Response of an orthotropic half-plane subjected to transient anti-plane loading with multiple edge cracks

M. Ayatollahi[1]

[1] Faculty of Engineering, University of Zanjan, P.O. Box 45195-313, Zanjan, Iran

Corresponding author's: <u>mo.ayatollahy@gmail.com</u>

Abstract. This study deals with the problem of multiple edge cracks in an elastic orthotropic half-plane under transient loading. The dislocation solution is utilized to derive integral equations for multiple interacting edge cracks in an orthotropic half-plane. These equations are solved numerically thereby obtaining the dislocation density function on the crack faces and stress intensity factors at crack tips. Numerical results are obtained to illustrate the variation of the dynamic stress intensity factors as a function of crack length and material properties.

1. Introduction

With the growing use of composites in great variety of engineering structures, in recent years the problems regarding their structured intensity and failure have been studied extensively. In the case of dynamic loads, two loading cases are of interest, harmonic loading and impact loading [1]. Various attempts have been made to analyze half-plane weakened by cracks under harmonic load. Impact loads applied on cracked structures may cause catastrophic failure. It is therefore, of great importance to investigate the transient response of cracked materials. The stationary semi-infinite crack under uniform step loading in the crack faces was considered first by Maue [2]. Loeber and Sih [3] investigated the dynamic stress intensity factor for a finite crack in the infinite plane under anti-plane deformation. Ma and Hou [4] investigated the response of an elastic solid containing a crack subjected to suddenly concentrated point loads acting at a finite distance from the crack tip. Wang et al. [5] obtained the dynamic stress intensity factor for mode III loading. They analyzed the anti-plane response of a non-homogeneous composite material containing several cracks subjected to dynamic impact. The failure behavior of fiber reinforced composites involving cracked matrix and imperfectly bounded fibers under dynamic anti-plane loading was investigated by Meguid et al. [6]. Results show the effect of the interaction between a main crack and a completely debounded fiber upon the dynamic stress intensity factors. The transient dynamic stress intensity factor was determined for an interface crack between two dissimilar isotropic viscoelastic bodies under impact loading by Wei et al. [7]. The transient response of an infinite orthotropic material with finite crack under point loading was studied by Rubio et al. [8]. Chen et al. [9] investigated the transient response of the internal crack in a functionally graded orthotropic strip. Results show the effects of the material parameter, the crack configuration and the orthotropic property on the dynamic stress intensity factors. Shul and Lee [10] considered a subsurface crack in a functionally graded coating layer on the layered half-space subjected to an anti-plane impact load. Lira-Vergaraand Rubio-Gonzalez [11] analyzed the transient response of an non-homogeneous orthotropic material with an interfacial finite crack under shear impact load. The problem of a homogeneous linear elastic body containing multiple collinear cracks under anti-plane dynamic load was considered by Wu et al. [12]. However, those previous solutions only considered the embedded or edge crack with simple shape and patterns. Monfared and Ayatollahi [13], investigated the scattering of anti-plane harmonic stress waves by multiple cracks in an orthotropic half-plane.

In this paper, we consider the problem of an orthotropic half-plane with multiple edge cracks The main purpose is toprovide an analytical treatment to investigate the transient behavior of orthotropic half-plane under transient load, which is important for the design and numerical simulation of composite structures. Based on the use of integral transforms, the problem is reduced to a singular integral equation, which can be solved using Chebyshev polynomial expansions. Numerical results are provided to show the effect of cracks interaction and material property on the stress intensity factors

2. Problem statements

We consider an orthotropic half-plane containing a screw dislocation under anti-plane deformation. A Cartesian coordinate is assumed in such a way that the x and y axes are taken as directions of principal material orthotropy and y-axis is along dislocation path. The constitutive equations for anti-plane problem of the orthotropic material are as follows:

$$\sigma_{zx}(x,y,t) = G_{zx}\frac{\partial w(x,y,t)}{\partial x}, \ \sigma_{zy}(x,y,t) = G_{zy}\frac{\partial w(x,y,t)}{\partial y}. \tag{1}$$

where G_{zx} and G_{zy} are elastic constants. In the absence of body force the equation of motion is:

$$\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{1}{f^2}\frac{\partial^2 w(x,y,t)}{\partial y^2} = s_l^2\frac{\partial^2 w(x,y,t)}{\partial t^2}. \tag{2}$$

where s_l^2 is wave slowness defined by $s_l = 1/c_l$ and c_l being the shear wave speed, $f = \sqrt{G_{zx}/G_{zy}}$, $c_l = \sqrt{G_{zx}/\rho}$ and ρ is the mass density. The conditions representing a Volterra-type screw dislocation located at the positive part of the y-axis in an orthotropic half-plane are

$$w(0^+,y,t) - w(0^-,y,t) = b_z(t)H(y), \sigma_{zx}(0^+,y,t) = \sigma_{zx}(0^-,y,t), \sigma_{zy}(x,h,t) = 0., \tag{3}$$

where, $H(.)$ is the Heaviside step function, $b_z(t)$ is the time dependent Burgers vector. The second Eq. (3)impliesthe continuity of traction crossing the dislocation line. Employing the symmetry of the problem with respect to the y-axis, Equation (2)may be solved in the $x > 0$. The boundary conditions for the half-plane are:

$$\sigma_{zy}(x,h,t) = 0, \ w(x,0^+,t) = w(x,0^-,t), \ \sigma_{zy}(x,0^+,t) = \sigma_{zy}(x,0^-,t), \ w(0,y,t) = \frac{1}{2}b_z(t)H(y). \tag{4}$$

The solution to equation (2) is achieved by means of integral transforms. The Laplace and Fourier sine transforms are applied to Eq. (2), assuming that the orthotropic half-plane is initially stationary, leads to:

$$\frac{d^2\bar{w}^*(\zeta,y,p)}{dy^2} - f^2(\zeta^2 + s_l^2 p^2)\bar{w}^* = 0., \tag{5}$$

The general solution of (5) is:

$$\bar{w}_1^*(\zeta,y,p) = A_1(\zeta,p)e^{-yf\beta} + B_1(\zeta,p)e^{yf\beta} + \frac{\zeta b_z(p)}{2\beta^2}, \qquad 0 < y < h$$

$$\bar{w}_2^*(\zeta,y,p) = B_2(\zeta,p)e^{yf\beta}. \qquad\qquad y < 0 \tag{6}$$

where $\beta = \sqrt{\zeta^2 + s_l^2 p^2}$, the functions $A_1(\zeta,p)$, $B_1(\zeta,p)$ and $B_2(\zeta,p)$ are unknown functions. By using the inversion of the Fourier sine transform of Eqs. (6), the anti-plane displacement in the Laplace domain can be obtained as:

$$\bar{w}_1(x,y,p) = -\frac{b_z(p)}{2\pi}\int_0^{+\infty}\frac{\zeta\sin(\zeta x)}{\beta^2}(e^{-yf\beta} + e^{-(2h-y)f\beta} - 2)d\zeta, \qquad 0 < y < h$$

$$\bar{w}_2(x,y,p) = \frac{b_z(p)}{2\pi}\int_0^{+\infty}\frac{\zeta\sin(\zeta x)}{\beta^2}(e^{yf\beta} - e^{-(2h-y)f\beta})d\zeta. \qquad y < 0 \tag{7}$$

From Eqs. (1) and (7), the stress components in Laplace domain may be written as:

$$\bar{\sigma}_{zx}(x,y,p) = -\frac{G_{zx}b_z(p)}{2\pi}\int_0^{+\infty}\frac{\zeta^2\cos(\zeta x)}{\beta^2}(e^{-yf\beta} + e^{-(2h-y)f\beta} - 2)d\zeta,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad 0 < y < h$$

$$\bar{\sigma}_{zy}(x,y,p) = \frac{fG_{zy}b_z(p)}{2\pi}\int_0^{+\infty}\frac{\zeta\sin(\zeta x)}{\beta}(e^{-yf\beta} - e^{-(2h-y)f\beta})d\zeta,$$

$$\bar{\sigma}_{zx}(x,y,p) = \frac{G_{zx}b_z(p)}{2\pi}\int_0^{+\infty}\frac{\zeta^2\cos(\zeta x)}{\beta^2}(e^{yf\beta}-e^{-(2h-y)f\beta})d\zeta$$

$$\bar{\sigma}_{zy}(x,y,p) = \frac{fG_{zy}b_z(p)}{2\pi}\int_0^{+\infty}\frac{\zeta\sin(\zeta x)}{\beta}(e^{yf\beta}-e^{-(2h-y)f\beta})d\zeta$$

$$y < 0 \qquad (8)$$

In order to specify the singular behavior of the stress components, the asymptotic behavior of the integrands in (8) should be examined. The singular parts of the kernels in the first relation of Eq. (8) can be separated after performing the appropriate asymptotic analysis and use the table of integral transforms [14]. Eq. (8) may be recast to more appropriate forms:

$$\bar{\sigma}_{zx}(x,y,p) = -\frac{G_{zx}b_z(p)}{2\pi}\left[\pi s_l p e^{-s_l p|x|} + \frac{yf}{x^2+y^2f^2} + \int_0^{+\infty}(\frac{\zeta^2}{\beta^2}(e^{-yf\beta}+e^{-(2h-y)f\beta})-e^{-yf\beta})\cos(\zeta x)d\zeta\right],$$

$$\bar{\sigma}_{zy}(x,y,p) = \frac{fG_{zy}s_l xp b_z(p)}{2\pi}\left[\frac{K_1(s_l p\sqrt{x^2+y^2f^2})}{\sqrt{x^2+y^2f^2}} - \frac{K_1(s_l p\sqrt{x^2+(2h-y)^2f^2})}{\sqrt{x^2+(2h-y)^2f^2}}\right]. \; 0 < y < h$$

$$\bar{\sigma}_{zx}(x,y,p) = -\frac{G_{zx}b_z(p)}{2\pi}\left[\frac{yf}{x^2+y^2f^2} - \int_0^{+\infty}(\frac{\zeta^2}{\beta^2}(e^{yf\beta}-e^{-(2h-y)f\beta})-e^{yf\zeta})\cos(\zeta x)d\zeta\right],$$

$$\bar{\sigma}_{zy}(x,y,p) = \frac{fG_{zy}s_l xp b_z(p)}{2\pi}\left[\frac{K_1(s_l p\sqrt{x^2+y^2f^2})}{\sqrt{x^2+y^2f^2}} - \frac{K_1(s_l p\sqrt{x^2+(2h-y)^2f^2})}{\sqrt{x^2+(2h-y)^2f^2}}\right]. \; y < 0. \qquad (9)$$

The details of the analysis will not be given in this paper. We observe that stress components are Cauchy singular at the dislocation position, which is a well-known feature of the stress fields caused by Volterra-type dislocations.

3. Orthotropic half-plane weakened by multiple edge cracks

The dislocation solutions accomplished in the foregoing section is extended to analyze orthotropic half-plane with several edge cracks. Cracks configurations are presented in parametric form as

$$x_i(\eta) = x_i + l_i\eta, \; -1 \le \eta \le 1$$
$$y_i(\eta) = y_i, \quad i = 1, 2, ..., N. \qquad (10)$$

To derive the integral equations for the crack problem, covering the crack faces by dislocations with unknown densities $b_z(\eta, p)$ in an orthotropic half-plane. By use of Buckner principle [15] the integral equation for N edge cracks is

$$\sigma_{zx}(x_i(\eta), y_i, p) = \sum_{j=1}^N l_i \int_{-1}^1 K_{ij}(s, \eta, p)B_{zj}(\eta, p)d\eta \quad -1 \le \eta \le 1, \; i = 1, 2, ..., N. \qquad (11)$$

Employing the definition of dislocation density function, the equations for the crack opening displacement across the jth crack reads:

$$w_j^-(s, p) - w_j^+(s, p) = \int_{-1}^s l_j B_{zj}(\eta, p)d\eta, \; j = 1, 2, ..., N. \qquad (12)$$

The numerical inversion of Laplace transform is carried out via Stehfest's method [16].

$$\sigma_{zxi}(x_i(\eta), y_i, \frac{\ln 2}{t}n) = \sum_{j=1}^N l_i \int_{-1}^1 K_{ij}(s, \eta, \frac{\ln 2}{t}n)B_{zj}(\eta, \frac{\ln 2}{t}n)d\eta, \qquad (13)$$

Stress fields for the cracks in orthotropic materials are singular at crack tips with a square-root singularity; Hence, the dislocation density functions for edge cracks can be represented by

$$B_{zj}(\eta, \frac{\ln 2}{t}n) = g_{zj}(\eta, \frac{\ln 2}{t}n)\sqrt{\frac{1-\eta}{1+\eta}}, \; -1 \le \eta \le 1, \; j = 1, ..., N. \qquad (14)$$

We substitute Eq. (14) into the Eq. (13) and apply the numerical technique developed by Erdogan [17] for the solution of integral equations to solve the resultant equations. The inverse Laplace transform of the solution becomes

$$g_{zj}(\eta,t) = \frac{\ln 2}{t} \sum_{n=1}^{M} v_n G_{zj}(\eta, \frac{\ln 2}{t} n), \quad -1 \leq \eta \leq 1, \ j = 1,...,N. \tag{15}$$

where $v_n = (-1)^{\frac{M}{2}+n} \sum_{k=[0.5(n+1)]}^{\min(\frac{M}{2},n)} \frac{k^{\frac{M}{2}}(2k)!}{(\frac{M}{2}-k)!k!(k-1)!(n-k)!(2k-n)!}.$ \hfill (16)

The stress intensity factor for *j*th edge crack is [18]:

$$(k_{III}(t))_j = \sqrt{l_j G_{zx} G_{zy}} \ g_{zj}(-1,t), \qquad j = 1,2,...,N. \tag{17}$$

4. Numerical results and discussion

In all the proceeding examples, The quantities of interest are the dimensionless stress intensity factors, $k(t)/k_0$. For conveni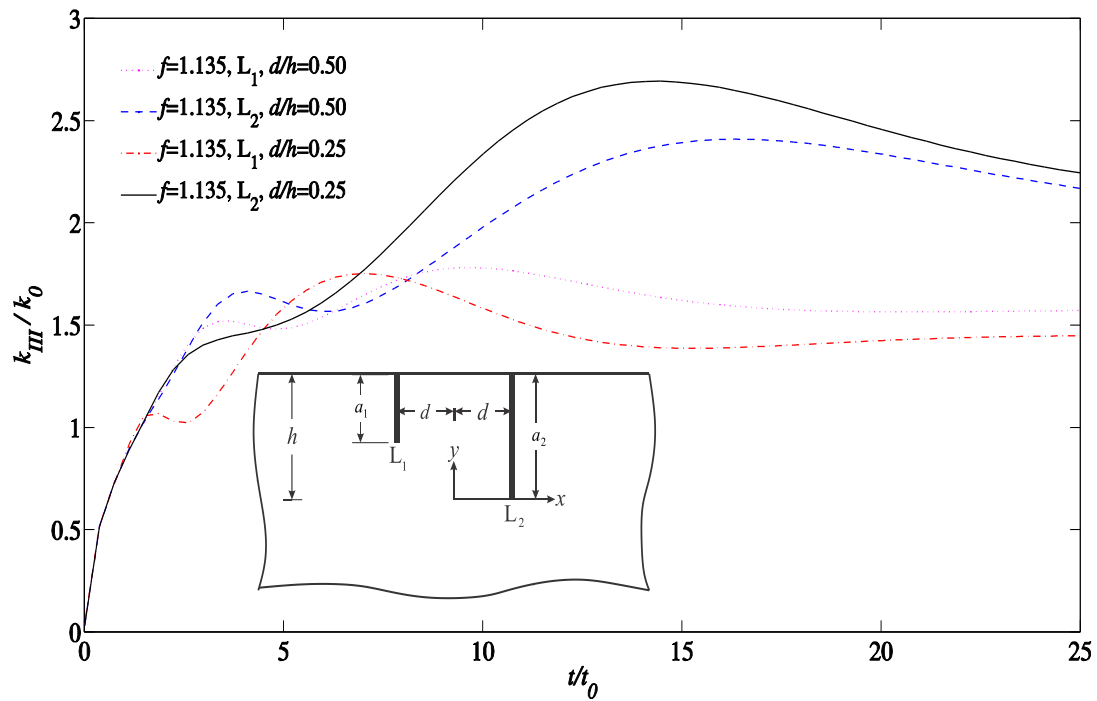ence, the dynamic stress intensity factors are normalized by $k_0 = \tau_0\sqrt{a/2}$, except example 2 where $k_0 = \tau_0\sqrt{a_1/2}$, for uniform shear traction.

In the first example, an orthotropic half-plane under constant shear traction $\tau_{zy} = \tau_0 H(t)$ weakened by an edge crack with length $a = 0.5h$ is analyzed, Fig. 1. The deviser for normalizing time is $t_0 = a/2s_l$, where *a* is the crack length. It is seen that, the dynamic stress intensity factor increases very quickly with time, reaching a pick then decreases in magnitude and tends to the quasi static solution for sufficiently large normalized time. From Fig. 1, we may conclude that dynamic stress intensity factor of the edge crack exhibit small variations versus $f = \sqrt{G_{zx}/G_{zy}}$.
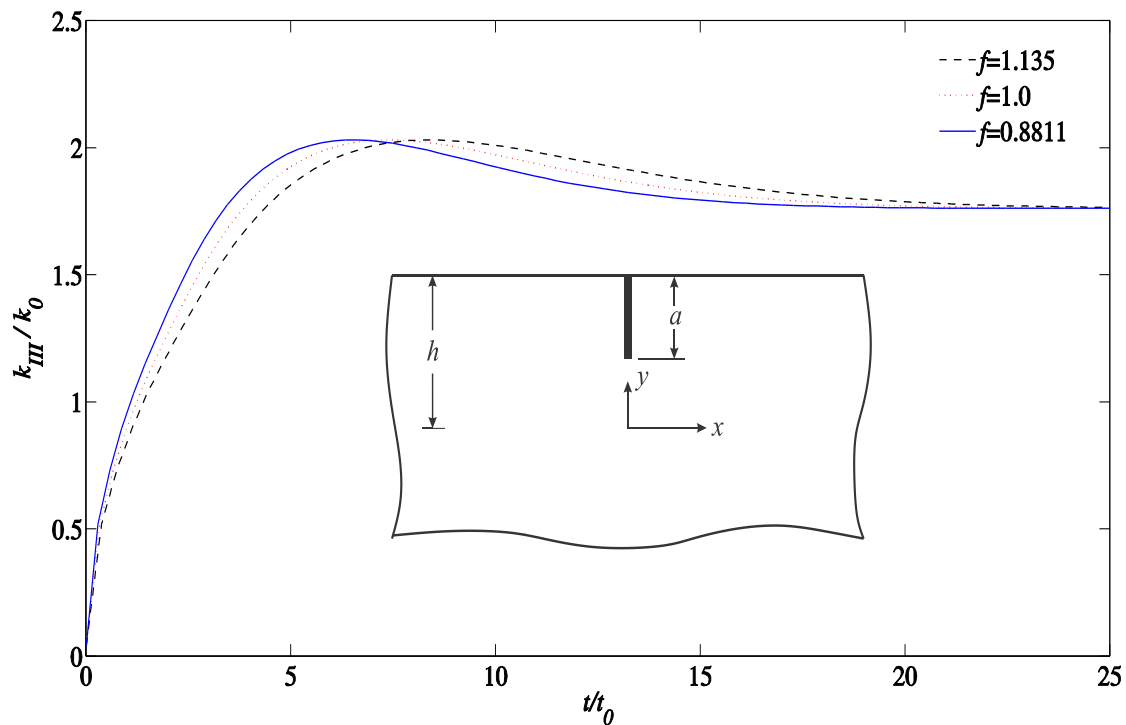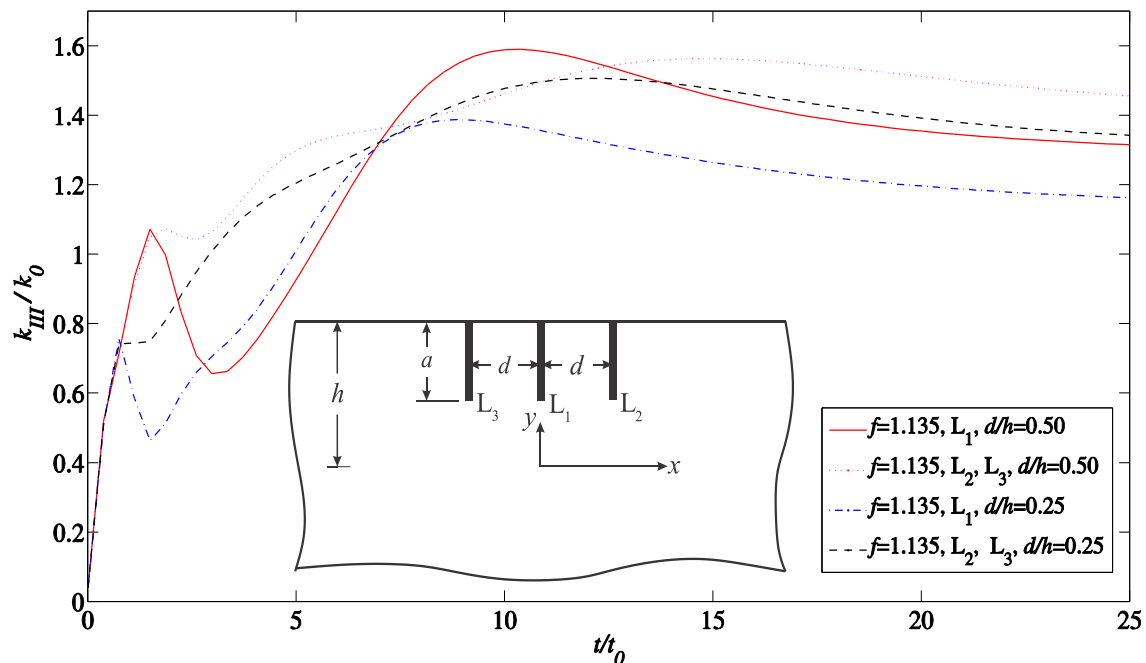

Fig. 1. Variation of normalized dynamic stress intensity factor of an edge crack.

In the next example, let us consider an orthotropic half-plane weakened by two unequal length edge cracks under anti-plane impact load, we study effects of the cracks location on dynamic stress intensity factors of crack tips Fig. 2. Note that, the variation of stress intensity factors of the tips L_2 is significant and the times at which the maxima of dynamic stress intensity factor occur for the tip L_1 are shorter than that the tip L_2.

Fig . 2. Variation of normalized dynamic stress intensity factor of two parallel edge cracks.

Fig. 3. Variation of normalized dynamic stress intensity factor of three identical parallel edge cracks.

Three equal-length cracks which are normal to the boundary are shown in Fig. 3. The problems are symmetric with respect to the *y*-axis. For other values of crack distance similar plots with different values of $k(t)/k_0$ may be obtained. As it was expected, the variation of stress intensity factors of three interacting crack tip namely L_1 is more pronounced than that of the tips L_2.

5. Conclusions

The dynamic response of an orthotropic half-plane with multiple edge cracks is studied analytically under the action of anti-plane impact loads. It may be observed that, once a solution is available for boundary loading with step function time dependence, extension to boundary loading with other time dependence is quite simple. According to the calculation results, it is found that the parameters such as the time, crack length and material properties have significant influences on the dynamic stress intensity factors. To show the applicability of the procedure more examples are solved and the stress intensity factors for multiple edge cracks are obtained.

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