

Self consistent models of deformed neutron stars in the framework of general relativity

Omair Zubairi¹ and Fridolin Weber^{2,3}

¹ Department of Sciences, Wentworth Institute of Technology, 550 Huntington Ave Boston, MA 02115, USA

² Department of Physics, San Diego State University, 5500 Campanile Drive, San Diego, CA 92182, USA

³ Center for Astrophysics and Space Sciences, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

E-mail: zubairio@wit.edu¹, fweber@mail.sdsu.edu^{2,3}

Abstract. Generally, over the last 75 years, since the publication of the well known papers from R. C. Tolman (Tolman 1939) and J. R. Oppenheimer and G. M. Volkoff (Oppenheimer and Volkoff 1939), standard models of non-rotating neutron stars are modeled with the assumption that they are perfect spheres. This assumption of perfect spherical symmetry is not correct if the matter inside of neutron stars is described by a non-isotropic equation of state (EoS). Particular classes of neutron stars such as Magnetars and neutron stars that contain color superconducting quark matter cores are expected to be deformed making them oblong spheroids. In this paper, we examine the deformity of these non-spherical neutron stars by deriving the stellar structure equations in the framework of general relativity. Using a non-isotropic model for the equation of state, these stellar structure equations are solved numerically in two dimensions. We then calculate stellar properties such as masses and radii along with pressure and energy-density profiles and investigate any changes from conventional spherical models.

1. Introduction

Non-rotating neutron stars are generally assumed to be perfect spheres, whose stellar properties such as masses and radii are described in the framework of general relativity by the Tolman-Oppenheimer-Volkoff (TOV) equation [1, 2]. Using a given model for the equation of state (EoS), describing the interior composition of a neutron star, this first order differential equation can be solved. However, the assumption of perfect spherical symmetry may not always be correct. It is well known that magnetic fields are present inside of neutron stars. In particular, if the magnetic field is strong (up to around 10^{18} Gauss in the core), such as for magnetars [3, 4, 5], and/or the pressure of the matter in the cores of neutron stars is non-isotropic, as predicted by some models of color superconducting quark matter [6], then deformation of neutron stars can occur [6, 7, 8, 9, 10, 11, 12].

In this work, we investigate the stellar structure of non-rotating deformed neutron stars by deriving TOV like stellar structure equations in the framework of general relativity whose mathematical form is similar to the conventional TOV equation for spherical symmetric stars. We then solve these equations numerically in two dimensions for a given EoS model and produce



stellar properties such as masses and radii along with pressure and energy-density profiles and investigate any changes from spherical models of neutron stars.

2. Non-Spherical Symmetry

In order to study deformed neutron stars, illustrated in Fig. 1, we first need to investigate the symmetry of these objects. Our assumption of deformity will lie within the geometry of these

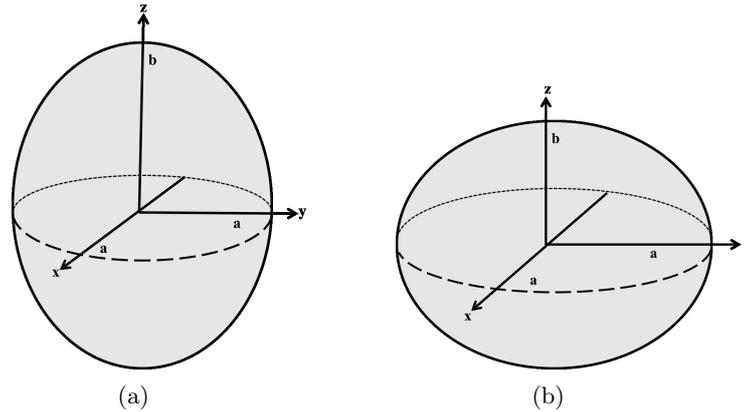


Figure 1. Schematic illustration of the geometry of an prolate (a) and oblate (b) spheroid.

objects. As shown in Fig. 1, the axial symmetry lies in the polar axis (z) being orthogonal with the equatorial axis ($x - y$). Mathematically, this can be described by the well known Weyl metric given by [13, 14],

$$ds^2 = e^{2\lambda} dt^2 - e^{-2\lambda} [e^{2\nu} (dr^2 + dz^2) + r^2 d\phi^2] = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

where t is the time component and r , z , and ϕ are the spatial components in cylindrical coordinates. The terms λ and ν are the unknown metric functions that depend on both the radial r and polar z directions such that $\lambda = \lambda(r, z)$ and $\nu = \nu(r, z)$.

Due to the fact that we have distinct polar and radial directions which are both coupled together, as described in Eq. (1), the hydrostatic equilibrium of deformed neutron stars will have to be in two dimensions. Having two dimensional stellar structure equations will allow us to utilize a model for the equation of state of neutron star matter that takes the pressure in both the radial and polar directions into considerations. Our first step is examine each component of Eq. (1) and determine which one makes any contributions to the stellar structure. For this purpose, we will need to make a few assumptions about the stellar configuration. First, we assume that the configuration is non-rotating and not pulsating. Second, we assume that the star is axially symmetric, as shown in Fig. 1. Hence, the only terms that make a contributions to the stellar configuration will be the radial and polar components.

3. A Parameterized Solution

One way of investigating deformity of neutron stars is by applying a parametrization on the polar radius in terms of the equatorial radius along with a deformation constant γ on the metric described by Eq. (1). Such a metric reads as

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-\gamma} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \quad (2)$$

described in [15]. In Eq. (2), γ determines the degree of deformation either in the oblate ($\gamma < 1$) or prolate case ($\gamma > 1$) (see figure 1), where the parametrization is described by $z = \gamma r$. Using Eq. (2), a parameterized TOV equation is then obtained [15]

$$\frac{dP}{dr} = - \frac{(\epsilon + P) \left(\frac{1}{2}r + 4\pi r^3 P - \frac{1}{2}r \left(1 - \frac{2m}{r}\right)^\gamma \right)}{r^2 \left(1 - \frac{2m}{r}\right)^\gamma}. \quad (3)$$

In the case when $\gamma = 1$, Eq. (3) simplifies to the well-known Tolman-Oppenheimer-Volkoff equation [1, 2] which describes the stellar structure of perfectly spherically symmetric object. The gravitational mass of a deformed neutron star parameterized with the deformation constant γ is then described by

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \gamma, \quad (4)$$

so that the total gravitational mass, M , of a deformed neutron star with an equatorial radius R follows as [14, 15] $M = \gamma m(R)$. For a given deformation, γ , this is a very useful and easy-to-use model for the description of deformed compact stars. Using a given isotropic EoS model, we numerically solve these parameterized stellar structure equations and generate stellar properties such as masses and radii along with pressure and energy-density profiles. Extending the work done by [15, 16], we include two more models for the EoS for a total of three isotropic models [17, 18, 19, 20, 21]. The nuclear properties which describe the key differences of each model is summarized in Table 1.

Table 1. Isotropic Equation of state models studied in this work.

Equation of state	Nuclear Composition	Stellar object
MIT Bag Model	u, d, s quarks	Strange Quark Star
Standard Baryonic Model	n, p, e, Σ , Λ , Ξ	Neutron Star
Quark-Hadron Model	n, p, e, Σ , Λ , Ξ , u, d, s quarks	Neutron Star

Using these three different EoS models, we numerically solve Eqs. (3) and (4) and produce mass-radius relations shown in Fig. 2. Using our parameterization of $z = \gamma r$, we can see a trend in all three EoS models—that is we see an increase in mass with increasing oblateness and a decrease in mass with increasing prolateness. When our deformation parameter is equal to one, we obtain the spherically symmetric solution back for original TOV equations.

In order to investigate the actual deformity (shape of the star), we calculate the pressure and energy-density profiles associated with the maximum masses shown in Fig. 2 and examine the convergence of the pressure for both the equatorial and polar directions. The results are illustrated in Fig. 3 for the maximum masses for the quark-hadron EoS model.

From our results shown in Fig. 3, we see the pressure vanishes at the surface for both equatorial and polar directions indicating the distinct shape of the star (oblate or prolate). Using this parameterized model gives us some insight into deformity and shows that it is important and can not be ignored; it thus requires a more detailed description of in two dimensions.

4. Weyl Metric Calculations

In order to investigate the deformation in two dimensions, we need to examine the components of Eq. (1) and see which ones make any contributions to the stellar structure. Due to our assumptions of a non-rotating, axial symmetric stellar configuration, the only components that

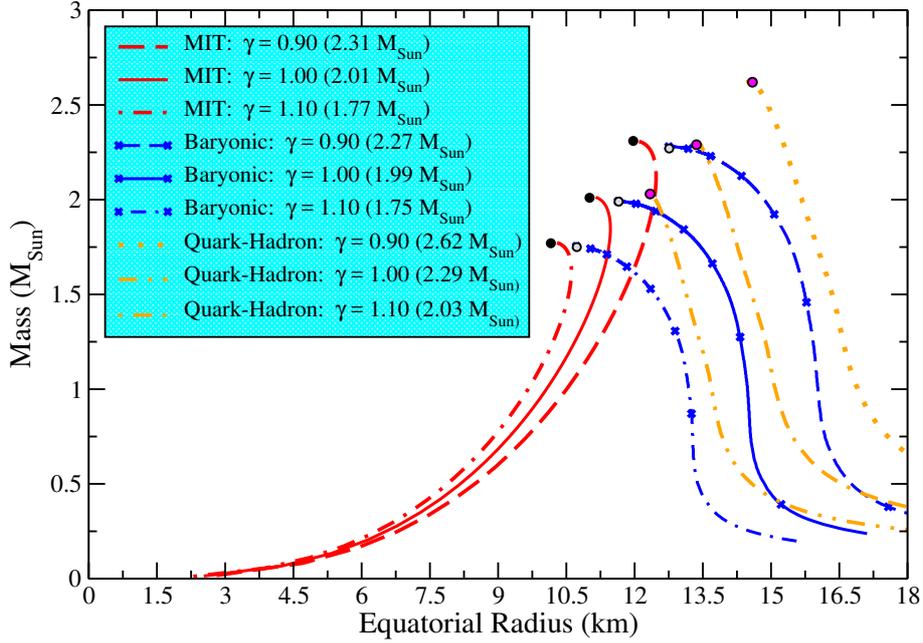


Figure 2. (Color-online) Mass-radius relationships calculated from Eqs. (3) and (4) for various isotropic nuclear equations of state (see also Refs. [22, 23]). The dots on each curve represent the maximum mass for each stellar sequence.

will make any contributions to the stellar structure of our deformed star will be the radial and polar components of Einstein's field equation,

$$G^{\mu}_{\nu} \equiv R^{\mu}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} R = -8\pi T^{\mu}_{\nu}, \quad (5)$$

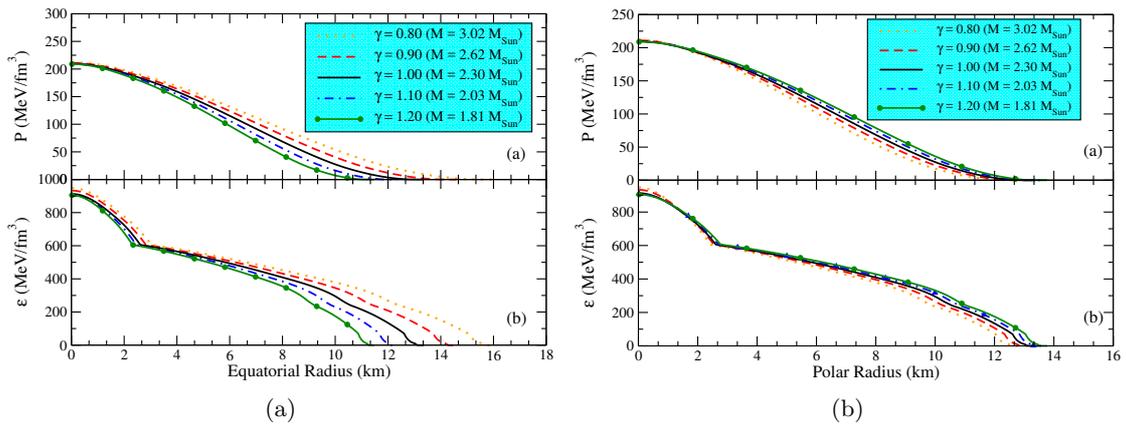


Figure 3. (Color-online) Pressure (top) and energy-density (bottom) profiles in the equatorial (left) and polar (right) direction for the maximum masses for the quark-hadron EoS model given in Fig. 2 (see also Refs. [22, 23]). As it is shown, a larger equatorial radius for $\gamma < 1$ results in a smaller polar radius thus giving us an oblate star and similarly when $\gamma > 1$, we get a larger polar radius than the equatorial radius resulting in a prolate star.

where G^μ_ν denotes the Einstein tensor, R^μ_ν is the Ricci tensor, R the scalar curvature, and T^μ_ν the energy-momentum tensor. We begin with the Weyl metric of Eq. (1), and calculate the Christoffel symbols to be

$$\Gamma^r_{tt} = \left[e^{-2\nu(r,z)+4\lambda(r,z)} \right] \partial_r \lambda(r, z), \quad (6)$$

$$\Gamma^z_{tt} = \left[e^{-2\nu(r,z)+4\lambda(r,z)} \right] \partial_z \lambda(r, z), \quad (7)$$

$$\Gamma^t_{tr} = \partial_r \lambda(r, z), \quad \Gamma^t_{tz} = \partial_z \lambda(r, z), \quad (8)$$

$$\Gamma^r_{rr} = \partial_r \nu(r, z) - \partial_r \lambda(r, z), \quad \Gamma^z_{rr} = \partial_z \lambda(r, z) - \partial_z \nu(r, z), \quad (9)$$

$$\Gamma^r_{rz} = \partial_z \nu(r, z) - \partial_z \lambda(r, z), \quad \Gamma^z_{rz} = \partial_r \nu(r, z) - \partial_r \lambda(r, z), \quad (10)$$

$$\Gamma^z_{rz} = \partial_r \nu(r, z) - \partial_r \lambda(r, z), \quad \Gamma^\phi_{r\phi} = (1 - r\partial_r \lambda(r, z)) / r, \quad (11)$$

$$\Gamma^r_{zz} = -\Gamma^r_{rr}, \quad \Gamma^z_{zz} = -\Gamma^z_{rr}, \quad \Gamma^\phi_{z\phi} = -\Gamma^t_{tz}, \quad (12)$$

$$\Gamma^r_{\phi\phi} = \left[e^{-2\nu(r,z)} \right] r (-1 + r\partial_r \lambda(r, z)), \quad (13)$$

$$\Gamma^z_{\phi\phi} = \left[e^{-2\nu(r,z)} \right] r^2 \partial_z \lambda(r, z), \quad (14)$$

where ∂_r and ∂_z denote partial derivatives with respect to r and z . Using these Christoffel symbols and from Eq. (5) we calculate the components of the Ricci tensor to be

$$R^t_t = \frac{1}{r} e^{-2\nu+2\lambda} (r\partial_r^2 \lambda + r\partial_z^2 \lambda + \partial_r \lambda), \quad (15)$$

$$R^r_r = -\frac{1}{r} e^{-2\nu+2\lambda} \left(r\partial_r^2 \lambda - 2(\partial_r \lambda)^2 - r\partial_z^2 \nu + r\partial_z^2 \lambda - r\partial_r^2 \nu + \partial_r \lambda + \partial_r \nu \right), \quad (16)$$

$$R^z_r = R^r_z = \frac{1}{r} e^{-2\nu+2\lambda} \left(2r(\partial_r \lambda)^2 - \partial_z \nu \right), \quad (17)$$

$$R^z_z = -\frac{1}{r} e^{-2\nu+2\lambda} \left(r\partial_r^2 \nu - 2(\partial_z \lambda)^2 + r\partial_r^2 \lambda - r\partial_z^2 \nu + r\partial_z^2 \lambda - \partial_r \nu + \partial_r \lambda \right), \quad (18)$$

$$R^\phi_\phi = \frac{1}{r} e^{2\nu+2\lambda} (r\partial_r^2 \lambda + r\partial_z^2 \lambda + \partial_r \lambda), \quad (19)$$

where ∂_r^2 and ∂_z^2 denote second partial derivatives with respect to r and z . The Ricci scalar R is then calculated to be

$$R = -\frac{1}{r} \left[2e^{-2\nu+2\lambda} \left(r\partial_r^2 \lambda - r(\partial_r \lambda)^2 - r\partial_z^2 \nu + r\partial_z^2 \lambda - r\partial_r^2 \nu + \partial_r \lambda - r(\partial_z \lambda)^2 \right) \right], \quad (20)$$

Using Eqs. (15) through (20), one obtains for the components of the Einstein tensor the following expressions,

$$G^t_t = \frac{1}{r} \left[e^{-2\nu+2\lambda} \left(2r\partial_r^2 \lambda + 2r\partial_z^2 \lambda + 2\partial_r \lambda - r(\partial_r \lambda)^2 - r\partial_z^2 \nu - r\partial_r^2 \nu - r(\partial_z \lambda)^2 \right) \right], \quad (21)$$

$$G^r_r = \frac{1}{r} e^{-2\nu+2\lambda} \left(r(\partial_r \lambda)^2 - \partial_r \nu - r(\partial_z \lambda)^2 \right), \quad (22)$$

$$G^z_r = G^r_z = \frac{1}{r} e^{-2\nu+2\lambda} ((2r\partial_r\lambda)(\partial_z\lambda) - \partial_z\nu), \quad (23)$$

$$G^z_z = -\frac{1}{r} e^{-2\nu+2\lambda} \left(r(\partial_r\lambda)^2 - \partial_r\nu - r(\partial_z\lambda)^2 \right), \quad (24)$$

$$G^\phi_\phi = -e^{-2\nu+2\lambda} \left((\partial_r\lambda)^2 + \partial_z^2\nu + \partial_r^2\nu + (\partial_z\lambda)^2 \right). \quad (25)$$

Due to the mathematical form of Eq. (1), there will be off-diagonal terms $T^r_z = T^z_r \neq 0$ in the energy-momentum tensor,

$$T^\mu_\lambda = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_\parallel & \tilde{P} & 0 \\ 0 & \tilde{P} & P_\perp & 0 \\ 0 & 0 & 0 & \tilde{P} \end{pmatrix}, \quad (26)$$

where ϵ is the energy-density, P_\parallel is the pressure in the radial direction, and P_\perp is the pressure in the polar direction. The \tilde{P} terms are the off-diagonal pressure contributions associated with the r - z direction of Eq. (1).

Before evaluating Einstein's field equation (Eq. (5)), we need to define expressions for the total gravitational mass. In cylindrical coordinates, the mass in the equatorial and polar directions will depend on both r and z coordinates such that the gravitational mass has the form of $m = m(r, z)$.

4.1. Mass Formalism

Since our symmetry is axial symmetric, we define two differential masses as shown in Fig. 4, where one of the differential masses corresponds to an infinitesimal cylinder of radius r , height z , and thickness dr , and a second differential mass for an infinitesimal slab of radius r and thickness dz . The differential masses shown in Fig. 4 are then described mathematically by the

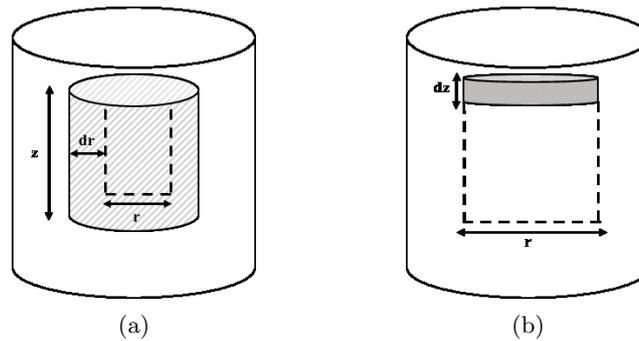


Figure 4. A graphical illustration of the differential masses in cylindrical coordinates for cross-sectional masses in the equatorial direction (a) and in the polar direction (b).

expressions,

$$\partial_r m(r, z) = 2\pi r z \epsilon(r, z), \quad (27)$$

$$\partial_z m(r, z) = \pi r^2 \epsilon(r, z). \quad (28)$$

Using Eqs. (27) and (28), we can now utilize Einstein's field equation $G^t_t = -8\pi T^t_t$, and thus Eq. (27) can be written as

$$\partial_r m(r, z) = \frac{rz}{4} G^t_t. \quad (29)$$

Using the other expressions for the Einstein tensor of Eqs. (21) through (25) along with the expression for the energy-momentum tensor given in Eq. (26) leads for Eq. (29) to

$$\partial_r m(r, z) = \partial_r \left(-e^{-2\nu+2\lambda} z (\partial_r \lambda) (\partial_r \nu - \partial_r \lambda) \right), \quad (30)$$

Applying an integral on both sides of Eq. (30), we obtain an expression that takes the form

$$\int_0^r (\partial_r m(r, z)) dr = - \int_0^r \partial_r \left(e^{-2\nu+2\lambda} z (\partial_r \lambda) (\partial_r \nu - \partial_r \lambda) \right) dr. \quad (31)$$

By examining Eq. (31), is clear that there is a serious inconsistency with the r -component, as the r -term has vanished from Eq. (31). In addition to the r -term vanishing, the unknown metric functions λ'_r and ν'_r are still present in Eq. (31); therefore, we can not solve for the r component ($e^{-2\nu+2\lambda}$) directly. Having no r -term indicates that there is no interior solution of Einstein's field equations that will match the exterior solution for the metric described by Eq. (1) and thus we have lost all the information about the radial coordinate of our deformed compact object.

5. A Self Consistent Model of Deformation

Due to the fact the all the information of the radial coordinate is lost from Eq. (31), we need another way to describe deformity. One way to investigate deformation is to look further into our parameterized one dimensional model.

Using our parametrization of the polar coordinate in terms of the radial coordinate $z = \gamma r$ and assuming our radial and polar step sizes are small (on the order of meters), we can look at the deformation constant γ as a differential ratio of the polar and radial step sizes described by

$$\gamma = \frac{dz}{dr}. \quad (32)$$

Hence, we can use Eq. (32), along with the parametrization $z = \gamma r$, and apply a transformation on Eq. (3) to obtain a relationship between pressure and the polar direction. Using Eq. (32), we find that

$$\frac{dP}{dz} = - \frac{(\epsilon + P) \left(\frac{z}{2\gamma} + 4\pi \left(\frac{z}{\gamma} \right)^3 P - \frac{z}{2\gamma} \left(1 - \frac{2m\gamma}{z} \right)^\gamma \right)}{\frac{z^2}{\gamma^3} \left(1 - \frac{2m\gamma}{z} \right)^\gamma}, \quad (33)$$

where now we can take distinct pressure gradients into account by assuming that the parallel pressure (P_{\parallel}) will be associated with dr and the perpendicular pressure (P_{\perp}) will be associated with dz . In addition, we will also have to take a look at the total gravitational mass m in Eqs. (3) and (33), since, both Eqs. (3) and (33) need to be coupled together. For the total gravitational mass, we utilize our mass elements described by

$$\partial_r m(r, z) = 2\pi r z \epsilon(r, z), \quad (34)$$

$$\partial_z m(r, z) = \pi r^2 \epsilon(r, z). \quad (35)$$

If we were to integrate then add the expressions given in Eqs. (34) and (35), we would obtain some total gravitational mass described as

$$\begin{aligned} m_{\text{total}}(r, z) &= 2\pi \epsilon(r, z) r^2 z, \\ &= 2m(r, z). \end{aligned} \quad (36)$$

However the expression for $m_{\text{total}}(r, z)$ does not correctly define the gravitational mass of a oblong spheroid, which is given by

$$M(r, z) = \frac{4}{3}\pi\epsilon(r, z)r^2z. \quad (37)$$

In order to obtain Eq. (37), we must subtract off another cross-sectional term from Eq. (36) that will insure the gravitational confinement of the star, leading to

$$\begin{aligned} M(r, z) &= 2\pi\epsilon(r, z)r^2z - \frac{2}{3}\pi\epsilon(r, z)r^2z, \\ &= 2m(r, z) - \frac{2}{3}m(r, z), \end{aligned} \quad (38)$$

however, we must break up the top expression in Eq. (38) equally to take the mass in the equatorial and polar directions separately but that are confined and result in a mass described by Eq. (37). Using the expressions from Eqs. (34) and (35) we find the gravitational mass to be

$$\begin{aligned} \mathcal{M}(r, z) &= 2\pi\epsilon(r, z)rz + \pi\epsilon(r, z)r^2 - \frac{1}{3}\pi\epsilon(r, z)r^2z, \\ &= \partial_r m(r, z) + \partial_z m(r, z) - \frac{1}{3}\pi\epsilon(r, z)r^2z, \end{aligned} \quad (39)$$

where we can define

$$\begin{aligned} \partial_r m(r, z) &= m_{\parallel}(r, z), \\ \partial_z m(r, z) &= m_{\perp}(r, z). \end{aligned} \quad (40)$$

The total gravitational mass is then given by

$$\mathcal{M}_{\text{total}}(r, z) = 2\pi\epsilon(r, z)r^2z - \frac{2}{3}\pi\epsilon(r, z)r^2z. \quad (41)$$

Using the expressions given in Eq. (39), we make substitutions into Eqs. (3) and Eq. (33) along with the substitutions for the parallel and perpendicular pressures and find that the hydrostatic equilibrium equations modify to

$$\frac{\partial P_{\parallel}}{\partial r} = -\frac{(\epsilon + P_{\parallel})\left(\frac{1}{2}r + 4\pi r^3 P_{\parallel} - \frac{1}{2}r\left(1 - \frac{2\mathcal{M}(r, z)}{r}\right)^{\gamma}\right)}{r^2\left(1 - \frac{2\mathcal{M}(r, z)}{r}\right)^{\gamma}}, \quad (42)$$

$$\frac{\partial P_{\perp}}{\partial z} = -\frac{(\epsilon + P_{\perp})\left(\frac{z}{2\gamma} + 4\pi\left(\frac{z}{\gamma}\right)^3 P_{\perp} - \frac{z}{2\gamma}\left(1 - \frac{2\mathcal{M}(r, z)\gamma}{z}\right)^{\gamma}\right)}{\frac{z^2}{\gamma^3}\left(1 - \frac{2\mathcal{M}(r, z)\gamma}{z}\right)^{\gamma}}, \quad (43)$$

where now the expressions given in Eqs. (42) and (43) are coupled together with the gravitational mass $\mathcal{M}(r, z)$ all governed by distinct parallel (P_{\parallel}) and perpendicular (P_{\perp}) pressure gradients. It is also important to note that now we do not need to deform the star by γ . For all of our calculations, we fixed γ , so that $\gamma = 1$. Having this fixed constant will allow the anisotropies in the equation of state to dictate the deformation.

Our self consistent two dimensional parameterized stellar model will involve solving the expressions in Eqs. (42) and (43) in conjunction with the expressions given in Eqs. (34) and

(35). The surface of the star will be defined when the pressure gradients both in the parallel and perpendicular directions vanish. That is, we will define the surface of the star by

$$P_{\parallel}(r = R) = 0, \quad (44)$$

$$P_{\perp}(z = Z) = 0, \quad (45)$$

where the total gravitational mass of the object is given by

$$M_{\text{Total}}(R, Z) = \mathcal{M}_{\text{total}}. \quad (46)$$

In order to solve our coupled system of equations, we now need an equation of state which has distinct pressure gradients in the parallel and perpendicular directions. The next section briefly describes the equation of state used in these two dimensional calculations.

6. Anisotropic Equation of State

In the two dimensional model, we modify the quark-hadron equation of state by changing the pressure by various amounts greater than and less than the spherical case. Modifying the equation of state this way will insure some symmetry in our calculations. We took data presented in [21], and changed the pressure terms by increasing the pressure by 4, 8, and 12 percent. We also decreased the pressure by 4, 8, and 12 percent as illustrated in Fig. 5. Using the data shown

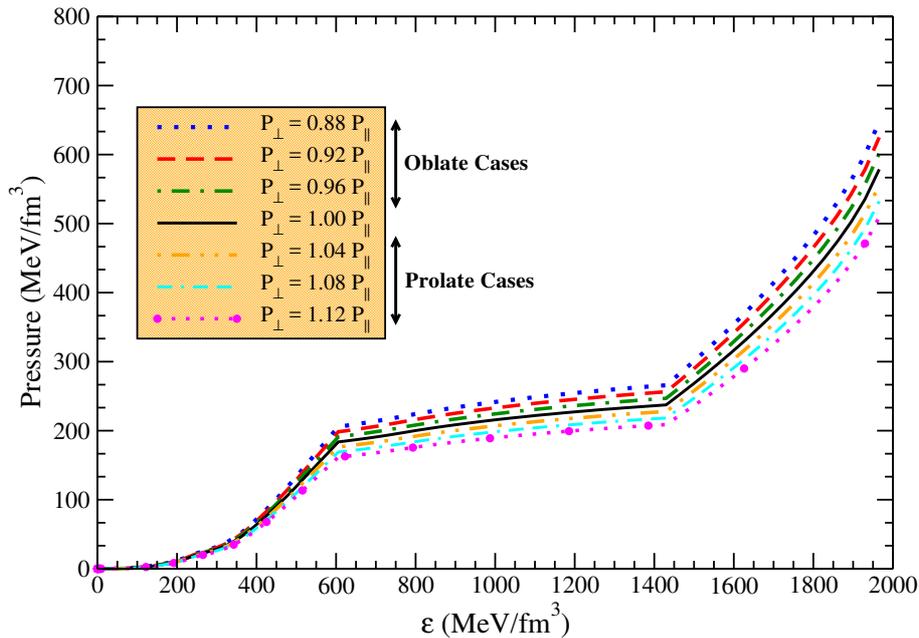


Figure 5. (Color-online) Anisotropic relationship of pressure and energy-density for a quark-hadron equation of state. In this model, the core is comprised of a mixed phase of up, down and strange quarks and hadronic matter. The pressure is varied by 4, 8, and 12 percent to mimic anisotropies in the EoS (for more details, see Refs. [22, 23]), thus giving us distinct pressure gradients.

in Fig. 5, we can now solve our two dimensional system of hydrostatic equilibrium for deformed neutron stars. Hence, with distinct pressure gradients, deformity should result; however, it should not be large. We have only deviated from the isotropic case by a few percent, and therefore the deformation of spheres to oblate and prolate stars should vary slightly.

7. Results and Conclusions

In contrast to traditional spherical models, solving our two dimensional model will require other strategies. In our two dimensional case, we cannot have mass-radius relationships to describe the maximum mass. This will not correctly give us any information of the deformation; however, we can however choose a central density and build our deformed star and calculate the pressure and energy-density profiles associated with that star. From the pressure profiles, we will be able to tell the deformity. Due to the non-isotropic equation of state, we can investigate the change in mass.

We begin with choosing a central density of $933 \text{ MeV}/\text{fm}^3$. This central density was chosen from the one-dimensional parameterized model for a spherical object [24]. Using this density, we calculate the pressure and energy-density profiles along with the total mass of the object. We first look at the pressure and energy-density profiles associated with increased pressures of 4, 8, and 12 percent. The results are shown in Figs. 6 and 7 respectively. The pressure profiles for each maximum mass star in Fig. 6 converge to zero at the stellar surfaces for both the equatorial and polar radii.

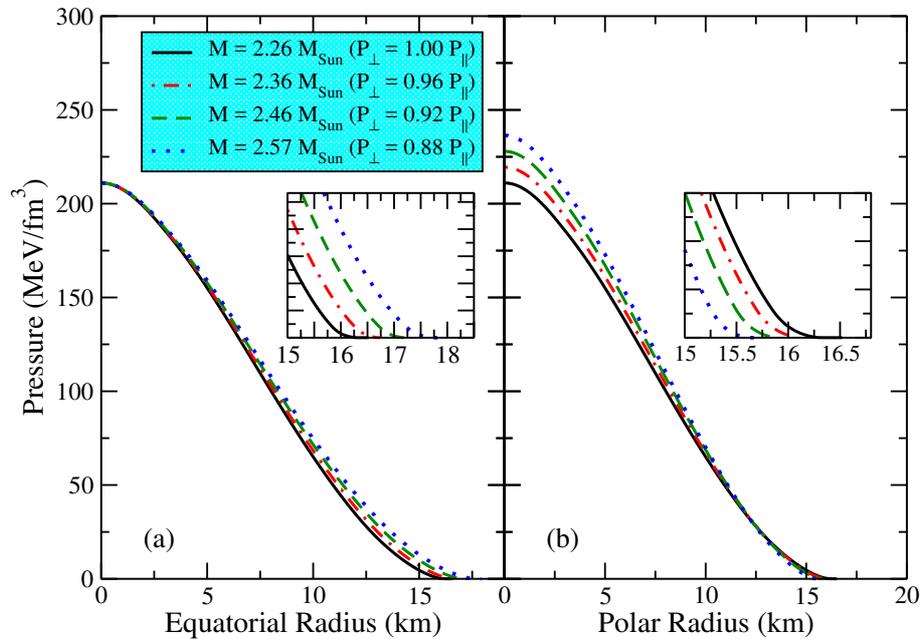


Figure 6. (Color-online) Pressure profiles for both the equatorial (a) and polar (b) directions for pressure gradients in the equatorial directions greater than 4, 8, and 12 percent (see also Refs. [22, 23]). The masses of such stars increase with oblateness.

As shown in Fig. 6, for no change in pressure, the mass is about $2.26 M_{\odot}$ which is in good agreement with results from [21, 24, 25] for the spherical case. From our calculations, the mass increases as the equatorial radii increases and the polar radius decreases. This increase in mass due to increased oblateness is consistent with our one dimensional parameterized model and validates our results presented using our parametrization. It is also important to note that from the pressure profiles shown in Fig. 6, the pressure decreases monotonically as the equatorial and polar radii increase to the surface and hence the pressure becomes zero as stated by our boundary conditions. The increase in mass is appreciable even though we have small changes in the pressure in our equation of state.

We next investigate the deformation in another way to obtain prolate spheroids and see if the masses decreases as we increase prolateness. We do this by calculating the pressure profiles

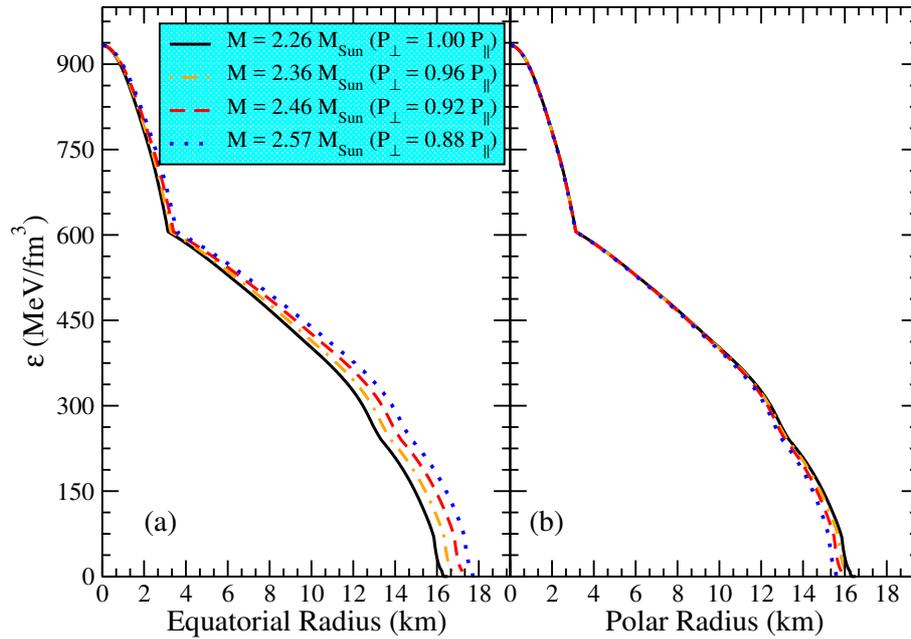


Figure 7. (Color-online) Energy-density profiles for both the equatorial (a) and polar (b) directions for pressure gradients in the equatorial directions (see Refs. [22, 23]) greater than the polar direction by 4, 8, and 12 percent.

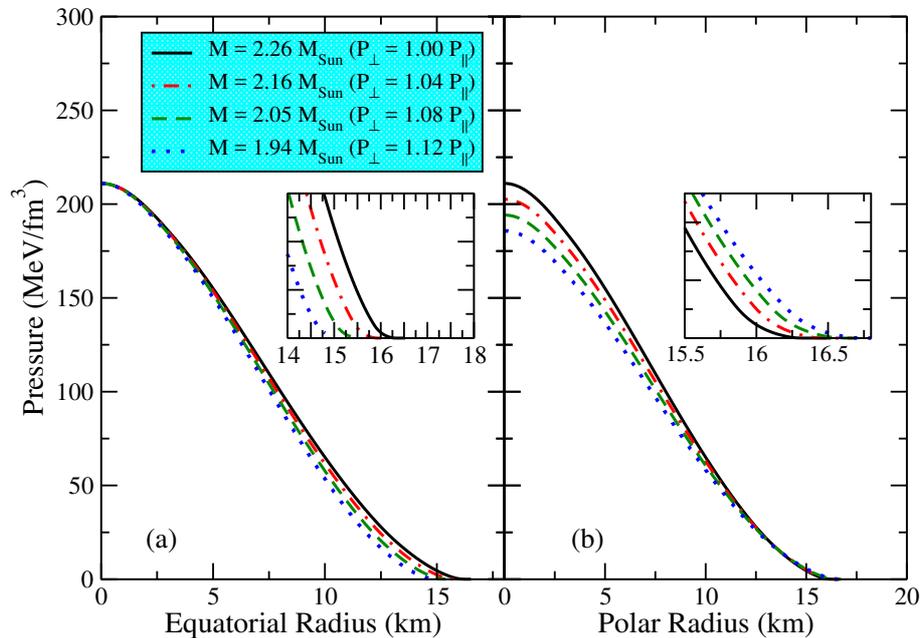


Figure 8. (Color-online) Pressure profiles for both the equatorial (a) and polar (b) directions for pressure gradients in the equatorial directions less than the polar direction by 4, 8, and 12 percent (see also Refs. [22, 23]). The masses of such stars decrease for increasing prolateness.

by decreasing the pressure by 4, 8 and 12 percent from our non-isotropic equation of state. However, we have to be mindful when applying the different pressure gradients. We do not

want to rotate the star by any means. Therefore, we decrease the pressure gradients in the equatorial directions, but keep the pressure in the polar direction fixed. This will insure we will obtain prolate spheroid but not an oblate spheroid just rotated by 90 degrees. The results of the pressure and energy-density profiles calculated this way are shown in Figs. 8 and 9 respectively. Just as in the oblate case, we see clearly that the mass decreases as we increase prolateness.

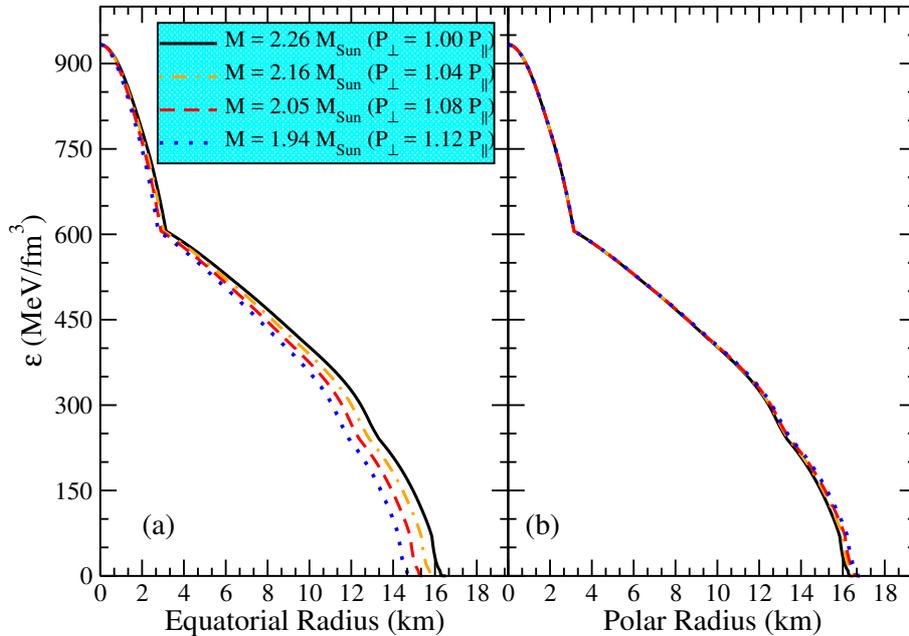


Figure 9. (Color-online) Energy-density profiles for both the equatorial (a) and polar (b) directions (see Refs. [22, 23]) for pressure gradients in the equatorial directions less than the polar direction by 4, 8, and 12 percent.

This is consistent with the results from [15, 24] for the one dimensional parameterized model for prolate stars. We see significant changes in mass due to increasing prolateness. It is important to note that the mass change (either increase or decrease) is roughly the same and it deviates slightly. This implies the two dimensional calculations are more sensitive to the anisotropies in the equation of state. A summary of the stellar properties resulting from our two dimensional model is provided in Table 2.

Table 2. Stellar properties of deformed neutron stars for the EoS given in Fig. 5.

Anisotropic EoS	R (km)	Z (km)	Total Mass (M_{\odot})	Shape
$P_{\perp} = 0.88P_{\parallel}$	17.92	15.77	2.57	Oblate Spheroid
$P_{\perp} = 0.92P_{\parallel}$	17.21	15.83	2.46	Oblate Spheroid
$P_{\perp} = 0.96P_{\parallel}$	16.73	16.06	2.36	Oblate Spheroid
$P_{\perp} = 1.00P_{\parallel}$	16.51	16.51	2.26	Sphere
$P_{\perp} = 1.04P_{\parallel}$	16.00	16.67	2.16	Prolate Spheroid
$P_{\perp} = 1.08P_{\parallel}$	15.39	16.73	2.05	Prolate Spheroid
$P_{\perp} = 1.12P_{\parallel}$	14.80	16.82	1.94	Prolate Spheroid

From our results given in Figs. 6 through 9 along with the values listed in Table 2, we clearly show that deformation plays a vital role in the stellar structure of neutron stars and we can not simply ignore this deformation. From our results, we conclude that anisotropies in the equation of state can deform the star either in the equatorial or polar directions resulting in both oblate or prolate spheroids. Depending on the deformation the mass could either increase or decrease. This change in mass could explain discrepancies in various equation of state models and help us better understand the internal structure of neutron stars.

Acknowledgments

This work was supported through the National Science Foundation under grant PHY-1411708. Additional computing resources were provided by the Computational Science Research Center at San Diego State University and the Department of Sciences at Wentworth Institute of Technology. Travel funding was provided by Wentworth Institute of Technology.

The authors would like to acknowledge Dr. Vivian de la Incera and Dr. Efrain Ferrer for their initial motivation on this work. The authors would also like to thank Dr. Martin Land for the invitation to speak at the biennial 2016 IARD conference.

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