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Estimation of dispersion curves by combining Effective Elastic Constants and SAFE Method: A case study in a plate under stress

J E Quiroga[1], L Mujica[3], R Villamizar[2], M Ruiz[3] and J Camacho[3]

[1]Escuela de Ingeniería Mecánica, [2]Escuela de Ingenierías Eléctrica, Electrónica y de Telecomunicaciones (E3T) , Universidad Industrial de Santander (UIS),Cra 27, Calle 9, C. Universitaria, Bucaramanga, Colombia.
3Universitat Politécnica de Catalunya (UPC) BARCELONATECH, Department of Mathematics, Escola d'Enginyeria de Barcelona Est. (EEBE), Campus Diagonal Besòs, Edifici A, C. Eduard Maristany, 10-14, 08019, Barcelona, Barcelona, Spain

Corresponding author: jabib@uis.edu.co

Abstract. This paper presents an approach to calculate dispersion curves for homogeneous and isotropic plates subject to stress, via Semi-Analytical Finite Element and the Effective Elastic Constants, since stresses in the waveguide modify the phase and group velocities of the lamb waves. In the proposed methodology an isotropic specimen subjected to anisotropic loading is emulated by proposing an equivalent stress-free anisotropic specimen. This approximation facilitates determining the dispersion curves by using the well-studied numerical solution for the stress-free cases. The lamb wave in anisotropic materials can be studied by means of the Effective Elastic Constants, which reduces the complexity of the numerical implementation. Finally, numerical data available in literature were used to validate the proposed methodology, where it could be demonstrated its effectiveness as approximated method.

1. Introduction

In last years, application of lamb waves in non-destructive tests is of a growing interest [1][2][3][4][5][6]. While bulk waves refers to waves propagated in an infinite media, guided waves require a boundary for their existence. Furthermore, when a guided wave is produced at least one dimension is smaller than the other ones. On the other hand, if the guided wave is propagating in a plate-like structure, the wave is referred as a Lamb wave. Propagation of lamb waves is a complex phenomenon, where phase and group velocity are frequency dependent and it can be graphically observed by means of called dispersion curves. The exact trace of the dispersion curves is obtained by using transcendent equations assembled by either the transfer matrix or the global matrix methods [7]. Recently, the Semi-Analytical Finite Element (SAFE) method has been proposed to compute rapidly and efficiently the dispersion curves for any constant cross section specimen. SAFE has been used to analyze wave modes in cylindrical waveguides [8][9][10][11], to calculate leaky lamb waves [12], to obtain the dispersion curves in a pipe elbow [13] and in materials with viscoelastic properties [14].

In SAFE methodology the waveguide is discretized over the cross section, while in an analytical solution is in the wave propagation direction. Based on a variational scheme by inserting the kinetic and potential energies into Hamilton´s equation, a system of linear equations can be constructed with the circular frequency and wave number as unknowns under stress-free condition. The unknowns can be solved using standard eigenvalue routines. Therefore, SAFE approach allows computing dispersion curves in waveguides with complex cross sections, such as: multilayered laminates [15] and rails [16][17][18][19], where it is often computationally complex to solve analytical solutions. For semi-infinite waveguides, SAFE has a better performance compared with the traditional FEM due to only the constant cross-section is considered, which reduces the computational cost. Additionally, SAFE does not have missing roots problem, found in matrix methods, when dispersion curves are computed [20].

The dispersion curves in stress-free isotropic and anisotropic specimens have been widely studied and can be adequately described based on the theory of elasticity. However, this formulation is insufficient to describe wave propagation in specimens under stress since small non-linearities in the stress-strain relationships become significant, which can be described by using the acoustoelasticity approach. It dictates the stress dependence of acoustic bulk wave velocities propagating in elastic media under stress. Few studies on dispersion of waves in a waveguide subjected to axial load using the acoustoelasticity have been reported [21][22][23]. On the other hand, the use of SAFE methodology for this problem is even more reduced. In [16][24][25] is demonstrated a proportionality between the stiffness matrix, required to describe the effect of axial load and the mass matrix, which makes the use of existing software (stress-free) trivial to obtain the dispersion curves for waveguides under stress.

Thus, this paper proposes the use of EEC to describe the anisotropy effect presents in the lamb wave dispersion curves when a load is applied along one direction in an isotropic plate, based on the SAFE scheme. Then, the isotropic specimen subject to loading in one direction is studied by proposing an equivalent stress-free anisotropic specimen.

2. Plate's SAFE Analytical Model

The SAFE methodology is in this case, implemented for an isotropic, homogenous stress-free plate (infinitely wide plate), where the wave propagates along direction *x* with wavenumber ξ_x and circular frequency ω (see fig. 1). Since, the cross-section lies in the *y–z* plane it is formulated as a two-dimensional problem (*x*,*z*) in the plane strain formulation. Thus, the harmonic displacement, stress and strain field components in cartesian coordinate at each point of the waveguide are expressed by equation (1)

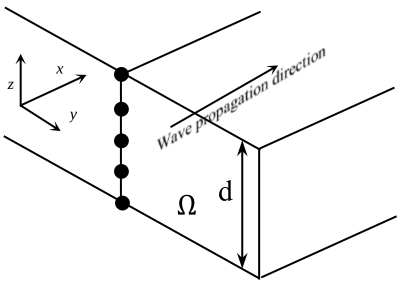

Figure 1. Schematic representation of the plate.

$$u = [u_x \; u_z]^T, \qquad \sigma = [\sigma_x \; \sigma_z \; \sigma_{xz}]^T, \qquad \varepsilon = [\varepsilon_x \; \varepsilon_z \; \gamma_{xz}]^T \tag{1}$$

The constitutive equation is given by equation (2)

$$\sigma = E\varepsilon \tag{2}$$

where, *E* is the elasticity matrix (real symmetric matrix for isotropic specimen) defined by equation (3)

$$E = \frac{2G}{(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \tag{3}$$

where G is the shear modulus and ν the Poisson´s ratio. The strain-displacement relationship can be compactly written as follows.

$$\varepsilon = Lu \tag{4}$$

where L is a three-dimensional differential operator defined as equation (5)

$$\mathrm{L} = \begin{bmatrix} \partial_x & 0 & \partial_z \\ 0 & \partial_z & \partial_x \end{bmatrix}^T \tag{5}$$

Thus, the compatibility equations can be written as equation (6)

$$\varepsilon = \left[L_x \frac{\partial}{\partial x} + L_z \frac{\partial}{\partial z}\right]^T u \tag{6}$$

where

$$\mathrm{L}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T, \quad \mathrm{L}_z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T \tag{7}$$

The cross-sectional domain of the plate, Ω, can be represented by a system of mono-dimensional finite elements with domain Ω_e, due to the simplification attributed to the waveguide symmetry. The displacement expressions, discretized over the element domain, can be written in terms of the shape functions $N_i(z)$ and the nodal unknown displacements, (U_{xi}, U_{yi}, U_{zi}), in cartesian coordinates. Such shape functions are considered in this formulation as unidimensional, isoparametrics and quadratics.

$$u^e_{(x,z)}(x,z,t) = \left[\sum_i^k N_i(z) U_{(x,z)i}\right]^e exp(-i(\xi x - \omega t)) = \mathbf{N}(z) f^e exp(-i(\xi x - \omega t)) \tag{8}$$

where

$$\mathbf{N}(z) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}, \quad f^e = [U_{x1}\ U_{z1}\ U_{x2}\ U_{z2}\ U_{x3}\ U_{z3}]^T \tag{9}$$

With $N_1 = \frac{\eta^2}{2} - (\frac{\eta}{2})$, $N_2 = 1 - \eta^2$ and $N_3 = \frac{\eta^2}{2} + (\frac{\eta}{2})$, represented in the local coordinates $\eta\ \epsilon[-1,1]$. Consequently, the Jacobian $|J|$ is equal to $l/2$, where l is the length of the element. For a maximum frequency of interest, the mesh criterion of [26] indicates the maximum element length, $L < 2\pi c_T/\beta\omega_{max}$, where c_T is the shear bulk wave velocity and β is 4 for the case of quadratic elements.

The strain vector in the element can be represented as a function of the nodal displacements:

$$\varepsilon^e = (\mathbf{B}_1 + i\xi\mathbf{B}_2) f^e exp(-i(\xi x - \omega t)) \tag{10}$$

Where $\mathbf{B}_1 = L_z(\partial N/\partial z)$, $\mathbf{B}_2 = L_x N$

By considering adiabatic wave propagation, the weak form of the governing balance equation can be obtained by applying Hamilton's principle and adding up the contributions of every element as in the standard FE method. The SAFE governing equation for the stress-free plate is obtained as equation (11)

$$[\xi^2\mathbf{K}_3 + i\xi\mathbf{K}_2 + \mathbf{K}_1 - \omega^2\mathbf{M}]U = 0 \tag{11}$$

With

$$\begin{aligned}
\mathbf{K}_1 &= \textstyle\sum_{e=1}^{N_{elem}} \int_{-1}^{1} \mathbf{B}_1^T\mathbf{C}\mathbf{B}_1\, |J|d\eta,\\
\mathbf{K}_2 &= \textstyle\sum_{e=1}^{N_{elem}} \mathbf{T}^T \left[\int_{-1}^{1} \mathbf{B}_1^T\mathbf{C}\mathbf{B}_2\, |J|d\eta - \int_{-1}^{1} \mathbf{B}_2^T\mathbf{C}\mathbf{B}_1\, |J|d\eta\right]\mathbf{T}\\
\mathbf{K}_3 &= \textstyle\sum_{e=1}^{N_{elem}} \int_{-1}^{1} \mathbf{B}_2^T\mathbf{C}\mathbf{B}_2\, |J|d\eta,\\
\mathbf{M} &= \textstyle\sum_{e=1}^{N_{elem}} \int_{-1}^{1} \mathbf{N}^T \rho\mathbf{N} \quad |J|d\eta,
\end{aligned} \tag{12}$$

where, $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are the global stiffness matrices and **M** is the global mass matrix, superscript *T* denotes the matrix transpose, N_{elem} is the total number discretized elements across d and *U* is the vector of global displacements (and also the eigenvectors) at particular circular frequency ω. Transformation matrix **T** is an orthogonal diagonal matrix introduced to eliminate the imaginary term in the eigenfunction. The u_z displacement components in **T** are equal to 1, while those of u_x are equal to the imaginary unit.

The estimation of dispersion curves is tackled by sweeping the wave number ξ in a set of real values. Thus, the eigenfunction can be solved as a standard eigenvalue problem in $\omega(\xi)$. Real, purely imaginary and complex eigenvalues are obtained, however only the real one values (that correspond to the propagating waves) are considered. In this case, the number of eigenvalues $\omega(\xi)$ obtained is the number of total Degrees Of Freedom (DOF) of the system, *N*. Thus, *N* propagating modes (ξ_m, ω_m) are determined for each $\mathbf{U}_N$ cross sectional wave structure. The phase velocity of the propagating *N* mode can be computed by $C_p^m = \omega/\xi^m$, while the group velocity C_g^m can be calculated using modal properties for conservatives materials, equation (13)[27].

$$C_g^m = \frac{\psi_L^m \mathbf{K}_{,\xi} \psi_R^m}{2\omega_m \psi_L^m \mathbf{M} \psi_R^m} \tag{13}$$

where ψ_L^M and ψ_R^M are the mth left and right eigenvectors of the eigenfunction, $\mathbf{K}_{,\xi}$ is $2\xi\mathbf{K}_3 + \mathbf{K}_2$ and M is the mass matrix.

3. Dispersion curves using EEC for a uniaxial stressed plate

Acoustoelasticity is a nonlinear phenomenon that explains speed changes in bulk waves (longitudinal and shear) as a function of applied stress, based on the continuum theory for small disturbances. The linearization of the relation between stress and strain is no longer valid for the case of ultrasound propagation in a media subject to stress and with finite deformations [22].

For the linear case, only the second order elastic constants λ and μ are needed as parameters to describe the linear constitutive relation stress-strain in an isotropic medium. However, for nonlinear characterization, additional third order elastic constants (TOECs) have to be added to the constitutive relation. Thus, the deformation energy, strain energy function *U*, has to be expressed in a third order form. On the other hand, the relation between strain energy function and stress is used to express the nonlinear relation between stress-strain, where this function is expressed as a power series in strains as equation (14)

$$T_{ij} = C_{ijkl}^{(2)} E_{kl} + C_{ijklmn}^{(3)} E_{kl} E_{mn} + \cdots \tag{14}$$

where T_{ij} is the second Piola-Kirchhoff stress tensor, E is the Lagrangian strain tensor and $C^{(2)}, C^{(3)}, ..$ are increasing order tensors and correspond to the coefficient of the power series expansion. $C^{(2)}_{ijkl}$ is the second order constants for the linear case, and $C^{(3)}_{ijklmn}$ represents the third order elastic constants, for the case of isotropic materials. This tensor can be represented in terms of the Murnaghan constants l, m and n.[28]. Some papers have shown the use of EECs to tackle the influence of the acoustoelasticity in the propagation of guided waves [29][30][31].

By using a Cartesian frame to describe the incremental stresses, strains and displacements in stressed media, three deformation states are defined. Following the notation used by [30], the unstressed frame is called the "natural state." The position of a material point is given by the position vector **x** whose "natural coordinates" are $\boldsymbol{x_1}(x_1, x_2, x_3,)$. "The initial state" is a finite deformation (applied or residual) in static equilibrium is then given by the position vector $\overline{\boldsymbol{x}}$, whose initial coordinates are $\overline{\boldsymbol{x_1}}(\overline{x_1}, \overline{x_2}, \overline{x_3})$. Finally, when a dynamic perturbation (wave propagation) is applied at the initial state, the point material reach the third state called the "final state." The position of the material point is then defined by the position vector $\widetilde{\boldsymbol{x}}$ whose "final coordinates" are $\tilde{x}_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. A common Cartesian frame (ξ_1, ξ_2, ξ_3) is used to refer to the position of material points of any of the three states. The physical quantities, which refer to the natural state, are denoted by the superscript O and those that refer to the initial state by the superscript I. The initial deformation and the final deformation are denoted by $\bar{u}$ and $\tilde{u}$, respectively. Equations (15) and (16) are the equations of motion with initial coordinates and natural coordinates, respectively [31],

$$\frac{\partial}{\partial \overline{x_j}}\left[\left(A_{ijkl} + \delta_{ik} t^I_{jl}\right)\frac{\partial \tilde{u}_k}{\partial \overline{x_l}}\right] = \rho^I \frac{\partial^2 \tilde{u}_I}{\partial t^2} \qquad (15)$$

$$\frac{\partial}{\partial x_j}\left[\left(\Gamma_{ijkl} + \delta_{ik} T^O_{jl}\right)\frac{\partial \tilde{u}_k}{\partial \overline{x_l}}\right] = \rho^O \frac{\partial^2 \tilde{u}_I}{\partial \overline{t^2}} \qquad (16)$$

where A_{ijkl} and Γ_{ijkl} are tensors that depend on the symmetry of the material and describe the relation between second and third order elastic constants and the displacements $\bar{u}$ with respect to x and $\bar{x}$ positons, ρ is the density, t^I is the Cauchy stress tensor, T^O is the Kirchhoff stress tensor or second Piola–Kirchhoff stress tensor and δ is the Kronecker delta function. In [31], a more detailed analysis of A_{ijkl} and Γ_{ijkl} is found, where it can be observed that variations of the wavespeed depend of the initial stress value and the initial displacement gradient $\partial u_i/\partial \xi$. Motion equation (16) in natural coordinates is equivalent to the equation of motion relative to a stress-free medium. In addition, the use of natural coordinates or material's specific frame avoids problems associated to the splitting terms in the EEC when initial coordinates are used. The similarity between these equations allows considering a stressed material as an unstressed material with EEC, which takes into account the disturbances linked to the presence of stress[30]. This description considers the following formalism:

$$C^a_{ijkl} = C_{ijkl} + \delta C_{ijkl} \qquad (17)$$

where, C^a_{ijkl} is the tensor of the EEC, C_{ijkl} is the tensor of the second order elastic constants for a stress-free material and δC_{ijkl} is the disturbance related to the presence of applied or residual stress. In this sense C^a_{ijkl} is equal to $\Gamma_{ijkl} + \delta_{ik} T^O_{jl}$, which corresponds to the tensor of the EEC in natural coordinates. Since, EEC allows the use of the second order approach in the case of stressed materials, they can be considered as a stress-free material presenting second-order elastic constants different from those of an unstressed material. Under these condition, the eigenfunction that satisfy the guided wave modal propagation can be written in the same form as shown in equation (18):

$$\left|K\left(\xi, C_{ijkl}\eta_j\eta_l\right) - M\omega^2\right| = 0 \qquad \left|K'\left(\xi', C^a_{ijkl}\eta_j\eta_l\right) - M{\omega'}^2\right| = 0, \tag{18}$$

where ω and ω' correspond to the ultrasonic wave circular frequency in unstressed and stressed media respectively, the tensors K and K' is the elasticity relation for unstressed and stressed materials and $\eta_j\eta_l$ are the direction cosines of the normal to the wavefront, i.e., $\xi_p = |\xi|\eta_p$. Thus, the symmetry of a material will change in the presence of static stress with respect to ultrasonic wave propagation. The terms of C^a_{ijkl} are listed in appendix A.

4. Numerical Validation

In order to validate the proposed approach, dispersion curves obtained by [31] for an aluminum isotropic plate ($\rho = 2800\ kg/m^3$) of thickness d=6.35 mm with material constants in GPa, $l = -252.2\,; m = -324.9\,; n = -351.2;\ \lambda = 54.9\,;\ \mu = 26.5$, were used. It is a benchmark data from an analytical model based on the acoustoelasticity principle. Figure 2 illustrates the used plate Cartesian frames, while figure 3 shows the dependence of phase velocity on the direction of propagation, by comparing the S1 modes about 600 kHz, propagating at varying angles with respect to the $x_1{}'$. Herein, analytical solution by [31] is represented by solid lines while the proposed approach by markers. Additionally, figure 4 presents the deviation of S1 mode between the proposed scheme and the previous analytical solution

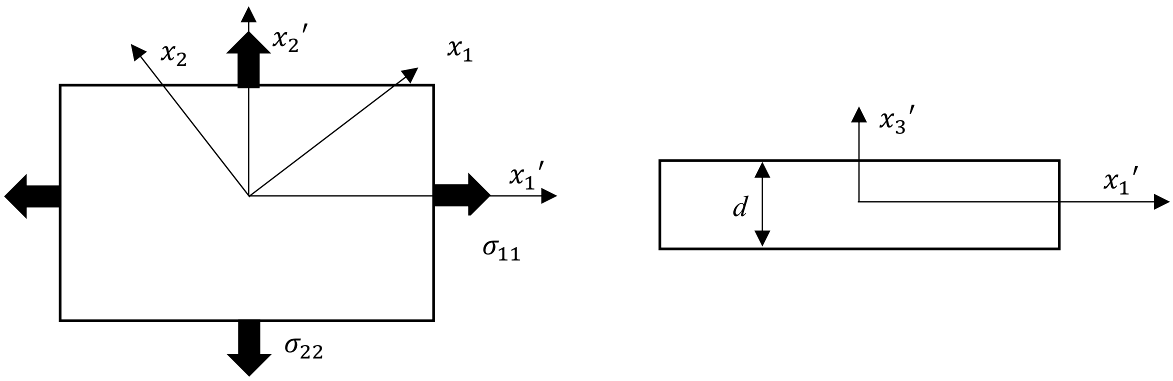

Figure 2. Plate's Cartesian frames.

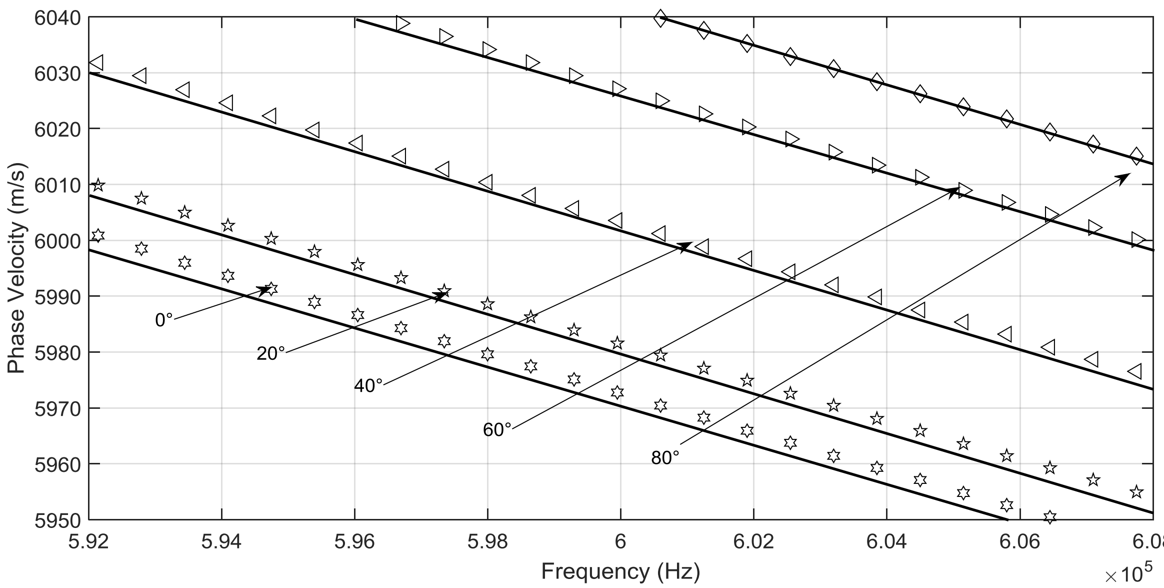

Figure 3. Comparison of angle dependence of S1 mode for a uniaxial load of 120 MPa.

Figure 5 presents the dispersion curves for the S1 mode about 600 kHz by applying different stresses values in order to observe the variations of phase velocity respect to the applied load, where all curves are at an angle of $\phi = 45°$. Moreover, figure 6 presents the deviation of S1 mode between the proposed scheme and the previous analytical solution in the case of different applied stresses.

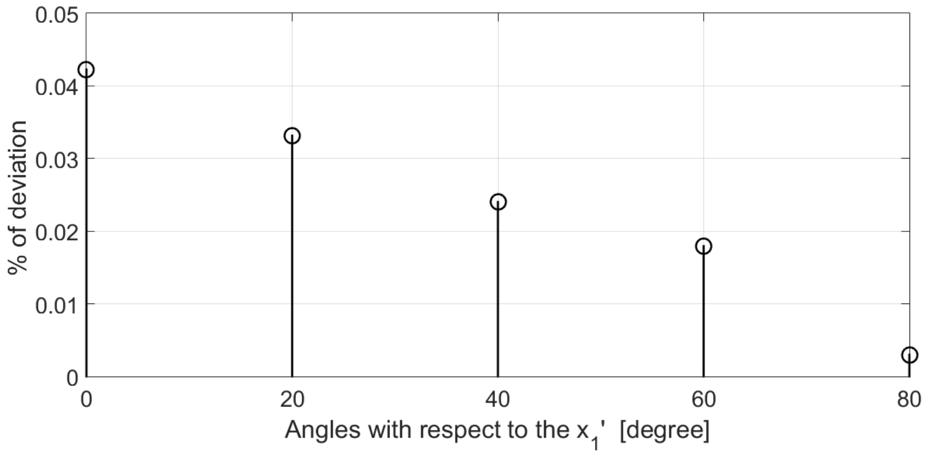

Figure 4. Deviations between proposed and analytical solution by [31] for S1 mode in [592-608] Khz

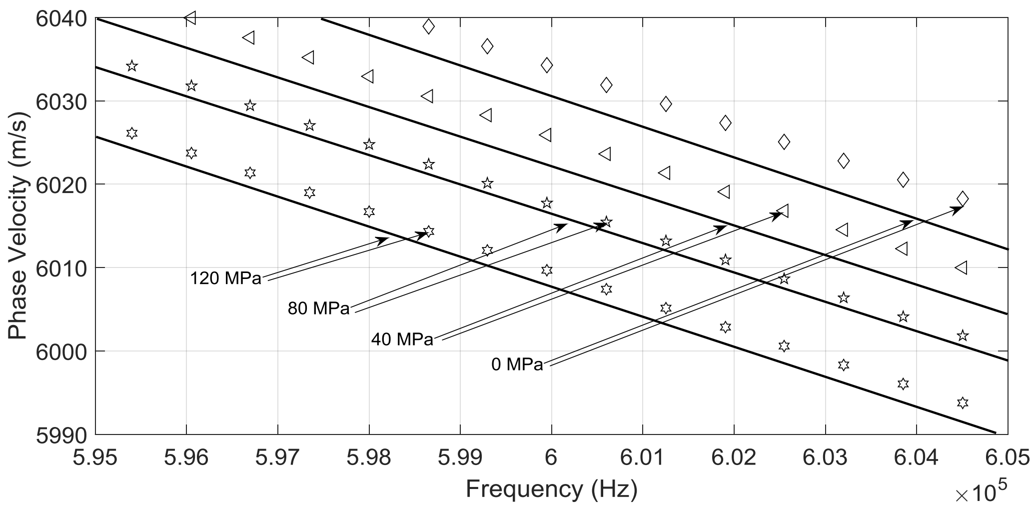

Figure 5. Comparison of stress dependence of S1 mode for a uniaxial load at φ = 45°. Analytical solution by [31] is represented by solid lines while the proposed approach by markers.

5. Results Analysis

According to figure 3, the dispersion curves of S1 mode (represented by markers) computed by means of EEC approach for a load of $\sigma_{11} = 120$ MPa are close to the analytical solution (based on the acoustoelasticity theory). However, this should not be generalized since the deviation of EECs from the theoretical model depends on the mode and frequency range under consideration [31].

In the case of the stress dependence (figure 4), it can be observed a good agreement in the range of analyzed dispersion curves between analytical model based on acoustoelasticity, and combined EEC´s and SAFE proposed model.

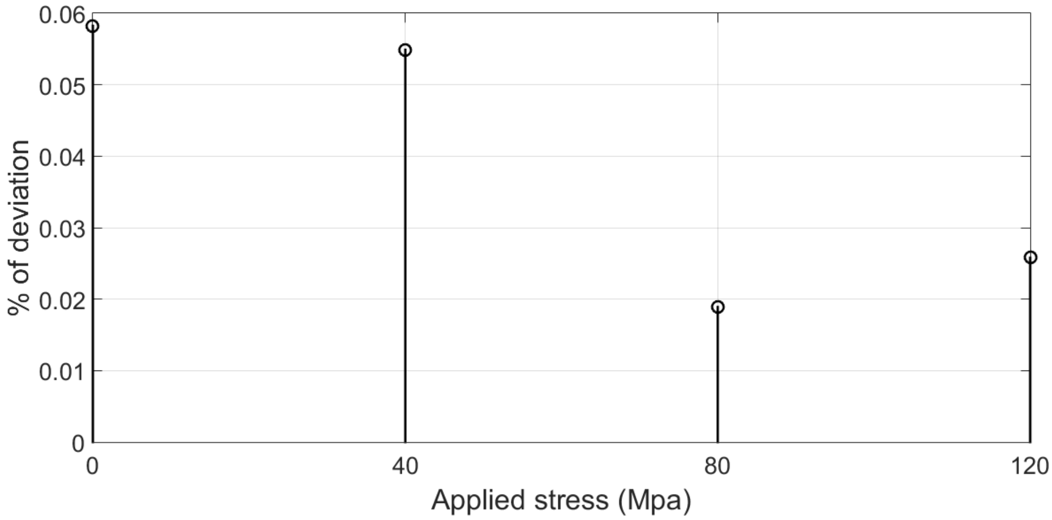

Figure 6. Deviations between proposed and analytical solution by [31] for S1 mode in [592-608] Khz

6. Conclusions

An approximate method for computing dispersion curves of lamb waves, phase and group velocity, in homogenous and isotropic plates under stress based on FEM and EEC is proposed in this work. The main contribution of the proposed approximate approach for estimating the dispersion curves of an isotropic and homogenous plate is accounting for the stress in the waveguide by using EEC combined with an SAFE scheme. These dispersion curves are a relevant tool for NDE/SHM applications, especially for identifying propagating modes and locating defects.

The proposed methodology was validated on several propagation angles and load conditions, which were previously examined by an analytical method based on the acoustoelasticity theory. It was found slight deviations by comparing the dispersion curves obtained by both solutions (proposed and previous method). It is remarked that a sufficient number of elements for discretizing the cross- section of the waveguide was ensured, thus, the observed variations can be attributed to limitations of the EEC representing the stressed state.

The main advantages of the proposed approximate approach on the available analytical methods are a low coding complexity and fast execution, with a decent accuracy compared to the previous analytical acoustoelasticity based model.

7. Acknowledges

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8. Appendix A

Expressions for the EEC proposed by [30]

$$C_{11}^{a} = k_2 + k_1[4\lambda + 10\mu + 4\nu_2 + 8\nu_3 + k_3(-2\mu + \nu_1 + 2\nu_2)] \tag{A.1}$$
$$C_{22}^{a} = C_{33}^{a} = k_2 + k_1[k_4 + k_3(\lambda + 2\mu + \nu_1 + 4\nu_2 + 4\nu_3)] \tag{A.2}$$
$$C_{44}^{a} = \mu + k_1[-2\mu - 2\nu_3 + k_3(\mu + \nu_2 + 2\nu_3)] \tag{A.3}$$
$$C_{55}^{a} = C_{66}^{a} = \mu + k_1[2\mu + \nu_3 + (1 - 2\nu)(\nu_2 + \nu_3)] \tag{A.4}$$
$$C_{12}^{a} = C_{13}^{a} = \lambda + k_1[\lambda + \nu_2 + k_3(\nu_2 + \nu_1)] \tag{A.5}$$
$$C_{23}^{a} = \lambda + k_1[-2\lambda - 2\nu_2 + k_3(\lambda + \nu_1 + 2\nu_2)] \tag{A.6}$$

Where λ and μ are the Lamé constants, ν_1, ν_2 and ν_3 are the Toupin and Berstein constants which are equivalent to $\nu_1 = n$, $\nu_2 = m - 0.5n$ and $\nu_3 = 1 - \nu_2$ in terms of Mourghanan (l, m, n) constants.

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