stated, Minkowski, section 11.6 in [1], was the first who derived these mathematically correct LT. He assumed that v, E and B are 1×4 matrices and F is a 4×4 matrix; their components are implicitly determined in the standard basis. He described how v and F separately transform under the LT A (the matrix of the LT is denoted as A)

$$v \longrightarrow v' = vA, \quad F \longrightarrow F' = A^{-1}FA. \tag{38}$$

Then, as shown by Minkowski, the mathematically correct LT of $E = vF$ is

$$E = vF \longrightarrow E' = (vA)(A^{-1}FA) = (vF)A = EA. \tag{39}$$

Thus, under the LT *both* quantities, the field-strength tensor F (4×4 matrix) *and the 4-velocity* v (1×4 matrix) are transformed and their product transforms as any 1×4 matrix transforms. It is already mentioned that this mathematically correct procedure is reinvented and generalized using 4D geometric quantities both in the tensor formalism and in the geometric algebra formalism in [16 - 21], [11].

The comparison with experiments in electromagnetism, the motional emf [17] and Secs. 7 - 7.2 here, the Faraday disk [18], and the Trouton-Noble experiment [2, 25], shows that the approach with 4D geometric quantities and their LT, i.e., the ISR, always agrees with the principle of relativity and it is in a true agreement (independent of the chosen inertial reference frame and of the chosen system of coordinates in it) with all experiments in electromagnetism. Also, it is shown in the mentioned papers that such a true agreement does not exist in the usual formulations of SR, e.g., [6, 15, 26], in which the electric and magnetic fields are represented by the 3D vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ that transform according to the UT. The same conclusion about the true agreement between the approach with 4D geometric quantities, the ISR, and the well-known experiments that test special relativity is obtained in [27]. There, in [27], and particularly in [23], it is explicitly shown that the relativity of simultaneity, the Lorentz contraction and the time dilation are not well-defined in 4D spacetime. They are not the intrinsic relativistic effects, because they depend on the chosen synchronization. But, *every synchronization is only a convention and physics must not depend on conventions. This true agreement of ISR with experiments directly proves the physical reality of 4D geometric quantities.*

Having briefly discussed the LT and the UT we go back to the discussion of Eqs. (29) (i.e., (32)) and (30). In all relatively moving inertial frames of reference and for any system of coordinates in them every term in the considered equations is always the same, because all CBGQs are the Lorentz invariant quantities, e.g., $j^{(C)\beta}\gamma_\beta = j'^{(C)\beta}\gamma'_\beta = j_r^{(C)\beta}\gamma_{r,\beta} = \ldots$. Observe that, in 4D spacetime, *only if all components, together with the associated basis vectors,* are taken into account in every term then all terms are invariant under the passive LT and thus Eqs. (29) ((32)) and (30) remain unchanged for different relatively moving inertial frames and for different systems of coordinates in them. Only in that case the physical quantities and the equations with them are correctly defined in 4D spacetime and the principle of relativity is naturally satisfied. This means that, in general, it is not allowed to consider separately some parts of 4D geometric quantities, or some parts of the equations with them, e.g., to take the part with γ_0 separately from those ones with γ_i in Eqs. (29) ((32)) and (30). Thus, for example, in 4D spacetime, only the whole current density $j^{(C)}$, the abstract vector from (27) and (31), or some of its representations, e.g., that one in the standard basis, the CBGQ, $j = j^{(C)\beta}\gamma_\beta$, is well-defined physical quantity, but not the charge density, $j^{(C)0}$ component, or the spatial components $j^{(C)i}$ taken alone. From the viewpoint of this 4D geometric approach the physical meaning of the charge density ρ is not well-defined. It is the temporal component j^0/c for one observer, but it transforms by the LT to the temporal component *and* the spatial component as well for the relatively moving observer. The same holds for the gradient operator ∂ and its CBGQ in the standard basis $\partial = \gamma^\beta \partial_\beta$ and for all other 4D geometric quantities.

This is particularly visible going to some nonstandard basis, like the $\{r_\mu\}$ basis, i.e. with the "radio," "r," synchronization, see, e.g., [23], Preprints in [27] and [24]. The "r" synchronization is commonly used in everyday life. If the observers who are at different distances from the studio clock set their clocks by the announcement from the studio then they have synchronized their clocks with the studio clock according to the "r" synchronization. The unit vectors in the $\{\gamma_\mu\}$ basis and the $\{r_\mu\}$ basis are connected as $r_0 = \gamma_0$, $r_i = \gamma_0 + \gamma_i$. The components of any vector are connected in the same way as the components of the position vector x are connected, $x_r^0 = x^0 - x^1 - x^2 - x^3$, $x_r^i = x^i$, e.g. for the components of vector E it also holds that $E_r^0 = E^0 - E^1 - E^2 - E^3$, $E_r^i = E^i$. The inverse relations are $\gamma_0 = r_0$, $\gamma_i = r_i - r_0$ and, e.g., for the components of the current density vector j, $j^0 = j_r^0 + j_r^1 + j_r^2 + j_r^3$, $j^i = j_r^i$. Thus, even in the same frame, the charge density in the $\{\gamma_\mu\}$ basis ($j^0 = c\rho$) loses its usual meaning; it is expressed by the sum of all components in the $\{r_\mu\}$ basis. However, observe that, as already stated, $j = j^\mu \gamma_\mu = j_r^\mu r_\mu$ and the same holds for $E = E^\mu \gamma_\mu = E_r^\mu r_\mu$, for B, for x, etc. This reveals that in the $\{r_\mu\}$ basis the space and time cannot be separated. Hence, in 4D spacetime the usual interpretations of the physical quantities, e.g., the charge density ρ and the current density as a 3-vector $\mathbf{j}$, are inappropriate.

An independent physical reality can be attributed either to the abstract geometric quantities, e.g., *vectors x, E, B, P, M , j, .. bivectors F, $\mathcal{M}$, .., or to their representations in different bases, the CBGQs, like $j^\mu \gamma_\mu$, $E_r^\mu r_\mu$, $M'^\beta \gamma'_\beta$,* etc.

Now, let us consider Eq. (29) in the case when $u = v$, i.e., the observer frame is comoving with medium. In that case it can be taken that $v = u = c\gamma_0$ ($u^\mu = v^\mu = (c, 0, 0, 0)$), i.e., that the observers who measure fields are at rest in a stationary medium. Then, Eq. (29) becomes

$$\begin{aligned} \partial_\alpha \{\varepsilon_0 [\delta^{\alpha\beta}{}_{\mu\nu} E^\mu (\gamma_0)^\nu + c\varepsilon^{\alpha\beta\mu\nu} (\gamma_0)_\mu B_\nu] & + \\ [\delta^{\alpha\beta}{}_{\mu\nu} P^\mu (\gamma_0)^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu} M_\mu (\gamma_0)_\nu]\}\gamma_\beta & = c^{-1} j^{(C)\beta} \gamma_\beta, \end{aligned} \tag{40}$$

The equation (40) can be also written in the "source representation" as

$$\begin{aligned} & \partial_\alpha \{\varepsilon_0 [\delta^{\alpha\beta}{}_{\mu\nu} E^\mu (\gamma_0)^\nu + c\varepsilon^{\alpha\beta\mu\nu} (\gamma_0)_\mu B_\nu]\}\gamma_\beta \\ = & \{c^{-1} j^{(C)\beta} - \partial_\alpha [\delta^{\alpha\beta}{}_{\mu\nu} P^\mu (\gamma_0)^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu} M_\mu (\gamma_0)_\nu]\}\gamma_\beta. \end{aligned} \tag{41}$$

As already mentioned, the sources of both fields together, E and B, are the true current density $j^{(C)}$ and the polarization and magnetization vectors, P and M respectively.

The observer frame is the γ_0-frame, $v = c\gamma_0$, which, with (19), yields that $E^0 = B^0 = 0$ and $E^i = F^{i0}$, $B^i = (1/2c)\varepsilon^{ijk0} F_{jk}$. Furthermore, in the considered case, the γ_0-frame coincides with the rest frame of the medium. Hence, in that frame and with (23), it also holds that $P^0 = M^0 = 0$, $P^i = \mathcal{M}^{i0}$, $M^i = (c/2)\varepsilon^{0ijk} \mathcal{M}_{jk}$. Then, Eq. (40) becomes

$$\begin{aligned} [\partial_k (E^k + P^k/\varepsilon_0) - j^{(C)0}/c\varepsilon_0]\gamma_0 + & \{c\varepsilon^{ijk0} \partial_j [(B_k - \mu_0 M_k] \\ -j^{(C)i}/c\varepsilon_0 - \partial_0 (E^i + P^i/\varepsilon_0)\}\gamma_i & = 0 \end{aligned} \tag{42}$$

In vacuum, Eq. (42) coincides with the first two terms, i.e., the terms with γ_0 and γ_i, in Eq. (8) in [18]. In the approach with 4D geometric quantities, i.e., in the ISR, it is not possible to make any further simplification. *In 4D spacetime, only the whole Eq.* (42) *is physically meaningful and there is no physical sense in some parts of it, for example, to take the part with γ_0 separately* *from those ones with* γ_i. Note that in the approach with 3D vectors there is not any Maxwell's equation that corresponds to Eq. (42).

Let us write Eq. (42) as $a^\alpha\gamma_\alpha = 0$, in which, as can be easily recognized, the coefficients a^α correspond to the usual Maxwell's equations in the component form. There are *two Maxwell equations* in the *component form*; the coefficient a^0 corresponds to the *component form* of the Gauss law for the electric field and the coefficients a^i correspond to the Ampère-Maxwell law in the *component form*. In [18], for the first time, a fundamental discovery is achieved that the usual Maxwell's equations with 3D vectors (for vacuum) are not covariant under the LT. In Sec. 2.3 in [18], the active LT (Eq. (16) in [18]) are applied to Eq. (8) in [18] (in vacuum, as already stated, our Eq. (42) corresponds to the first two terms of Eq. (8) from [18]). There, in that section, it is obtained that the coefficient a^0, which corresponds to the *component form* of the Gauss law for the electric field, does not transform by the LT again to the Gauss law but to a'^0, $a'^0 = \gamma a^0 - \beta\gamma a^1$, which is a combination of the Gauss law and a part of the Ampère-Maxwell law (a^1). (In our case, for the material medium, $a^0 = \partial_k(E^k + P^k/\varepsilon_0) - j^{(C)0}/c\varepsilon_0$.) If the Lorentz transformed Eq. (42), similarly as in Eqs. (21)-(24) in [18], is expressed in terms of Lorentz transformed derivatives and Lorentz transformed vectors E, B, P, M and j, then we find the same equation for a'^0 as it is Eq. (24) in [18], $a'^0 = \{[\gamma(\partial'_k E'^k) - j'^0/c\varepsilon_0] + \beta\gamma[\partial'_1 E'^0 + c(\partial'_2 B'_3 - \partial'_3 B'_2)]\}$, but E'^α has to be replaced by $E'^\alpha + P'^\alpha/\varepsilon_0$, B'_k by $B'_k - \mu_0 M'_k$, and j'^0 by $j'^{(C)0}$. The same discussion holds here as it is presented after Eq. (24) in [18]. From that discussion, and the above mentioned replacements, one concludes that the LT do not transform the Gauss law into the "primed" Gauss law but into quite different law; a'^0 contains the time component $E'^0 + P'^0/\varepsilon_0$, whereas $E^0 = P^0 = 0$, and also the new "Gauss law" includes the derivatives of the magnetic field. The same situation happens with other Lorentz transformed terms, which again explicitly shows that *the Lorentz transformed Maxwell's equations are not of the same form as the original ones.* Hence, contrary to all previous considerations, and contrary to the general opinion, *the usual Maxwell's equations are not Lorentz covariant equations either in vacuum or in a material medium.* This result proves in another way that *in 4D spacetime* only the whole Eq. (42) is physically meaningful and not its separate parts. Remember that Eq. (42) is derived from Eq. (40), i.e. from (29), for which it also holds that $a^\beta\gamma_\beta = a'^\beta\gamma'_\beta = a_r^\beta\gamma_{r,\beta} = \ldots$. Here, as before, the primed quantities are the Lorentz transforms of the unprimed ones in the $\{\gamma_\mu\}$ basis, whereas the last expression refers to the $\{r_\mu\}$ basis with the "r" synchronization.

Let us see how Eq. (42) could be compared with the usual form of Maxwell's equations for stationary media, which deals with 3D vectors, e.g., [15, 26, 28, 29], etc. Obviously, the comparison will be possible *only* if the term with γ_0 is considered separately from those ones with γ_i and if in Eq. (42) *only the components are taken into account.* But, as explained above, *from the viewpoint of the geometric approach,* i.e., *the ISR, such a procedure is not correct in 4D spacetime.*

If in Eq. (42), i.e., in $a^0\gamma_0 + a^i\gamma_i = 0$, one takes that $a^0 = 0$, and multiply the spatial components of E, P and $j^{(C)}$ from a^0 by *the unit 3D vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, then the term with γ_0 will become the equation

$$\nabla \cdot \varepsilon_0 \mathbf{E}(\mathbf{r}, t) = \rho^{(C)}(\mathbf{r}, t) - \nabla \cdot \mathbf{P}(\mathbf{r}, t), \tag{43}$$

what is Eq. (4.5) (3) in [15], or Eq. (9-6)(1) in [28], or Eq. (4.139) in [29], etc. In the same way it will be obtained that the terms with γ_i will become the equation

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0[\mathbf{j}^{(C)}(\mathbf{r}, t) + \partial(\varepsilon_0\mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r},t))/\partial t + \nabla \times \mathbf{M}(\mathbf{r}, t)], \tag{44}$$

what is Eq. (4.5) (2) in [15], or Eq. (9-6)(4) in [28], or Eq. (4.142) in [29], etc. The way in which the equations with 3D vectors (43) and (44) are constructed clearly shows that Eq. (42) is essentially different than Eqs. (43) and (44).

Similarly, we find that in the γ_0-frame Eq. (30) becomes

$$(c^2\partial_k B^k)\gamma_5\gamma_0 - (c\partial_0 B^i + \varepsilon^{ijk0}\partial_j E_k)\gamma_5\gamma_i = 0. \tag{45}$$

The equation (45) coincides, without any changes, with the last two terms, i.e., the terms with $\gamma_5\gamma_0$ and $\gamma_5\gamma_i$, in the equation for vacuum, Eq. (8) in [18]. As already stated, in (45), Faraday's law and the law that expresses the absence of magnetic charge are *inseparably* connected in one law. But, as in the discussion of Eq. (42), we can make the comparison of Eq. (45) with the usual form of Maxwell's equations for stationary media, which deals with 3D vectors. The equation with 3D vectors that expresses the absence of magnetic charge

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{46}$$

can be constructed from the term with $\gamma_5\gamma_0$ in (45) in the same way as Eqs. (43) and (44) are constructed from Eq. (42). The obtained Eq. (46) is Eq. (4.5) (4) in [15], or Eq. (9-6)(2) in [28], or Eq. (4.140) in [29], or Eq. (7.55)(ii) in [26], etc. Similarly, the terms with $\gamma_5\gamma_i$ will give the equation with 3D vectors, Faraday's law,

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial \mathbf{B}(\mathbf{r}, t)/\partial t, \tag{47}$$

what is Eq. (4.5) (1) in [15], or Eq. (9-6)(3) in [28], or Eq. (4.141) in [29], or Eq. (7.55)(iii) in [26], etc. (The components of vectors E, B, P, M with superscripts (E^i, B^i, P^i, M^i) from (42) are identified with the components of the usual 3D vectors and $\varepsilon^{0123} = 1$.) As already mentioned, Maxwell's equations in terms of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{M}$, Eqs. (4.5) in [15], i.e., (43), (44), (46) and (47) here, are said to be in the Ampèrian form.

If, as usual, the electric displacement 3D vector $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ has been introduced together with the magnetic field intensity 3D vector $\mathbf{H} = (1/\mu_0)\mathbf{B} - \mathbf{M}$ then Eqs. (43) and (44) become

$$\begin{aligned} \nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho^{(C)}(\mathbf{r}, t), \\ \nabla \times \mathbf{H}(\mathbf{r}, t) &= \mathbf{j}^{(C)}(\mathbf{r}, t) + \partial \mathbf{D}/\partial t. \end{aligned} \tag{48}$$

In (48), the first equation is Eq. (9-7)(1) in [28], or Eq. (4.139a) in [29], or Eq. (7.55)(i) in [26], etc., whereas the second one is Eq. (9-7)(4) in [28], or Eq. (4.142a) in [29], or Eq. (7.55)(iv) in [26], etc.

According to this discussion there is an essential difference between Maxwell's equations (43), (44), or (48), with the 3D vectors and Eq. (42), i.e., the equations from which (42) is derived, (40), (29) and (27). In the 4D geometric approach, i.e., in the ISR, there is *one law*, Eq. (42), i.e., (40), or (29), or (27), whereas in the approach with 3D vectors there are *two laws*, Eqs. (43) and (44), or (48). In order to obtain two laws (43), (44), or (48), from Eq. (42) we had to make several steps. First, the term with γ_0 is taken separately from those ones with γ_i, then only the components in these terms are taken into account and finally the components are multiplied by the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. But, in 4D spacetime, as explained above, these steps are not mathematically correct. This consideration clearly shows that Eq. (42), i.e., Eqs. (40), or (29), or (27), from which (42) is derived, is not equivalent to Eqs. (43), (44), or (48). The equation (42) is more general and, strictly speaking, it is not possible to obtain Eqs. (43), (44), or (48) from (42) by a mathematically correct procedure in 4D spacetime. The same consideration holds

IOP Conf. Series: Journal of Physics: Conf. Series **845** (2017) 012013 doi:10.1088/1742-6596/845/1/012013

in the same measure for the relation between (45) and the equations with 3D vectors (46) and (47).

Furthermore, in the usual approach with 3D vectors one can speak about the static case. Then, Eqs. (43) and (44), or the first and the second equation in (48), are completely decoupled, i.e., in the static case the electric and magnetic fields as 3D vectors are decoupled. In the 4D geometric approach such a decoupling is never possible, because there is only one law in which there are both together E and B as vectors. The same consideration holds for Maxwell's equations (46) and (47) with 3D vectors and Eq. (45), i.e., the equations from which (45) is derived, (30) and (28).

The most important difference is the following. The quantities entering into (42) and (45) are representations in the standard basis of the abstract 4D quantities from Eq. (26), i.e., Eqs. (27) and (28). All these quantities are correctly defined in 4D spacetime and they correctly transform under the LT, e.g., for the components (36), or for the vector E, Eq. (8) in [5], i.e., Eq. (13) in [11], whereas it is not the case with quantities appearing in (43),(44), (46), (47) and (48), which transform according to the UT, e.g., (33), of the 3D vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$.

5. Observers are at rest in the laboratory frame, but material medium is moving. Comparison with the usual formulation with 3-vectors

Let us examine Eqs. (29) and (30) in the case of a moving material medium, but the observers are at rest in the laboratory frame, which will be denoted as the S frame. Then, in S, $v = c\gamma_0$, $v^\mu = (c, 0, 0, 0))$. Thus the laboratory frame is the γ_0 -frame in which it holds that $E^0 = B^0 = 0$ and $E^i = F^{i0}$, $B^i = (1/2c)\varepsilon^{ijk0}F_{jk}$. Obviously, Eq. (30) becomes the same as Eq. (45), i.e., the same as in vacuum and the whole discussion about the comparison of (45) with Maxwell's equations with 3D vectors remains unchanged. But, it is not so for Eq. (29).

In the γ_0-frame the considered medium is moving with velocity u , $u \neq c\gamma_0$, i.e., some of u^i are $\neq 0$. The rest frame of the medium will be denoted as the S' frame. For the sake of comparison with the usual formulation we present the considered equation in an expanded form in which the term with γ_0 and the terms with γ_i are explicitly written. Then, in the laboratory frame, which is the γ_0-frame, Eq. (29) becomes

$$\begin{aligned}
\{\partial_k[\varepsilon_0 E^k + c^{-1}(P^k u^0 - P^0 u^k) + c^{-2}\varepsilon^{kij0}M_i u_j] - c^{-1}j^{(C)0}\}\gamma_0 &\quad + \\
\{-c^{-1}j^{(C)i} + (c\mu_0)^{-1}\varepsilon^{ijk0}\partial_j B_k - \varepsilon_0\partial_0 E^i &\quad + \\
c^{-1}\partial_\mu(P^\mu u^i - P^i u^\mu) - c^{-2}\varepsilon^{i\mu\alpha\beta}\partial_\mu M_\alpha u_\beta]\}\gamma_i &= 0
\end{aligned} \tag{49}$$

Again, as in the discussion of Eq. (42), it can be argued that *in 4D spacetime, only the whole Eq.* (49) *is physically meaningful and there is no physical sense in some parts of it, for example, to take the part with* γ_0 *separately from those ones with* γ_i. Observe that in (49) there are terms with P^0 and M^0, which cannot exist in the usual formulation with 3D vectors.

What will be obtained from (49) for the case of low velocities of the medium, i.e., for $\beta_u \ll 1$, $\gamma_u = (1-\beta_u^2)^{-1/2} \simeq 1$, where, in S and in the $\{\gamma_\mu\}$ basis, $u = u^\nu\gamma_\nu$, $u^\nu = (\gamma_u c, \gamma_u U^1, \gamma_u U^2, \gamma_u U^3)$, U^k are the same as the components of the 3-velocity $\mathbf{U}$ and $\beta_u = |\mathbf{U}|/c$. To determine and compare P^0 and P^k in S *we use the LT of* P'^μ *from* S', the rest frame of the medium, and, for simplicity, it is taken that the medium, the S' frame, is moving along the common $+x^1$, x'^1 axes, i.e., $u^\nu = (\gamma_u c, \gamma_u U^1, 0, 0)$. In S' , $P'^\mu = (0, P'^1, P'^2, P'^3)$. Then, using the LT, the same as in Eq. (36), $P^\mu = (\beta_u\gamma_u P'^1, \gamma_u P'^1, P'^2, P'^3)$. Since $\beta_u \ll 1$ and $\gamma_u \simeq 1$ it follows that $P^0 \ll P^1$

and $P^k u^0 - P^0 u^k$ in (49) becomes $\simeq cP^k$, i.e., in that approximation $P^0 u^k$ can be neglected relative to $P^k u^0$. In the same way it can be concluded that $M^0 u^k$ can be neglected relative to $M^k u^0$. Therefore, with these approximations, Eq. (49) can be written as

$$\begin{aligned}\partial_k \varepsilon_0 E^k \gamma_0 + (c\mu_0)^{-1} \varepsilon^{ijk0} \partial_j B_k \gamma_i \simeq [c^{-1} j^{(C)0} - \partial_k P^k - \\ c^{-2} \varepsilon^{kij0} \partial_k M_i U_j] \gamma_0 + [c^{-1} j^{(C)i} + \partial_0 (\varepsilon_0 E^i + P^i) \quad + \\ c^{-1}((U^k \partial_k) P^i - U^i (\partial_k P^k)) + c^{-2} \varepsilon^{ijk0} (\partial_0 M_j U_k + c \partial_j M_k] \gamma_i, \end{aligned} \tag{50}$$

Notice that (50) *is obtained from* (49) *using the LT of the vectors* $P^\mu \gamma_\mu$ and $M^\mu \gamma_\mu$ *and not the UT of the 3D vectors* $\mathbf{P}$ and $\mathbf{M}$, Eq. (35). We see that $D^k = \varepsilon_0 E^k + P^k$ *and* $H^k = B^k/\mu_0 - M^k$ *can be introduced into Eq.* (50), *whereas such replacement is not possible for Eq.* (49), *due to the existence of the terms with* P^0 *and* M^0 *in* (49). Then the part with γ_0 in (50) becomes

$$\partial_k D^k \gamma_0 = [c^{-1} j^{(C)0} - c^{-2} \varepsilon^{kij0} \partial_k M_i U_j] \gamma_0, \tag{51}$$

whereas the part with γ_i is

$$\varepsilon^{ijk0} \partial_j H_k \gamma_i = [j^{(C)i} + c\partial_0 D^i + ((U^k \partial_k) P^i - U^i (\partial_k P^k)) + c^{-1} \varepsilon^{ijk0} \partial_0 M_j U_k] \gamma_i. \tag{52}$$

The equation (50), i.e., Eqs. (51) and (52), can be compared with the usual form of Maxwell's equations for moving media, which deal with 3D vectors, e.g., [15, 28, 29], etc. Again, as in Sec. 4, the comparison will be possible *only* if the term with γ_0 is considered separately from those ones with γ_i, as in (42) and (45) and if in Eq. (50) *only the components are taken into account.* As before, we argue that *from the viewpoint of the geometric approach,* i.e., *the ISR, such a procedure is not correct in 4D spacetime.* If, as in Sec. 4, in Eq. (50), i.e., in $a^0 \gamma_0 + a^i \gamma_i = 0$, one takes that $a^0 = 0$, and multiply the spatial components of E, P, M and $j^{(C)}$ from a^0 by *the unit 3D vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ then the term with γ_0 will become the equation

$$\nabla \cdot \varepsilon_0 \mathbf{E}(\mathbf{r}, t) = \rho^{(C)}(\mathbf{r}, t) - \nabla \cdot [\mathbf{P}(\mathbf{r}, t) - c^{-2} (\mathbf{M}(\mathbf{r}, t) \times \mathbf{U})]. \tag{53}$$

Usually, as, e.g., in [28] (the derivation of Eqs. (9-18) (1-4)) the case of: "a non-magnetized medium moving with a velocity $\mathbf{u}$ which is small compared with velocity of light," is considered. This means that in (49) one has to take not only $\beta_u \ll 1$, which leads to (50), but also $M_i = 0$. Then, instead of the part with γ_0 from (50), i.e., Eq. (51), one gets the equation $\partial_k D^k \gamma_0 = \rho^{(C)} \gamma_0$. In the formulation with 3D vectors that equation corresponds to, e.g., Eq. (9-18)(1) in [28], or Eq. (7.55)(i) in [26], $\nabla \cdot \mathbf{D} = \rho^{(C)}$.

In the same way, the terms with γ_i from (50), i.e., Eq. (52), can be compared with the usual form of Maxwell's equations for moving media, which deals with 3D vectors, e.g., Eq. (9-18) (4) in [28], or the equations in Problem 6.8 in [29]. Then, the terms with γ_i from (50) correspond to the following equation with 3D vectors

$$\begin{aligned}\nabla \times \mathbf{B}(\mathbf{r}, t) \quad &= \quad \mu_0 [\mathbf{j}^{(C)}(\mathbf{r}, t) + \partial(\varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t))/\partial t + \nabla \times (\mathbf{P}(\mathbf{r}, t) \times \mathbf{U}) \\ &\quad + (1/c^2) \partial(\mathbf{U} \times \mathbf{M}(\mathbf{r}, t))/\partial t + \nabla \times \mathbf{M}(\mathbf{r}, t)], \end{aligned} \tag{54}$$

or, from (52), the equation (54) can be written in the following form

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}^{(C)}(\mathbf{r}, t) + \partial \mathbf{D}(\mathbf{r}, t)/\partial t + (1/c^2) \partial(\mathbf{U} \times \mathbf{M}(\mathbf{r}, t))/\partial t + \nabla \times (\mathbf{P}(\mathbf{r}, t) \times \mathbf{U}), \tag{55}$$

where the 3D vectors $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = (1/\mu_0)\mathbf{B} - \mathbf{M}$ have been introduced. Taking in Eq. (50) that not only $\beta_u \ll 1$ but that $M_i = 0$ as well, i.e., that a non-magnetized medium is considered, then instead of Eq. (55) we find

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0[\mathbf{j}^{(C)}(\mathbf{r}, t) + \partial \mathbf{D}(\mathbf{r}, t)/\partial t + \nabla \times (\mathbf{P}(\mathbf{r}, t) \times \mathbf{U})]. \tag{56}$$

This equation is the fourth equation in Problem 6.8 in [29]. It differs from Eq. (9-18) (4) in [28], which contains an additional term $\mu_0 \rho^{(C)}\mathbf{U}$. As seen from (56) the appearance of that additional term is not justified.

The whole consideration on the difference between Maxwell's equations with 3D vectors and the equations with 4D quantities that is presented at the end of Sec. 4 holds in the same measure here. However, the difference between these two approaches (3D vectors versus 4D geometric quantities) is even bigger for the case examined in this section. Namely, due to the existence of the terms with P^0 and M^0 in (49) that equation *cannot* be compared with Maxwell's equations with 3D vectors. The comparison can be made only for low velocities of the medium when Eq. (49) reduces to Eq. (50).

There is also an additional difference between Maxwell's equations with 3D vectors, e.g., Eqs. (9-18) in [28], and our Eqs. (49) and (50). It is stated in [28] (under Eqs. (9-18)): "Note that Maxwell's equations for moving (nonmagnetic) media in the form given by Eq. (9-18) (4) are "mixed," i.e., the sources $\mathbf{j}_{true}$, $\mathbf{P}$, ρ_{true}, are measured in the moving medium, while the fields are given in the stationary frame." On the other hand, as already stated above, *all quantities* in (49) and (50) are determined in the laboratory frame, which is the γ_0-frame. Moreover, as already explained, all quantities in (49) and (50) are correctly defined in 4D spacetime and they correctly transform under the LT, which is not the case with Maxwell's equations with 3D vectors.

6. Comparison with Galilean Electromagnetism

At this place it is worth mentioning that in a recent review [30] under the title "Forty years of Galilean Electromagnetism (1973 - 2013)" and in the references therein, e.g., [31, 32], it is argued that, [30], "Galilean Electromagnetism is precisely the low-velocity limit of Special Relativity when applied to Classical Electromagnetism." There, it is also stated that in a Galilean limit the usual Maxwell-Minkowski equations with the 3D vectors **E**, **B**, **D** and **H**, Eq. (1) in [30], have to be replaced by two Galilean limits, the magnetic and electric limits, i.e., with two sets of low-velocity formulae, Eqs. (16) and (18) in [30], respectively. Furthermore, in [30], it is considered that the UT, Eqs. (33) and (35) here and the similar ones for **D** and **H**, Eq. (3) in [30] or in [31], are the relativistically correct LT, but that in Galilean approximation they have to be replaced by two sets of low-velocity formulae, the magnetic and electric limits, which are presented, e.g., in Sec. 3 in [30]. The same happens with Minkowski's constitutive equations, Eq. (9), i.e., Eq. (10) in [30] or in [31], which are replaced by the Galilean magnetic constitutive equations, Eq. (13) in [30], and the Galilean electric constitutive equations, Eq. (14) in [30]. In addition, it is considered in [30], as in almost all other usual approaches, that the contraction of lengths and the dilation of time are "the phenomena inherent to Special Relativity."

However, in 4D spacetime the physical quantities are represented in a mathematically correct way by 4D geometric quantities that properly transform under the LT, e.g., (36), and not by 3D quantities that transform by the UT, (33) and (35) and the similar UT for **D** and **H**, Eq. (3) in [30] or in [31]. As discussed in Sec. 4, according the UT, (33) and (35), i.e., Eq. (3) in [30] or in [31], e.g., the transformed $\mathbf{E}'$ is expressed by the mixture of the 3D vectors **E** and

B. Using that essential feature of the UT of the fields as 3D vectors a Galilean limit of the field transformations is derived in two steps in, e.g., [32]. First, the quasi-static approximation, $\beta \ll 1$, is taken and then the assumption on the relative magnitude of $|\mathbf{E}|$ and c$|\mathbf{B}|$ is taken into account. If the magnetic field is dominant then the magnetic limit, Eq. (7) in [32] is obtained, whereas in the opposite case the electric limit, Eq. (8) in [32], is obtained. On the other hand, as discussed in Sec. 2, the essential feature of the mathematically correct LT of 4D fields, like (36), is that the LT transform, e.g., the electric field vector only to the electric field vector; there is no mixing with the magnetic field vector. This means that both Galilean limits, the magnetic limit and the electric limit of the field transformations, Eqs. (7) and (8) in [32] are meaningless in 4D spacetime in which the fields are represented by 4D geometric quantities that correctly transform under the LT.

In [5] the constitutive relations are formulated in terms of coordinate-free quantities that correctly transform under the LT. First, in Sec. 3 they are formulated as the relations between $\mathcal{M}$ and F, Eqs. (11) and (12) in [5]. Then, using the decompositions of F (17) and $\mathcal{M}$ (21) the basic constitutive relations for $P(x)$ and $M(x)$, Eqs. (13) and (14) in [5], are obtained. They show how $P(x)$ and $M(x)$ depend on E, B and two different velocity vectors, u and v. These constitutive relations differ from all previous expressions and they are not reported in any previous approach. In Secs. 5 - 5.2 in [5] it is explained that Minkowski's constitutive relations, Eqs. (23) and (24), or (25) in [5], i.e., Eq. (11) in [30], or Eq. (10) in [31], are not the *relativistic* constitutive equations. They are the relations with 3D vectors that transform according to the UT, like (33) and (35), and these transformations, as already stated several times, are not the LT. Therefore, the constitutive relations, Eqs. (13) and Eq. (14) in [30] are not any kind of a quasi-static approximation of the relativistically correct constitutive relations. As stated in [5], there is only one mathematically correct quasi-static approximation of the constitutive relations for $P(x)$ and $M(x)$, which is given by Eqs. (19) and (21) in [5] in which for the low velocities of the medium it is only taken that $\beta_u \ll 1$, i.e., $\gamma_u \simeq 1$, see Secs. 4, 5 - 5.2 in [5].

It can be concluded from the preceding discussion that both Galilean limits, the Galilean magnetic Maxwell - Minkowski equations and the Galilean electric Maxwell - Minkowski equations, e.g., Eqs. (16) and (18) in [30], respectively, are ill-defined in 4D spacetime; they are not a quasi-static approximation of the relativistically correct field equations. As shown in Sec. 4 and in this section the usual Maxwell-Minkowski equations with the 3D vectors **E**, **B**, **D** and **H**, Eq. (1) in [30], *are not* the relativistic form of Maxwell's equations, because in 4D spacetime there is no room for 3D vectors and their UT. The mathematically correct field equation is, e.g., Eq. (49) and the equations from which (49) is derived and there is only one its quasi-static approximation, i.e., the approximation for the case of low velocities of the medium, it is Eq. (50).

Furthermore, as discussed in Sec. 4, it is exactly proved in [27] and in [23] that the relativity of simultaneity, the Lorentz contraction and the time dilation *are ill-defined in 4D spacetime*, because they depend on the chosen synchronization. Hence, they are not, as argued in [30] "the phenomena inherent to Special Relativity." This dependence on chosen synchronization holds in the same measure for the usual Maxwell-Minkowski equations, Eq. (1) in [30], for the UT of the fields as 3D vectors, Eq. (3) in [30] or in [31], for Minkowski's constitutive relations, Eq. (11) in [30] and for the whole Galilean Electromagnetism including the comparison with experiments that is presented in Secs. 5 - 8.2 in [30].

7. Motional electromotive force in the approaches with 3D quantities and with 4D geometric quantities

On the other hand, as already mentioned in Sec. 4, the approach with 4D geometric quantities and their LT, always agrees with the principle of relativity and it is in a true agreement with all experiments in electromagnetism and all experiments that test SR, see [17, 18, 2, 25, 27]. In Secs. 7.1 and 7.2, instead of to discuss the experiments which are presented in Secs. 5 - 8.2 in [30], we shall briefly present the discussion of the motional electromotive force (emf) that is exposed in Secs. 5 - 5.2 in [17]. However, in Sec. 7.2, some changes relative to Sec. 5.2 in [17] will be introduced (F instead of E and B). It is a nice example that illustrates the fundamental difference between the LT, e.g., (36), and the UT (33) and (35), i.e., between the approach with 4D geometric quantities and the usual approach with 3D vectors.

7.1. Motional emf with 3D quantities

The motional emf is produced in an electrical circuit when a circuit or part of a circuit moves in a magnetic field. In Sec. 5.1 in [17] the emf ε is calculated using 3D quantities, the 3D Lorentz force $\mathbf{F}_L = q\mathbf{E} + q\mathbf{U} \times \mathbf{B}$, the 3D $\mathbf{E}$ and $\mathbf{B}$ and their UT (33) as in Secs. 6.4 in [29], 7.2 in [33], 9-5 in [28], 5.6 in [15] and in all other calculations in the usual approaches. The emf ε of a complete circuit is defined by means of the Lorentz force $\mathbf{F}_L$ that acts on a charge q, which is at rest relative to the section $\mathbf{dl}$ of the circuit,

$$\varepsilon = \oint (\mathbf{F}_L/q) \cdot \mathbf{dl}, \tag{57}$$

Eq. (26) in [17]. The important remark is that it is implicitly assumed in these equations for ε that the integral is taken over the whole circuit at the same moment of time in S, say $t = 0$. Let us take that in the laboratory frame S a conducting bar is moving in a steady uniform magnetic field (3-vector) $\mathbf{B} = -B\mathbf{k}$ with velocity 3-vector $\mathbf{U}$ parallel to the x axis. The length of the bar is l and it moves parallel to the y axis. There is no external applied electric field in S, $\mathbf{E}$ =0. Since in S $\mathbf{E}$ =0 and the components of $\mathbf{B}$ are $(0, 0, -B)$ the emf ε is determined by the contribution of the magnetic part of $\mathbf{F}_L$, i.e., $q\mathbf{U} \times \mathbf{B}$, as $\varepsilon = \int_o^l UBdy = UBl$, which is Eq. (27) in [17].

In S' the conducting bar is at rest. Then, *according to the UT* (33) of the 3D $\mathbf{E}$ and $\mathbf{B}$ the observer in the S' frame "sees" $E'_y = \gamma UB$ and $B'_z = -\gamma B$. Hence, *in* S', there is not only the magnetic field but *an electric field* $\mathbf{E}'$ as well. The contribution of the magnetic part (due to B'_z) of the Lorentz force $\mathbf{F}'_L$ to the emf ε' is zero and only the contribution of the electric part (due to E'_y) of the Lorentz force remains, which is $\varepsilon' = \int_o^l \gamma UBdy = \gamma UBl$, Eq. (29) in [17]. Observe that the integral is taken at the same moment of time t' in S', which can be arbitrarily chosen, say $t' = 0$, or $t' = 10s$, ... The moments of time t in S and t' in S' are not connected in any way. The LT cannot transform the moment of time t in S again, exclusively, to some t' in S'. According to the LT, to one t in S will correspond many t' in S' depending on the spatial position in S'; $t = \gamma(t' + Ux'/c^2)$. This remark clearly shows that the usual definition of ε, Eq. (57), is not relativistically correct definition. Obviously, *the emf* ε' *in* S' *is not equal to the emf* ε, *determined in* S;

$$\varepsilon = UBl,\ \varepsilon' = \gamma UBl, \quad \varepsilon' \neq \varepsilon. \tag{58}$$

Consequently, *the principle of relativity is not satisfied;* the emf obtained by the application of the UT (33) is different for relatively moving 4D observers. This explicitly shows that the conventional calculation of ε and the UT (33) of the 3D $\mathbf{E}$ and $\mathbf{B}$ are not relativistically correct, i.e., the UT (33) are not the LT. The fact that ε and ε' do not significantly differ for low velocities, $U \ll c$, is completely irrelevant; the principle of relativity is not satisfied in the usual approach. Of course, the same holds for the Galilean Electromagnetism.

7.2. Motional emf with 4D geometric quantities

In Sec. 5.2 in [17] the emf ε is calculated using 4D geometric quantities, the vectors E and B, the vector of the Lorentz force K, etc. There, it is found that in the approach with 4D geometric quantities and their LT, e.g., (36), the same value for ε is always obtained, $\varepsilon = \gamma UBl$, which means that the principle of relativity is naturally satisfied. Here, we shall obtain the same result as in Sec. 5.2 in [17] but dealing with F and not with its decompositions (17) and (18). The Lorentz force K is defined as an abstract vector and as a CBGQ in the standard basis by the following relations

$$K = (q/c)F \cdot u, \quad K = K^{\mu}\gamma_{\mu} = (q/c)(F^{\mu\nu}u_{\nu})\gamma_{\mu}. \tag{59}$$

where u is the velocity vector of the considered charge. *The emf ε is defined as an invariant 4D quantity, the Lorentz scalar*,

$$\varepsilon = \int_{\Gamma}(K/q) \cdot dl, \quad \varepsilon = \int_{\Gamma}(K^{\mu}/q)dl_{\mu} = (1/c)\int_{\Gamma}F^{\mu\nu}u_{\nu}dl_{\mu}, \tag{60}$$

where *vector dl* is the infinitesimal spacetime length and Γ is the spacetime curve.

In the laboratory frame S with the standard basis in it the components of u and dl are $u^{\mu} = (\gamma c, \gamma U, 0, 0)$, $dl^{\mu} = (0, 0, dl^2 = dy, 0)$. It can be seen from Eq. (20) that in the considered case all components $F^{\mu\nu}$ are zero except F^{21} (F^{12}), which is $F^{21} = cB^3$. This result and Eq. (59) yield that the components of the Lorentz force K are $K^0 = K^1 = K^3 = 0$, but $K^2 = (q/c)F^{21}(-\gamma U)$. Hence, the emf ε is

$$\varepsilon = (1/c)\int_0^l F^{21}(-\gamma U)dy = (1/c)F^{21}(-\gamma U)l. \tag{61}$$

This result can be compared with that one from Sec. 5.2 in [17], $K^2 = \gamma qUB$ and $\varepsilon = \gamma UBl$, using the relations $F^{21} = cB^3$, and $B^3 = B_z = -B$. Then

$$\varepsilon = (1/c)\int_0^l F^{21}(-\gamma U)dy = \gamma UBl, \tag{62}$$

which is the same as in [17]. In contrast to the usual approaches with 3D quantities the expression for ε (60) is independent of the chosen reference frame and of the chosen basis in it; ε is the same in S and in the relatively moving S' frame,

$$\varepsilon = \int_{\Gamma}(K^{\mu}/q)dl_{\mu} = \int_{\Gamma}(K'^{\mu}/q)dl'_{\mu} = \gamma UBl, \tag{63}$$

which is Eq. (37) in [17]. This result for ε can be checked directly performing the LT of all vectors from S to S'. Observe that in S' the 3-velocity $\mathbf{U}$ is zero, but the velocity vector u *is not*, $u = c\gamma_0$, i.e., $u'^{\mu} = (c, 0, 0, 0)$. The same value for ε will be obtained if another basis, e.g., the $\{r_{\mu}\}$ basis, will be used in both frames. Obviously, *in the 4D geometric approach, the principle of relativity is naturally satisfied.*

The result that the conventional theory with the 3D $\mathbf{E}$ and $\mathbf{B}$ and their UT (33) yields different values for ε for relatively moving inertial observers, $\varepsilon = UBl$ in S and $\varepsilon' = \gamma UBl$ in S', whereas ISR, i.e., the approach with 4D geometric quantities and their LT, e.g., (36), yields always the same value for ε, $\varepsilon = \gamma UBl$, Eq. (63), is very strong evidence that the usual approach is not relativistically correct. It is for the experimentalists to find the way to measure the emf ε with a great precision and to see that in the laboratory frame $\varepsilon = \gamma UBl$ and not simply $\varepsilon = UBl$.

Observe that in this calculation with 4D geometric quantities all quantities are invariant under the passive LT, e.g., $u = u^\nu \gamma_\nu = u'^\nu \gamma'_\nu = u_r^\nu r_\nu = u_r'^\nu r'_\nu$, $K = K^\nu \gamma_\nu = K'^\nu \gamma'_\nu = K_r^\nu r_\nu = K_r'^\nu r'_\nu$, etc., which means that the observers in S and S' are "looking" at the same quantity, for example, the Lorentz force K. This consideration shows that ISR, i.e., the approach with 4D geometric quantities and their LT (36) is substantially different than the usual approaches with the 3D quantities and their UT (33). The former is completely suited to the symmetry of 4D spacetime, which is not the case with the latter.

There are many similar examples in the literature; it will be always found that there is a fundamental difference between the UT of the 3D $\mathbf{E}$ and $\mathbf{B}$ (33) and the LT of 4D geometric quantities representing the electric and magnetic fields, e.g., (36), and that the 4D geometric approach, i.e., ISR, correctly describes electromagnetic phenomena in all relatively moving 4D inertial frames of reference. One important experiment, the Faraday disk, which leads to the same conclusions, is considered in detail in [18].

8. Discussion and conclusions

There are several important differences between the field equations reported here and all others in previous literature including the modern textbook on classical electrodynamics [34], which uses the calculus of exterior forms.

First, instead of dealing with the electromagnetic excitation $\mathcal{H}$ (14) and the field equation with it (15) we exclusively deal with the equations (7) for the electromagnetic field F and a matter field $\mathcal{M}$ as the primary equations for electromagnetism in moving media. As discussed in Secs. 2 and 3, the expression for $\mathcal{H}$ (14) in terms of F and $\mathcal{M}$ is in some sense unsatisfactory, since F and $\mathcal{M}$ are physically different kind of entities. Furthermore, what is particularly important, in general, two different velocity vectors, v - the velocity of the observers and u - the velocity of the moving medium, enter into the decompositions of F and $\mathcal{M}$, Eqs. (17) and (21), respectively. For this reason we also do not deal with the decomposition of $\mathcal{H}$ (25) into the electric and magnetic excitations D and H, respectively, where $D = \varepsilon_0 E + P$ and $H = (1/\mu_0)B - M$. As stated in Sec. 3, such a decomposition as (25) is possible if only one velocity, the velocity of the medium u, is taken into account, or the case $u = v$ is considered, or both decompositions (17) and (21) are made with the same velocity vector, either u or v. Recently, the last case is considered in [35], but with F and $\mathcal{H}$.

The second important difference refers to the interpretation of the field equations. The basic field equation (26) contains two different velocities u and v. It is also written as Eqs. (27) and (28), i.e., with CBGQs, (29) and (30), respectively. From these equations it is visible that *in 4D spacetime*, in contrast to the formulation of electromagnetism in terms of Maxwell's equations with the 3D vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{M}$, *there are no two laws, the Ampère-Maxwell law and Gauss's law, but only one law, that is expressed by Eq.* (27) ((29)), i.e., *Eq.* (31) ((32)), *and the same for Eq.* (28) ((30)) and *Faraday's law and the law that expresses the absence of magnetic charge.*

Furthermore, the interesting results are obtained in Secs. 4 and 5. There, the field equations, written in the standard basis (29), i.e., (32), and (30), are compared with the usual form (with 3D vectors) of Maxwell's equations for moving media. In Sec. 4, it is shown that the comparison is possible *only* if the term with γ_0 is considered *separately* from those ones with γ_i and if in Eqs. (42) and (45) *only the components* are taken into account. In order to get the usual equations with 3D vectors, (43),(44), (46), (47) and (48), these components have to be multiplied by *the unit 3D vectors* **i**, **j**, **k**. Moreover, as shown in Sec. 5, such a procedure is not applicable to Eq. (49), but only to Eq. (50), which is derived from (49) for the case of low velocities of the medium. As explained in Sec. 4, *the above mentioned steps in the comparison are not mathematically correct in 4D spacetime.* Hence, in 4D spacetime, the equations (26), (27), (28), (29), (30), ... , with the 4D geometric quantities E, B, P and M that correctly transform under the LT, e.g., for the components (36), or for the vector E, Eq. (8) in [5], i.e., Eq. (13) in [11], *are not equivalent* to the usual Maxwell equations (43), (44), (46), (47), (48), (53), (54), ... , with the 3D vectors **E**, **B**, **P** and **M** that transform according to UT, e.g., Eqs. (33) and (35).

It will be important for physics to examine the theoretical and experimental consequences of the results that are obtained in this paper. An interesting consequence that can be experimentally examined is already mentioned in connection with Eq. (31). There, it is stated that if we have a magnetization M, a permanent magnet, moving *or stationary*, but with a negligible permanent polarization P and without $j^{(C)}$, then, as can be seen from (31), M will induce *both* B *and* E. In Sec. 8 in [22] the existence of the electric field from a stationary permanent magnet is investigated in detail and that field is used in the consideration of the "charge-magnet paradox" in [22] and [36].

At this place it is worth mentioning a recent paper, [37], under the title "Nature of Electric and Magnetic Fields; How the Fields Transform." In that paper, in 3.1 and 3.3 the mathematically correct proofs are given that the electric and magnetic fields, $E(x)$ and $B(x)$, respectively are properly defined vectors on 4D spacetime and not the usual 3D vectors **E** and **B**. Furthermore, it is proved in [37] that the correct LT of the electric field are given by (36) and not by the UT of the 3D vectors Eqs. (11.148) and (11.149) in [1]. The proof 3.3 from [37] is based on the mathematical theorem presented in Sec. 3 here, i.e., it is given by the relations (17) - (20) here. The proof 3.1 is a simple but very strong mathematical argument, which is stated by Oziewicz, e.g., in [38]: *What is essential for the number of components of a vector field is the number of variables on which that vector field depends, i.e., the dimension of its domain.* In general, the dimension of a vector field that is defined on a n-dimensional space is equal - n. *The electric and magnetic fields are defined on a 4D space, i.e., the spacetime. They are always functions of the position vector* x. *This means that they are not the usual 3D fields, but they are properly defined vectors on 4D spacetime*, $E(x)$ and $B(x)$. The same holds for the polarization vector $P(x)$ and the magnetization vector $M(x)$. Since E, B, P and M are 4D vectors they must transform *under the LT as any other vector transforms*, e.g., *the electric field vector must transform again to the electric field vector* like in (36). In [39] the same result is obtained for the electric field as a bivector and for the magnetic field as well.

Acknowledgments

It is a pleasure to acknowledge to Larry Horwitz and Martin Land for inviting me to the IARD conferences and for their continuous support of my work. I am also very grateful to Zbigniew Oziewicz for numerous and very useful discussions during years and to him and to Alex Gersten for the continuous support of my work.

References

[1] Minkowski H 1908 *Nachr. Ges. Wiss. Göttingen* 53; Reprinted in: 1910 *Math. Ann.* **68** 472; English translation in: M N Saha and S N Bose 1920 *The Principle of Relativity: Original Papers by A. Einstein and H. Minkowski* (Calcutta: Calcutta University Press)

[2] Ivezić T 2005 *Found. Phys. Lett.* **18** 401

[3] Einstein A 1905 *Annalen der Physik* **17** 891; English translation in: Perrett W and Jeffery G B 1952 *The Principle of Relativity* (New York: Dover)

[4] Hestenes D 1966 *Space-Time Algebra (*New York: Gordon & Breach); Hestenes D and Sobczyk G 1984 *Clifford Algebra to Geometric Calculus* (Dordrecht: Reidel); Doran C and Lasenby A 2003 *Geometric algebra for physicists* (Cambridge: Cambridge University Press)

[5] Ivezić T 2012 *Int. J. Mod. Phys. B* **26** 1250040

[6] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley)

[7] Hehl F W 2008 *Annalen der Physik* **17** 691

[8] Obukhov Y N 2008 *Annalen der Physik* **17** 830

[9] Itin Y and Friedman Y 2008 *Annalen der Physik* **17** 769

[10] Ludvigsen M 1999 *General Relativity, A Geometric Approach* (Cambridge: Cambridge University Press); Sonego S and Abramowicz M A J 1998 *J. Math. Phys.* **39** 3158

[11] Ivezić T 2010 *Phys. Scr.* **82** 055007

[12] Møller C 1972 *The Theory of Relativity* 2nd edn (Oxford: Clarendon Press)

[13] Hillion P 1993 *Phys. Rev. E* **48** 3060;

[14] Dereli T, Gratus J and Tucker R W 2007 *Phys. Lett. A* **361** 190

[15] Bladel J Van 1984 *Relativity and Engineering* (Berlin: Springer-Verlag)

[16] Ivezić T 2003 *Found. Phys.* **33** 1339

[17] Ivezić T 2005 *Found. Phys.* Lett. **18** 301

[18] Ivezić T 2005 *Found. Phys.* **35** 1585

[19] Ivezić T 2008 *Fizika A* **17** 1

[20] Ivezić T 2008 *arXiv:* 0809.5277

[21] Ivezić T 2007 *Phys. Rev. Lett.* **98** 108901

[22] Ivezić T 2013 *J. Phys.: Conf. Ser.* **437** 012014

[23] Ivezić T 2001 *Found. Phys.* **31** 1139

[24] Ivezić T 2010 *Phys. Scr.* **81** 025001

[25] Ivezić T 2007 *Found. Phys.* **37** 747

[26] Griffiths D J 2013 *Introduction to Electrodynamics* 4th edn (Boston: Pearson)

[27] Ivezić T 2002 *Found. Phys. Lett.* **15** 27; Ivezić T 2001 *Preprint* physics/0103026; Ivezić T 2001 *Preprint* physics/0101091

[28] Panofsky W K H and Phillips M 1962 *Classical electricity and magnetism* 2nd edn (Reading: Addison-Wesley)

[29] Rosser W G W 1968 *Classical Electromagnetism via Relativity* (New York: Plenum)

[30] Rousseaux G 2013 *Eur. Phys. J. Plus* **128** 81

[31] Rousseaux G 2008 *Europhys. Lett.* **84** 20002

[32] de Montigny M and Rousseaux G 2006 *Eur. Phys. J. Plus* **27** 755

[33] Purcell E M 1985 *Electricity and Magnetism* 2nd edn (New York: McGraw-Hill)

[34] Hehl F W and Obukhov Yu N 2003 *Foundations of Classical Electrodynamics: Charge, flux, and metric* (Boston: Birkhäuser)

[35] Goto Shin-itiro, Tucker R W and Walton T J 2011 *Proc. Roy. Soc. A* **467** 59

[36] Ivezić T 2012 *Preprint* 1212.4684

[37] Ivezić T 2015 *Preprint* 1508.04802

[38] Oziewicz Z 2011 *J. Phys.: Conf. Ser.* **330** 012012

[39] Ivezić T 2008 *Fizika A* **17** 1