A complete characterization of relativistic uniform acceleration

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Abstract. We use the Frenet frame to define and completely characterize "uniform acceleration" in flat spacetime. We extend the definition to arbitrary curved spacetime and provide an example in Schwarzschild spacetime.

1. Introduction

Einstein's intuitive definition of *uniform acceleration* is "constant acceleration in the instantaneously co-moving inertial frame." This definition is found widely in the literature, as early as [1] and [2], again in [3], and as recently as [4] and [5]. This definition is natural, since the acceleration in the comoving frame is "precisely the push we feel when sitting in an accelerating rocket" or automobile. Similarly, "by the equivalence principle, the gravitational field in our terrestrial lab is the negative of our proper acceleration, our instantaneous rest frame being an imagined Einstein cabin falling with acceleration g" ([6], page 71).

Historically, however, there have been many difficulties in capturing this definition *mathematically.* Einstein considered the equation

$$F = \frac{dp}{d\tau}, \tag{1}$$

where F is the four-force, p is the four-momentum, and τ is proper time (see [7]). Unfortunately, when F is a constant, as in a homogeneous gravitational field, equation (1) has no solution! This follows from the fact that the four-velocity and the four-acceleration are orthogonal. This was noticed by Planck, who wrote to Einstein about it. This, in turn, prompted Einstein to submit a "correction" [8] to [9]. In the correction, he states that the "concept 'uniformly accelerated' needs further clarification."

In [10, 11], working in flat spacetime, we considered motions which are solutions to

$$c\frac{d\lambda^\mu_{(\kappa)}}{d\tau} = \Omega^\mu{}_\nu \lambda^\nu_{(\kappa)}, \tag{2}$$

where $\{K_\tau\}$ is a one-parameter family, indexed by proper time τ, of instantaneously comoving inertial frames, with orthonormal bases $\Lambda(\tau) = \{\lambda_{(\kappa)}(\tau) : \kappa = 0, 1, 2, 3\}$, $u(\tau) = \lambda_{(0)}(\tau)$ is

the four-velocity of the motion, and $\Lambda^{-1}\Omega\Lambda$ is constant along the worldline. Equation (2) is fully Lorentz covariant and admits four Lorentz-invariant classes of solutions. The class of *translational acceleration* is the smallest Lorentz invariant class which includes *hyperbolic motion*. In the *null acceleration* class, the worldline of the motion is *cubic* in the time. *Rotational acceleration* covariantly extends pure rotational motion. *General acceleration* is obtained when the translational component of the acceleration is parallel to the axis of rotation.

In [10, 11], we showed that all four classes of solutions to (2) have constant acceleration in the comoving frame. At that time, however, it was not clear whether equation (2) captures *all* uniformly accelerated motions. In other words, do there exist motions which have constant acceleration in the instantaneously co-moving inertial frame but do *not* satisfy equation (2)? In section 5, we settle this question by showing that, in flat spacetime, a motion has constant acceleration in the co-moving frame if and only if it satisfies equation (2).

Having completely characterized uniform acceleration in flat spacetime, we show how to extend the definition of uniform acceleration to *curved spacetime*. We do this with the help of the so-called *Frenet frame*. While the authors of [10, 11, 12, 13] define uniform acceleration *with respect to an inertial frame*, the Frenet frame is more suited for working on a *manifold* and resembles the approach of Mashhoon [14]. It was already shown in [10] that in flat spacetime, the inertial frame and the Frenet frame approaches are, in some sense, equivalent.

The Frenet frame or *Frenet basis* at each point along a smooth future-pointing timelike curve is an orthonormal basis of four-vectors belonging to the tangent space at the point in question. In section 3, working in arbitrary curved spacetime, we provide the explicit construction of the Frenet frame and the derivation of the *Frenet equations*. We show in section 4 that the Frenet equations extend the *geodesic equation*. Despite the fact that the Frenet equations are coupled, we show how to *parallel transport* the individual vectors of the Frenet basis. A detailed example in Schwarzschild spacetime is provided in section 6.

Basic definitions and notation are established in section 2. A discussion appears in section 7. Sections 3 and 4 follow closely the development in [13]. Most of section 6 appeared in [15].

2. Notation

Consider a time-orientable four-dimensional differential manifold $\mathcal{M}$ endowed with a metric $g_{\mu\nu}$ of Lorentzian signature $(+,-,-,-)$. A tangent vector v at a given point of M is *timelike* if $g(v,v) > 0$, *spacelike* if $g(v,v) < 0$, and *null* if $g(v,v) = 0$. Let $\gamma : I \to \mathcal{M}$, I an open interval of $\mathbb{R}$, be a smooth future-pointing timelike curve, parameterized by the arclength $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$. Note that $ds = cd\tau$. Moreover, under a change of coordinates $x^\mu \to x^{\mu'}$, we have $ds = ds'$. In a local system of coordinates x^μ, the curve $\gamma(s)$ is described by a set of four functions $x^\mu(s)$. We assume that the metric satisfies the metricity condition $dg_{\mu\nu}/ds = 0$.

At every point of γ, the *four-velocity* u^μ is defined by

$$u^\mu = \frac{dx^\mu}{ds} \tag{3}$$

and has unit length:

$$u^2 = g_{\mu\nu}u^\mu u^\nu = 1. \tag{4}$$

The *covariant derivative* $\frac{D}{ds}$ of a four-vector w^μ along $\gamma(s)$ is defined [16] by

$$\frac{Dw^\mu}{ds} = \frac{dw^\mu}{ds} + \Gamma^\mu{}_{\alpha\nu}w^\alpha u^\nu, \tag{5}$$

where the Christoffel symbols $\Gamma^\mu{}_{\alpha\nu}$ are defined by

$$\Gamma^\mu{}_{\alpha\nu} = \frac{1}{2}g^{\rho\mu}(g_{\rho\alpha,\nu} + g_{\rho\nu,\alpha} - g_{\alpha\nu,\rho}). \tag{6}$$

For ease of notation, we often write $\dot{w}$ instead of $\frac{Dw}{ds}$.

The *four-acceleration* a^μ is the covariant derivative of the four-velocity:

$$a = c^2 \dot{u}. \tag{7}$$

Differentiating $u^2 = 1$, we have

$$\dot{u} \cdot u = 0, \tag{8}$$

meaning that the four-acceleration is orthogonal to the four-velocity. Since the four-velocity is timelike, the four-acceleration is spacelike.

The *plane of simultaneity* or the *restspace* at the point $\gamma(s)$ is the linear subspace of four-vectors w such that $w \cdot u(s) = 0$.

3. The Frenet Frame

We now construct the Frenet basis $\{\lambda_{(0)}, \lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}\}$, $\lambda_{(\alpha)} = \lambda_{(\alpha)}(s)$, of the tangent space at the point $\gamma(s)$. The construction uses covariant differentiation and the Gram-Schmidt orthonormalization procedure. In the event that the Gram-Schmidt process breaks down, we are free to complete the orthonormal basis arbitrarily. The orthonormality condition means that

$$\lambda_{(\alpha)} \cdot \lambda_{(\beta)} = \eta_{\alpha\beta}, \tag{9}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric $\operatorname{diag}(1, -1, -1, -1)$. Differentiating (9), we obtain

$$\dot{\lambda}_{(\alpha)} \cdot \lambda_{(\beta)} = -\lambda_{(\alpha)} \cdot \dot{\lambda}_{(\beta)}. \tag{10}$$

In particular,

$$\lambda_{(\alpha)} \cdot \dot{\lambda}_{(\alpha)} = 0. \tag{11}$$

First, let $\lambda_{(0)}(s) = u(s)$. The four-acceleration $a = \dot{u}$ is spacelike and orthogonal to the four-velocity u. We assume that $\dot{u} \neq 0$ for all s. Set $\kappa = \sqrt{-a^2}$ and define

$$\lambda_{(1)}(s) = \frac{a(s)}{\kappa}. \tag{12}$$

The unit vector $\lambda_{(1)}(s)$ gives the direction of the four-acceleration. The scalar $\kappa(s)$ is the magnitude of the four-acceleration and is also called the curve's *curvature*. From (12), we trivially get the first Frenet equation

$$c^2 \dot{\lambda}_{(0)} = \kappa \lambda_{(1)}. \tag{13}$$

Using the Gram-Schmidt procedure, (9) and (10), we construct a vector $v_{(2)}$ which is orthogonal to both $\lambda_{(0)}$ and $\lambda_{(1)}$:

$$v_{(2)} = c^2 \dot{\lambda}_{(1)} - (c^2 \dot{\lambda}_{(1)} \cdot \lambda_{(0)}) \lambda_{(0)} = c^2 \dot{\lambda}_{(1)} - \kappa \lambda_{(0)}. \tag{14}$$

Let $\tau_1 = \sqrt{-(v_{(2)})^2} > 0$, and define

$$\lambda_{(2)} = \frac{v_{(2)}}{\tau_1}. \tag{15}$$

Then, from (14), we have

$$c^2 \dot{\lambda}_{(1)} = \kappa \lambda_{(0)} + \tau_1 \lambda_{(2)}. \tag{16}$$

Similarly, we construct a vector $v_{(3)}$ orthogonal to $\lambda_{(0)}, \lambda_{(1)}$ and $\lambda_{(2)}$:

$$v_{(3)} = c^2 \dot{\lambda}_{(2)} - (c^2 \dot{\lambda}_{(2)} \cdot \lambda_{(0)})\lambda_{(0)} + (c^2 \dot{\lambda}_{(2)} \cdot \lambda_{(1)})\lambda_{(1)}. \tag{17}$$

Now (10), (13) and (9) imply that $\dot{\lambda}_{(2)} \cdot \lambda_{(0)} = 0$, while (10), (16) and (9) imply that $\dot{\lambda}_{(2)} \cdot \lambda_{(1)} = \tau_1$. Let $\tau_2 = \sqrt{-(v_{(3)})^2} > 0$, and define

$$\lambda_{(3)} = \frac{v_{(3)}}{\tau_2}. \tag{18}$$

From (17) we now obtain

$$c^2 \dot{\lambda}_{(2)} = -\tau_1 \lambda_{(1)} + \tau_2 \lambda_{(3)}. \tag{19}$$

Finally, using (13), (16), (10) and (11), we get that $\dot{\lambda}_{(3)}$ is parallel to $\lambda_{(2)}$. Then, using (10), (19) and (9), we get $\lambda_{(2)} \cdot \dot{\lambda}_{(3)} = \tau_2$. Hence,

$$c^2 \dot{\lambda}_{(3)} = -\tau_2 \lambda_{(2)}. \tag{20}$$

The $\lambda_{(\alpha)}$'s constitute the *Frenet basis* of the trajectory, and the equations (13), (16), (19) and (20) are called the *Frenet equations*. They are a system of coupled ordinary differential equations and can be written compactly as

$$c^2 \dot{\lambda}_{(\alpha)}(s) = \lambda_{(\beta)}(s) A^{(\beta)}_{(\alpha)}(s), \tag{21}$$

where

$$A^{(\beta)}_{(\alpha)}(s) = \begin{pmatrix} 0 & \kappa(s) & 0 & 0 \\ \kappa(s) & 0 & -\tau_1(s) & 0 \\ 0 & \tau_1(s) & 0 & -\tau_2(s) \\ 0 & 0 & \tau_2(s) & 0 \end{pmatrix}. \tag{22}$$

Note that $A(s)$ is a matrix of *scalars*. Under a change of coordinates $x^\mu \to x^{\mu'}$, we have $ds' = ds$, and therefore, $\kappa(s') = \kappa(s)$ and likewise for $\tau_1(s)$ and $\tau_2(s)$. Thus, its two indices are coordinate-free, so we place them in parentheses. We call $A(s)$ the *acceleration matrix*. Note that the lower index on the λ's is a *label* and not a tensorial index.

The physical meaning of $\kappa(s)$ and $\tau_1(s), \tau_2(s)$ is as follows. An observer on $\gamma(s)$ experiences *linear acceleration* of magnitude $\kappa(s)$ in the direction of $\lambda_{(1)}$. The magnitude and the direction of this acceleration can be measured by an accelerometer carried by an observer moving along $\gamma(s)$. The torsion is defined by a 3D vector $\boldsymbol{\omega} = -\tau_2 \lambda_{(1)} - \tau_1 \lambda_{(3)}$, which can be measured by the precession of gyroscopes carried by the observer. The vector $\boldsymbol{\omega}$ is the axis of the observer's rotational acceleration, with magnitude $\sqrt{\tau_1^2 + \tau_2^2}$. τ_2 is the component of the torsion parallel to the direction of linear acceleration, while τ_1 is the component in the orthogonal direction. The Frenet basis and κ, τ_1 and τ_2 are definable *locally* and are completely determined by the trajectory.

If κ is constant and $\tau_1 = 0$, then equations (13) and (16) imply that

$$\dddot{\lambda}_{(0)} = \frac{\kappa^2}{c^4} \lambda_{(0)}, \tag{23}$$

which is equivalent to [17], equation (6). Equation (23) describes hyperbolic motion. However, to obtain a fully Lorentz covariant theory, we cannot assume that $\tau_1 = 0$. This is because hyperbolic motion itself is not covariant, as shown in [11, 12]. Note that equation (23) is third order in the coordinates x^μ. For a unique solution, one need only specify the initial position $\gamma(0)$, the initial four-velocity $u(0)$, and the initial linear acceleration $a(0)$.

4. Analysis of the Frenet equations

Our first goal is to show that the Frenet equations extend the *geodesic equation*

$$\frac{du^\mu(s)}{ds} + \Gamma^\mu{}_{\sigma\rho}u^\sigma(s)u^\rho(s) = 0. \tag{24}$$

First, use (5), (21) and $\lambda_{(0)}(s) = u(s)$ to write the Frenet equations as

$$c^2\frac{d\lambda^\mu_{(\alpha)}(s)}{ds} + c^2\Gamma^\mu{}_{\sigma\rho}\lambda^\sigma_{(\alpha)}(s)\lambda^\rho_{(0)}(s) = \lambda^\mu_{(\beta)}(s)A^{(\beta)}{}_{(\alpha)}. \tag{25}$$

Now we show that equation (25) extends the geodesic equation. Along a geodesic, there is zero acceleration, so $A \equiv 0$. Thus, (25) becomes

$$\frac{d\lambda^\mu_{(\alpha)}(s)}{ds} + \Gamma^\mu{}_{\sigma\rho}\lambda^\sigma_{(\alpha)}(s)\lambda^\rho_{(0)}(s) = 0. \tag{26}$$

Setting $\alpha = 0$ and using $\lambda_{(0)}(s) = u(s)$, we obtain the geodesic equation (24).

We now show how to parallel transport the Frenet basis vectors $\lambda_{(\kappa)}$. Recall that a four-vector w^μ is said to be *parallel transported* along $\gamma(s)$ by the Levi-Civita connection if

$$\frac{Dw^\mu}{ds} = 0. \tag{27}$$

If the four-acceleration $a(s) = c^2\frac{Du(s)}{ds}$ is nonzero, then the four-velocity u is *not* parallel transported along $\gamma(s)$ by the Levi-Civita connection. This means that the Frenet basis is not parallel transported. The covariant derivative along the curve does not preserve the restspaces of an accelerating particle.

A more appropriate transport is defined using the *generalized Fermi-Walker derivative* (*GFW*) $\frac{\widehat{D}w^\mu}{ds}$ of a four vector w^μ (see [16], [17]):

$$\frac{\widehat{D}w^\mu}{ds} = c^2\frac{Dw^\mu}{ds} - \Omega^\mu{}_\nu w^\nu, \tag{28}$$

where $\Omega_{\mu\nu}$ is a rank 2 tensor, defined along $\gamma(s)$. Hehl [16] shows that the metric compatibility condition, which follows from the transport of the orthonormal basis, implies that $\Omega_{\mu\nu}$ is antisymmetric.

Let $\Lambda(s)$ be the 4×4 matrix whose ith column consists of the components of the vector $\lambda_{(i)}(s)$ in the local basis. Then the Frenet equations (21) can be written as

$$c^2\frac{D\Lambda(s)}{ds} = \Lambda(s)A(s). \tag{29}$$

In order to ensure parallel transport of the Frenet basis vectors $\lambda_{(\kappa)}$, we choose

$$\Omega(s) = \Lambda(s)A(s)\Lambda^{-1}(s). \tag{30}$$

$\Omega(s)$ is an antisymmetric tensor with the same Lorentz invariants as the matrix $A(s)$. In fact, $\Omega(s)$ is the acceleration matrix $A(s)$ computed in the initial comoving frame. Hence, we refer to $\Omega(s)$ as the *pullback* of the acceleration matrix $A(s)$ along the worldline. Multiplying the right side of (29) by $\Lambda^{-1}(s)\Lambda(s)$, we obtain

$$c^2\frac{D\Lambda(s)}{ds} = \Omega(s)\Lambda(s). \tag{31}$$

Clearly, the Frenet basis is *GFW parallel transported*. Using (31) and (28), we have

$$\frac{\widehat{D}\lambda_{(\kappa)}(s)}{ds} = c^2 \frac{D\lambda_{(\kappa)}(s)}{ds} - \Omega(s)\lambda_{(\kappa)}(s) = 0. \tag{32}$$

In equation (29), the matrix Λ is multiplied by A on the *right*. This means that the time evolution of each basis vector depends on *all* of the basis vectors. However, to have parallel transport, each basis vector must be transported without referring to the other basis vectors. This explains why, in equation (29), Λ is multiplied by Ω on the *left*.

5. Uniform Acceleration

Einstein's intuitive definition of *uniform acceleration* is "constant acceleration in the instantaneously co-moving inertial frame." Historically, however, there have been many difficulties in capturing this definition *mathematically*. For more details, see [12].

In [10, 11], working in flat spacetime, we considered motions which are solutions to

$$c\frac{d\lambda_{(\kappa)}^{\mu}}{d\tau} = \Omega^{\mu}{}_{\nu}\lambda_{(\kappa)}^{\nu}, \tag{33}$$

where $\{K_\tau\}$ is a one-parameter family, indexed by proper time τ, of instantaneously comoving inertial frames, with orthonormal bases $\Lambda(\tau) = \{\lambda_{(\kappa)}(\tau) : \kappa = 0, 1, 2, 3\}$, $u(\tau) = \lambda_{(0)}(\tau)$ is the four-velocity of the motion, and $\Lambda^{-1}\Omega\Lambda$ is constant along the worldline.

In the $1 + 3$ decomposition of flat spacetime, the tensor Ω of equation (33) has the form

$$\Omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \mathbf{g}^T \\ \mathbf{g} & -c\mathbf{B} \end{pmatrix}. \tag{34}$$

where $\mathbf{g}$ is a 3D vector with physical dimension of acceleration, $\boldsymbol{\omega}$ is a 3D vector with physical dimension $1/$time, the superscript T denotes matrix transposition, and, for any 3D vector $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)$,

$$\mathbf{B} = \varepsilon_{ijk}\omega^k,$$

where ε_{ijk} is the Levi-Civita tensor. The factor c in Ω provides the necessary physical dimension of acceleration. The vector $\mathbf{g}$ represents *linear* acceleration. If $\boldsymbol{\omega} = 0$, we obtain constant linear acceleration in a fixed direction, otherwise known as *hyperbolic motion*. The vector $\boldsymbol{\omega}$ is the angular velocity of the motion. If $\mathbf{g} = 0$, we obtain *pure rotational motion* with constant angular velocity.

In [10, 11], we showed that the solutions to (33) have constant acceleration in the comoving frame. At that time, however, it was not clear whether equation (33) captures *all* uniformly accelerated motions. In other words, do there exist motions which have constant acceleration in the instantaneously co-moving inertial frame but do *not* satisfy equation (33)? With the help of the Frenet frame, we settle this issue.

Theorem 1. *In flat spacetime with proper time τ, the following are equivalent:*

(i) *The future-pointing, timelike worldline $\gamma(\tau)$ has constant acceleration in the instantaneously co-moving inertial frame,*

(ii) *There exist an antisymmetric tensor $\Omega_{\mu\nu}(\tau)$ and a one-parameter family $\{K_\tau\}$ of instantaneously comoving inertial frames, with orthonormal bases $\Lambda(\tau) = \{\lambda_{(\kappa)}(\tau)\}$ such that*

  (a) $\Lambda^{-1}\Omega\Lambda$ *is constant with respect to τ,*

(b) $c\frac{d}{d\tau}\lambda^{\mu}_{(\kappa)}(\tau) = \Omega^{\mu}{}_{\nu}\lambda^{\nu}_{(\kappa)}(\tau)$,

(c) $u(\tau) = \lambda_{(0)}(\tau)$ is the four-velocity of $\gamma(\tau)$.

The implication $(ii) \Rightarrow (i)$ was proven in [10, 11].

To prove $(i) \Rightarrow (ii)$, let $\gamma(\tau)$ be the worldline of a motion with constant acceleration in the instantaneously co-moving inertial frame. Let $\Lambda(\tau)$ and $A(\tau)$ be as in the construction of section 3. Since the acceleration is constant in the co-moving frame, we have $A(s) \equiv A$. Using (29) and $ds = cd\tau$, and noting that covariant derivatives becomes normal derivatives, since we are working in flat spacetime, we have

$$c\frac{d\Lambda(\tau)}{d\tau} = \Lambda(\tau)A. \tag{35}$$

Then $(a), (b)$ and (c) of (ii) are satisfied with

$$\lambda^{\mu}_{(\kappa)} = \Lambda^{\mu}_{(\kappa)} \quad \text{and} \quad \Omega^{\mu}{}_{\nu} = \Lambda^{\mu}_{(\alpha)}A^{(\alpha)}_{(\beta)}(\Lambda^{-1})^{(\beta)}_{\nu}. \tag{36}$$

This proves the theorem.

Theorem 1, which is valid in flat spacetime, motivates the following extension of the definition of uniform acceleration to curved spacetime.

Definition 1. *Let $\gamma(s)$ be a timelike curve, with acceleration matrix $A(s)$. We say that γ represents* uniformly accelerated motion *if*

$$\frac{dA(s)}{ds} = 0. \tag{37}$$

We turn now to an example of uniform acceleration in the Schwarzschild metric.

6. Uniform Acceleration in Schwarzschild spacetime

In this section, we provide an example of uniform acceleration in *Schwarzschild spacetime*, with the metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right)c^2dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2d\Omega^2, \tag{38}$$

where r_s is the Schwarzschild radius and $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$. Here θ is the colatitude (= the angle from north) and φ is longitude.

The known (see [18, 19]) nonzero Christoffel symbols are

$$\Gamma^0{}_{01} = \frac{r_s}{2r(r - r_s)} \quad , \quad \Gamma^1{}_{00} = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right) \quad , \quad \Gamma^1{}_{11} = \frac{-r_s}{2r(r - r_s)}$$

$$\Gamma^1{}_{22} = r_s - r \quad , \quad \Gamma^1{}_{33} = (r_s - r)sin^2\theta \quad , \quad \Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \quad , \quad \Gamma^3{}_{13} = \frac{1}{r} \quad , \quad \Gamma^3{}_{23} = \frac{\cos\theta}{\sin\theta}. \tag{39}$$

For our example, we consider motion in the (t, r) plane and set $\theta = \frac{\pi}{2}, \varphi = 0$. Then

$$\Gamma^1{}_{33} = r_s - r \quad , \quad \Gamma^2{}_{33} = \Gamma^3{}_{23} = 0.$$

Let the acceleration matrix $A = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, representing hyperbolic motion in the radial direction. Equation (25), with $\alpha = 0, \mu = 2$, is

$$c^2 \frac{d\lambda^2_{(0)}}{ds} + \frac{2c^2}{r}\lambda^1_{(0)}\lambda^2_{(0)} = \kappa\lambda^2_{(1)}.$$

Since $\theta = \frac{\pi}{2}$, we have $\lambda^2_{(0)} = \frac{d\theta}{ds} = 0$. Thus, $\lambda^2_{(1)} = 0$. Similarly, since $\varphi = 0$, we have $\lambda^3_{(0)} = \lambda^3_{(1)} = 0$.

Since the $\lambda_{(\alpha)}$'s form an orthonormal basis, we have

$$\left(1 - \frac{r_s}{r}\right)\lambda^0_{(0)}\lambda^0_{(1)} - \left(1 - \frac{r_s}{r}\right)^{-1}\lambda^1_{(0)}\lambda^1_{(1)} = 0, \tag{40}$$

$$\left(1 - \frac{r_s}{r}\right)\left(\lambda^0_{(0)}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\left(\lambda^1_{(0)}\right)^2 = 1, \tag{41}$$

$$\left(1 - \frac{r_s}{r}\right)\left(\lambda^0_{(1)}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\left(\lambda^1_{(1)}\right)^2 = -1. \tag{42}$$

Next, we write $\lambda^0_{(0)}, \lambda^0_{(1)}, \lambda^1_{(1)}$ in terms of $\lambda^1_{(0)}$. From (41), we get

$$\lambda^0_{(0)} = \sqrt{\left(\frac{r}{r - r_s}\right)\left(1 + \frac{r}{r - r_s}\left(\lambda^1_{(0)}\right)^2\right)}. \tag{43}$$

From (42), we get

$$\lambda^0_{(1)} = \sqrt{\left(\frac{r}{r - r_s}\right)\left(-1 + \frac{r}{r - r_s}\left(\lambda^1_{(1)}\right)^2\right)}. \tag{44}$$

Substituting (43) and (44) into (40), we get

$$\lambda^1_{(1)} = \sqrt{1 - \frac{r_s}{r} + \left(\lambda^1_{(0)}\right)^2}. \tag{45}$$

Substituting (45) into (44) yields

$$\lambda^0_{(1)} = \frac{r}{r - r_s}\lambda^1_{(0)}. \tag{46}$$

Equation (25), with $\alpha = 0, \mu = 1$, is, after dividing by c^2,

$$\frac{d\lambda^1_{(0)}}{ds} + \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)\left(\lambda^0_{(0)}\right)^2 - \frac{r_s}{2r(r - r_s)}\left(\lambda^1_{(0)}\right)^2 = \frac{\kappa}{c^2}\lambda^1_{(1)}. \tag{47}$$

Substituting (43) and (45) into this equation, we obtain

$$\left\{ \begin{array}{c} \frac{d\lambda^1_{(0)}}{ds} + \frac{r_s}{2r^2} - \frac{\kappa}{c^2}\sqrt{1 - \frac{r_s}{r} + \left(\lambda^1_{(0)}\right)^2} = 0 \\ \\ \frac{dr}{ds} = \lambda^1_{(0)} \end{array} \right\}, \tag{48}$$

or, equivalently,

$$\frac{d^2r}{ds^2} + \frac{r_s}{2r^2} - \frac{\kappa}{c^2}\sqrt{1 - \frac{r_s}{r} + \left(\frac{dr}{ds}\right)^2} = 0. \tag{49}$$

In the particular case $\kappa = 0$, then, writing $\dot{r}$ for $\frac{dr}{ds}$ and $\ddot{r}$ for $\frac{d^2r}{ds^2}$, equation (49) becomes

$$\ddot{r} + \frac{r_s}{2r^2} = 0. \tag{50}$$

Multiplying by $2\dot{r}$, we obtain

$$2\dot{r}\ddot{r} + \frac{\dot{r}r_s}{r^2} = 0. \tag{51}$$

Integrating, we obtain

$$\dot{r}^2 - \frac{r_s}{r} = \text{ constant.} \tag{52}$$

The quantity $E = \dot{r}^2 - \frac{r_s}{r}$ is the *total dimensionless energy*. It is the total energy divided by the maximal kinetic energy $\frac{mc^2}{2}$. Hence, the total energy is conserved, as expected along a geodesic.

We consider now the general case of equation (49) ($\kappa \neq 0$). Differentiating E by t, we have

$$\dot{E} = 2\dot{r}\ddot{r} + \frac{\dot{r}r_s}{r^2}. \tag{53}$$

Dividing by $2\dot{r}$ and using (49), we obtain

$$\frac{\dot{E}}{2\dot{r}} = \frac{\kappa}{c^2}\sqrt{1 + E}. \tag{54}$$

Separating variables and integrating, we have

$$\sqrt{1 + E} = \frac{\kappa r}{c^2} + C, \tag{55}$$

where C is a constant of integration. Squaring and using the definition of E, we obtain

$$\dot{r}^2 = \frac{r\left(\frac{\kappa r}{c^2} + C\right)^2 + r_s - r}{r}. \tag{56}$$

We now show that there are no *bounded orbits*. Define

$$f(r) = r\left(\frac{\kappa r}{c^2} + C\right)^2 + r_s - r. \tag{57}$$

To have a bounded orbit, say between r_1 and r_2, with $0 < r_1 < r_2$, we must have $f(r_1) = f(r_2) = 0$ and $f(r) > 0$ for $r_1 < r < r_2$. However, $f(r)$ is a cubic polynomial, $f(0) > 0$ and $\lim_{r\to\infty} f(r) = +\infty$. This implies that f has at most two zeroes for $r > 0$ and between these two zeroes, $f(r) < 0$. Hence, there are no bounded orbits.

Figure 1 compares solutions for $r(s)$ in flat spacetime and Schwarzschild spacetime. Here, the Schwarzschild radius $r_s = 3000 \ km$. In order to see the difference between the trajectories in flat spacetime and Schwarzschild spacetime, we must use relatively small values of r, since, for large r, the Schwarzschild metric is approximately flat. Thus, we take $r(0) = 2r_s$. Similarly, we use a high acceleration of $\kappa = 8 \times 10^{16} \ ms^{-2}$. Otherwise, the trajectories will be indistinguishable. The need for high acceleration can be seen from the first equation of (48), in which κ is divided by c^2. We take the initial velocity to be $\dot{r}(0) = 0$.

For small values of s, the curves are indistinguishable. For larger s, the curves start to separate, reflecting the difference between flat and curved spacetime. Comparing curves (b) and (d) (respectively, (c) and (e)), we see that an object in Schwarzschild spacetime moves away from

(respectively, toward) the attracting mass more slowly than in flat spacetime. Nevertheless, the trajectories are asymptotically parallel. To check this, first use (56) to compute

$$\lim_{r\to\infty}\frac{\dot{r}}{r} = \sqrt{\frac{r\left(\frac{\kappa r}{c^2}+C\right)^2 + r_s - r}{r^3}} = \frac{\kappa}{c^2}. \tag{58}$$

For the flat spacetime solution, we have

$$r(s) = \frac{c^2}{\kappa}\left(\cosh\left(\frac{\kappa s}{c^2}\right)-1\right) + 2r_s. \tag{59}$$

Hence, $\dot{r}(s) = \sinh\left(\frac{\kappa s}{c^2}\right)$. Since $\lim_{x\to\infty}\tanh(x)=1$, we have

$$\lim_{s\to\infty}\frac{\dot{r}}{r} = \frac{\kappa}{c^2}. \tag{60}$$

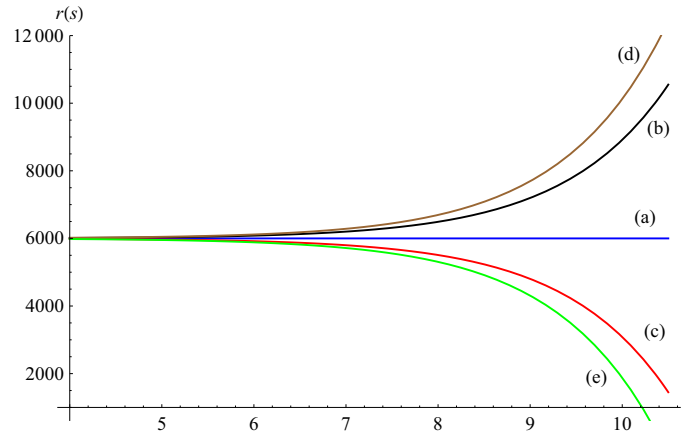**Figure 1.** Solutions for $r(s)$, with $r_s = 3000$m, $r(0) = 2r_s$, $\dot{r}(0) = 0$: (a) $\kappa = 0$ (a geodesic) (b) $\kappa = 8\times 10^{16} ms^{-2}$, compared to flat spacetime solution (d), (c) $\kappa = -8\times 10^{16} ms^{-2}$, compared to flat spacetime solution (e)

7. Discussion

Uniformly accelerated systems are important because only in *these* systems can all of the rest clocks in the system be synchronized to each other. It was shown in [13] that the rate of a rest clock in a uniformly accelerated system is constant in time but varies with position within the system. The time dilation between rest clocks at different positions can be explained by the difference in the potential energy between the two positions. This is similar to the known "gravitational time shift" for inertial systems with gravity, which, by the Equivalence Principle, are equivalent to accelerating systems. In a uniformly accelerated system, we may synchronize two rest clocks by sending two light signals back and forth between the two clocks. The first light signal is used to synchronize the common $t = 0$, and the second light signal is used to adjust for the different *rates* of the clocks. However, once the *rates* have been synchronized, they will remain synchronized, because the rate of each clock is constant. If the system is *not* uniformly accelerated, then, in general, the rate of each rest clock *changes* with time. Therefore,

it will not be enough to synchronize the rates of two clocks only once since their relative time dilation changes with time.

We plan to extend the example in Schwarzschild spacetime to include planetary motion. One may consider the gravitational pull of, say, Jupiter, on the Earth to be uniform acceleration. A solution of our equations would then predict the perturbation on the Earth's orbit caused by Jupiter. Alternatively, one could predict the perturbation of the Moon's orbit around the Earth caused by the Sun.

Our theory of uniform acceleration may prove useful in both *Relativistic Newtonian Dynamics (RND)* [20, 21, 22] and *Extended Relativity (ER)* [23, 24]. The dynamics of Special Relativity describes the influence of velocity, or *kinetic energy*, on spacetime. RND, on the other hand, modifies Newtonian dynamics by considering the effect on spacetime due to *potential energy*. This theory has successfully and accurately predicted the precession of Mercury *without* General Relativity [20]. RND also handles the trajectories of binary stars and accurately predicts the Hulse-Taylor pulsar's periastron advance [22]. Currently, however, we have only the three-dimensional version of RND. We plan to apply the ideas of [12] to derive a fully Lorentz covariant 4D version.

ER extends Special Relativity by examining the influence of high accelerations on relativistic dynamics. *ER* has been successfully applied to both the hydrogen atom [23] and the harmonic oscillator [24]. Thus far, however, only one-dimensional acceleration has been treated. The next step is to extend *ER* to full Lorentz covariance by treating first the case of *rotations*.

Rotations, even at constant angular velocity, are not trivial. The search for the spacetime transformations between a disk, rotating with constant angular velocity, and a (non-rotating) inertial lab frame has a long and rich history, dating back to Einstein and continuing to the present day. The recent book [25] makes it clear that *there is still no universally accepted theory.* More specifically, as pointed out in [26], a good and complete theory must have a *global metric* and a *unique time at each event.* There is no such theory known today which also agrees with recent experiments [27, 28]. Moreover, many of the current approaches make arbitrary assumptions about the form of the transformation of the *radial* coordinate. There is no theory today which derives the transformations from first principles.

We are currently applying our theory of uniform acceleration to the case of rotations. We hope to obtain explicit spacetime transformations using only the basic tenets of Special Relativity, the inherent symmetries of the problem, and the results of [10, 11, 13]. We conjecture that we can avoid both the *time gap* and the *horizon problem*.

In [10, 11, 13], working in flat spacetime, we derived spacetime transformations, velocity transformations, and acceleration transformations from a uniformly accelerated system to an inertial frame. We plan to extend these transformations to *curved* spacetime. We also want to determine whether the spacetime transformations between uniformly accelerated systems form a *group*. If yes, we want to characterize this group, which will be an extension of the Lorentz group.

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