Minimum constitutive relation error based static identification of beams using force method

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Abstract. A new static identification approach based on the minimum constitutive relation error (CRE) principle for beam structures is introduced. The exact stiffness and the exact bending moment are shown to make the CRE minimal for given displacements to beam damages. A two-step substitution algorithm—a force-method step for the bending moment and a constitutive-relation step for the stiffness—is developed and its convergence is rigorously derived. Identifiability is further discussed and the stiffness in the undeformed region is found to be unidentifiable. An extra set of static measurements is complemented to remedy the drawback. Convergence and robustness are finally verified through numerical examples.

1. Introduction

This work focuses on the damage identification method which is performed based on static test data through a minimum constitutive relation error (CRE) procedure [1]. For structures such as beams, static tests are easily performed and accurate enough to give useful results as dynamic tests [2-10]. According to the recent variational theory of Geymonat and Pagano [11], constitutive parameters and stresses can be identified through minimization of the CRE as long as full-field displacement measurements are readily acquired under given loads and this idea has been proposed very recently by Florentin and Lubineau [12-16] for parameter identification in plane elasticity problems. This method is also quite tailored for beam structures, to which this paper is just devoted, for the reason that global displacement measurements are easily available and the standard force method [17] based on the minimum complementary energy principle is easily conducted and it often requires substantially less degrees of freedom than conventional displacement-based finite element models. Practical problems such as algorithm implementation, convergence, identifiability and effect of measurement noises will also be discussed in detail.

The remainder of the paper is organized as follows. The minimum CRE principle for inverse identification of a Bernoulli-Euler beam model is simply introduced and the convergence and identifiability of the proposed approach are also discussed in Section 2. Numerical tests are performed in Section 3 and final conclusions are drawn in Section 4.

2. Formulations of the problem

2.1. Reference problem

Let us consider a one-dimensional Bernoulli-Euler beam defined in interval $X = [x_s, x_e]$. The flexural stiffness of the beam is assumed to be $k(x) = EI(x)$. Loading conditions are distributed loads $q(x)$ and concentrated loads F^{ex}, M^{ex} along with boundary displacements w_p, θ_p on the prescribed displacement boundary. Then the beam model can be described as a combination of the following three parts:

- Kinematic constraints

$$w \in \mathcal{U} \, ; \tag{1}$$

- Equilibrium equations

Find bending moment $M \in \mathfrak{I}$ with

$$\mathfrak{I} = \{ M : V = M', M'' = q(x), V \text{ and } M \text{ are in equilibrium with } F^{ex} \text{ and } M^{ex} \}, \tag{2}$$

where V is a shear force and x^{ex} is the position where concentrated loads F^{ex}, M^{ex} are enforced;

- Constitutive relation

$$M = kw'' \tag{3}$$

with $k \in \mathcal{C} = \{ k \in L_2(X) : \exists 0 < c_0 < C_0, \text{ such that } c_0 \le k(x) \le C_0 \}$ representing a positive stiffness field.

Generally, $\mathcal{U}$, $\mathfrak{I}$, $\mathcal{C}$ that will be used frequently in what follows are called the spaces of (kinematically) admissible displacement field, (statically) admissible bending moment field and (constitutively) admissible elastic stiffness field, respectively.

In this paper, identification of damage in beams is performed with the help of *static measurements* which contain the information of static loads and full-field displacement measurements. The objective is to identify the damaged stiffness $k(x) \in \mathcal{C}$, given the displacement measurements $w \in \mathcal{U}$ and the respective static loading conditions $\mathfrak{I}$; that is to say, the inverse identification problem reads: find $(k, M) \in \mathcal{C} \times \mathfrak{I}$ with given $w \in \mathcal{U}$ which is a reverse of the forward problem: find $(w, M) \in \mathcal{U} \times \mathfrak{I}$ with known $k \in \mathcal{C}$. Evidently, by solving the inverse problem, one can not only identify the damage, but also reconstruct the stiffness and bending moment distribution.

2.2. The minimum-CRE principle and identification algorithm

Consider an admissible solution trio $(\tilde{w}, \tilde{M}, \tilde{k})$ which satisfies

$$\tilde{w} \in \mathcal{U}, \tilde{M} \in \mathfrak{I}, \tilde{k} \in \mathcal{C} . \tag{4}$$

To measure the distance of the admissible solution trio to the exact solution trio in energy product, an error in constitutive relation is introduced

$$e_{CRE}(\tilde{w}, \tilde{M}, \tilde{k}) = \frac{1}{2} \int_X \frac{(\tilde{M} - \tilde{k}\tilde{w}'')^2}{\tilde{k}} \, dx \tag{5}$$

which is termed the constitutive relation error (CRE) [1] (or constitutive equation gap (CEG) [12]).

For the present inverse problem, the displacement field $w \in \mathcal{U}$, the spaces $\mathcal{C}$ and $\mathfrak{I}$ for search of the stiffness k and the bending moment M, respectively, are already known. Minimization of the CRE can then be interpreted as the following procedure,

$$(M, k) = \arg \min_{\tilde{M} \in \mathfrak{I}, \tilde{k} \in \mathcal{C}} F(\tilde{M}, \tilde{k}); \ F(\tilde{M}, \tilde{k}) := e_{CRE}(w, \tilde{M}, \tilde{k}) , \tag{6}$$

which is known as *the minimum CRE principle* [11] for inverse identification. For practical implementation, a two-step substitution algorithm is developed.

Firstly, minimization of the CRE function over $\tilde{M}$ yields a minimum complementary energy problem that should be solved by the force method. Therefore, the procedure is simply designated as

$$\tilde{M} = Force\_Method(\tilde{k}, \mathfrak{I}, \mathcal{U})$$

$$:= \arg\min_{\tilde{M} \in \mathfrak{I}} \{\frac{1}{2} \int_X \frac{\tilde{M}^2}{\tilde{k}} dx - \sum_p (\theta_p \tilde{M}|_p + w_p (\tilde{M})'|_p)\} \tag{7}$$

and this step is called the *force-method* step.

Secondly, minimization of the CRE over $\tilde{k}$ gives

$$\int_X \left\{ \frac{\tilde{M}^2}{\tilde{k}^2} - (w'')^2 \right\} \delta\tilde{k} dx = 0. \tag{8}$$

Practically, it is sufficient to assume that the stiffness $\tilde{k}$ is piecewise constant, that is to say,

$$X = \bigcup_{i=1}^N X_i, \forall 1 \le i \ne j \le N, X_i \text{ and } X_j \text{ are non-overlapping,}$$

$$\tilde{k}(x) = \{\tilde{k}_i, x \in X_i\} \tag{9}$$

where N is the number of non-overlapping pieces, or *elements*. Under this circumstance, equation (8) is reduced into

$$\int_{X_i} \left\{ \frac{\tilde{M}^2}{\tilde{k}_i^2} - (w'')^2 \right\} \delta\tilde{k}_i dx = 0, i = 1, 2, ..., N \tag{10}$$

and then, one has

$$\tilde{k}_i = \sqrt{\frac{\int_{X_i} \tilde{M}^2 dx}{\int_{X_i} (w'')^2 dx}}, i = 1, 2, ..., N . \tag{11}$$

It turns out that equation (11) looks like the direct use of the constitutive relation in equation (3). Thus, this step is named the *constitutive-relation* step. After integration of both equations (7) and (11), a two-step iterative algorithm for inverse identification of the damaged stiffness could be established.

2.3. Identifiability
In fact, convergence of the algorithm requires the uniqueness of the minimizer and therefore, the algorithm may not work well in the case of $\mu(w''(x) = 0, x \in X) > 0$. Let us assume that the measured displacements do not deform on a certain region, e.g., $w'' = 0$ on X_l with $l \in \{1, 2, ..., N\}$ being a number. Thus, the CRE is slightly modified into

$$e_{CRE}^{\text{mod}}(\tilde{w}, \tilde{M}, \tilde{k}) = \frac{1}{2} \int_{X \setminus X_l} \frac{(\tilde{M} - \tilde{k}\tilde{w}'')^2}{\tilde{k}} dx \tag{12}$$

that is to say, to identify the stiffness of a beam at a certain region, practical loads must render this region deformable, i.e. $w'' \ne 0$. If the stiffness of an undeformed region must be identified, another static load should be enforced on the beam so that this region is deformed and another set of static measurements is obtained. To be distinct, denote the extra measured displacement field by $w_1 \in \mathcal{U}_1$ and the corresponding bending moment is sought in $\tilde{M}_1 \in \mathfrak{I}_1$. Then, the inverse identification procedure can proceed in an analogous way to equation (13),

$$(M_1, M, k) = \arg \min_{\tilde{M} \in \mathfrak{I}, \tilde{M}_1 \in \mathfrak{I}_1, \tilde{k} \in \mathcal{C}_h} \left\{ re_{CRE}(w, \tilde{M}, \tilde{k}) + r_1 e_{CRE}(w_1, \tilde{M}_1, \tilde{k}) \right\} \tag{13}$$

with $r, r_1 \geq 0$ and $r + r_1 = 1$. The parameters r and r_1 are determined by the reliability of practical measurements. In a word, the strategy to introduce additional sets of static measurements can help improve and enhance the identifiability of the algorithm, and provide remedy for possible unidentifiability of the proposed approach accordingly. In fact, the strategy is also effective for other inverse identification problems [17].

3. Numerical tests

To show the effectiveness of the proposed damage identification approach, two beams are studied. They are a simply supported beam and a propped cantilever beam. A Gaussian noise is added to the simulated response as

$$\text{noise}(x) = randn \cdot a \cdot u(x) \tag{14}$$

where $randn$ is the standard Gaussian random distribution with zero mean and unit standard deviation, a is the applied noise level and $u(x)$ is the simulated displacement at a certain point x. For application of the two-step substitution algorithm, the convergence tolerance is practically set to $\text{TOL} = 0.001$.
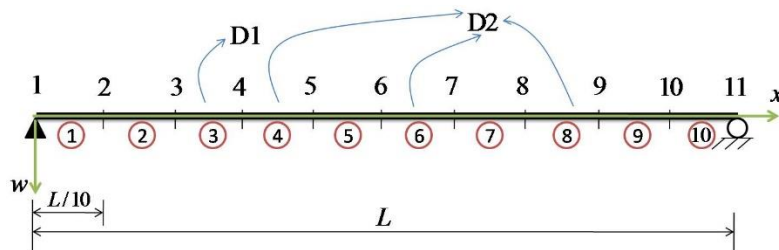

Figure 1. Geometry and damage location of simply supported beam

3.1. Example 1—a simply supported beam

Damage identification of a simply supported beam is studied herein. There are eleven uniformly distributed measurement points which divide the beam into ten elements as enumerated in **Figure 1**. Parameters of the intact beam are: length $L = 6\,\text{m}$, Young's modulus $E = 1 \times 10^{10}\,\text{Pa}$ and the rectangle section is of size $0.1\text{m} \times 0.1\text{m}$; that is to say, inertial moment of the beam is $I = 0.08333 \times 10^{-4}\,\text{m}^4$ and the bending stiffness of the intact beam is $k^0 = EI = 8.33 \times 10^4\,\text{N} \cdot \text{m}^2$ over the whole beam.

Two damage cases D1 and D2, having single damage region and multiple damage regions, respectively, are considered:
- Case D1: stiffness reduced to 50% in element 3;
- Case D2: stiffness reduced to 50%, 30% and 70% in elements 4, 6 and 8, respectively.

Moreover, two different load cases are enforced separately as
- Case L1: concentrated force $F = 1\,\text{kN}$ at the middle point (point 6); and
- Case L2: uniformly distributed load $q = 0.5\,\text{kN/m}$ over the whole beam,

such that the effect of load types on damage identification is examined. The combinatory of the two damage cases D1 and D2 with the two load cases L1 and L2 provides four damage identification scenarios (I1:D1+L1; I2:D1+L2; I3:D2+L1; I4:D2+L2). The displacements under the four scenarios (I1, I2, I3 and I4) are measured numerically by finite element computations, since for an elastic Bernoulli-Euler beam with piecewise constant stiffness, finite element analysis with cubic Hermit shape functions can give rise to the exact nodal displacements if every element is of constant stiffness [18].

For this statically determinate beam, the exact bending moment is directly known, a single step in equation (11) would complete the identification of the damaged stiffness. For the sake of convenience, the damage index DAI is introduced to represent the scaled damage in the beam, i.e.

$$\text{DAI}_i = k_i^{damage} / k_i^0 \tag{15}$$

for the i th element where k_i^{damage} is the damaged stiffness in element i and k_i^0 is the undamaged initial stiffness. Then, damage indices for the four scenarios are computed and listed in **Table 1**. Obviously, the damage is perfectly identified and reproduced for all the four scenarios. Nevertheless, it is seen that the concentrated load (in scenarios I1 and I3) can lead to slightly better damage identification results than the uniformly distributed load (in scenarios I2 and I4); this is reasonable since Hermit interpolation of the exact pointwise displacements can lead to the exact displacement field for a concentrated load, but only gives an approximate displacement field w for a uniformly distributed load.

Table 1. Identified damage indices of simply supported beam

Element number		1	2	3	4	5	6	7	8	9	10
Damage identification scenarios	I1	1.0000	1.0000	0.5000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	I2	1.0001	1.0000	0.5000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0001
	I3	1.0000	1.0000	1.0000	0.5000	1.0000	0.3000	1.0000	0.7000	1.0000	1.0000
	I4	1.0001	1.0000	1.0000	0.5000	1.0000	0.3000	1.0000	0.7000	1.0000	1.0001

In addition, to investigate the robustness of the proposed approach, measurement noises of levels 1%, 2%, 5% and 10% are taken into consideration for scenarios I1 and I4. The identified results are displayed in **Figure 2**. It is seen that when measurement noise is not greater than 5%, the damage can be well identified; while if noises rise to 10%, the proposed approach may not work well for a statically determinate beam structure. To examine how identified results are affected by the number of measurement points, the damage case D1—50% stiffness reduction over interval $1.2\text{m} \le x < 1.8\text{m}$ of the beam— under concentrated moments $M_0= -1\text{kNm}$ at the left end and $M_L=1\text{kNm}$ at the right end is further considered for 11 and 21 uniformly distributed measurement points and under measurement noises of levels 0%, 2% and 5%. As shown in **Figure 3**, the identified results are almost of the same quality under either 10 or 20 measurement elements. Thus, it is deduced that the quality of the identified results depends mainly on the accuracy of measurement rather than the number of measurement points.
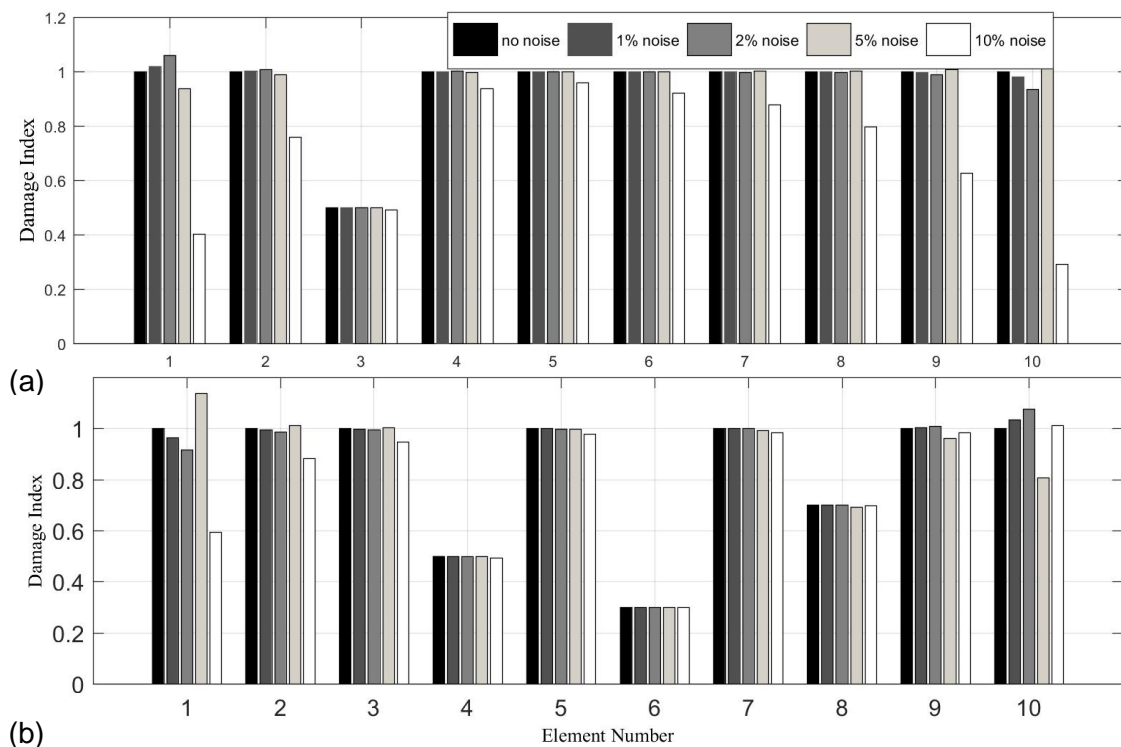

Figure 2. Damage identification with measurement noises in simply supported beam for: (a) scenario I1 and (b) scenario I2
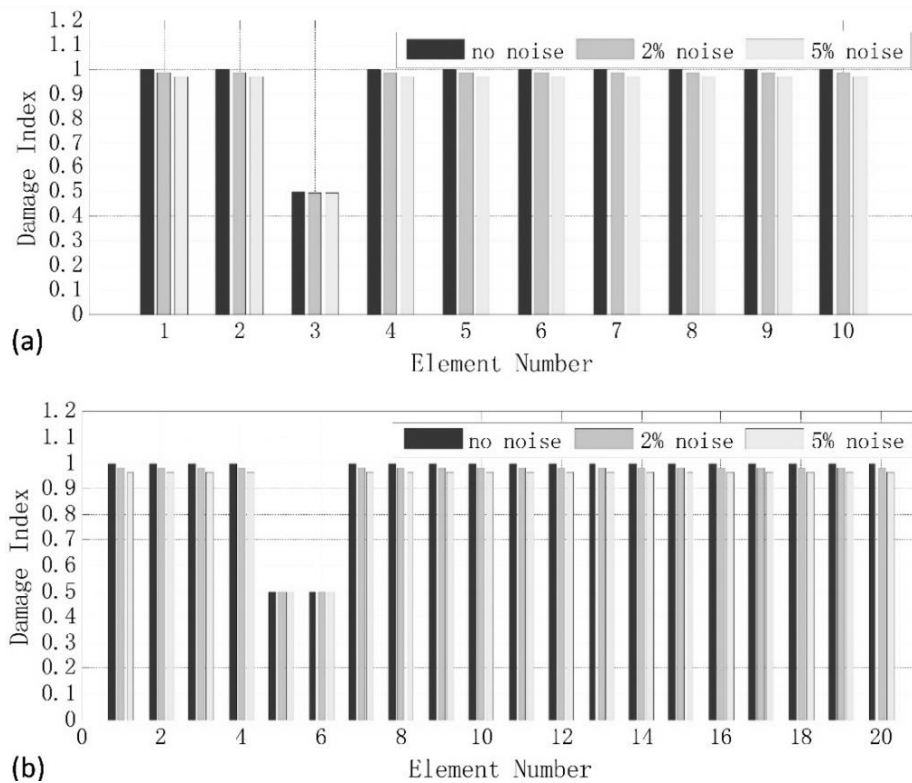
Figure 3. Damage identification with measurement noises in simply supported beam under (a) 10 measurement elements and (b) 20 measurement elements
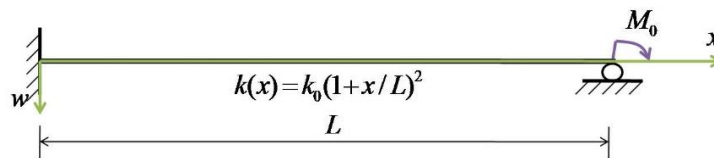

Figure 4. Geometry and loads for a propped cantilever beam

3.2. Example 2—a propped cantilever beam

A propped cantilever beam of continuously varying stiffness $k(x) = k_0(1 + x/L)^2$ with $k_0 = 1\text{kN} \cdot \text{m}^2$ (see **Figure 4)** is studied. The geometric parameter is $L = 1\text{m}$ and the load is a concentrated moment $M_0 = 1\text{kN} \cdot \text{m}$ at the right end. For practical measurements, points are uniformly distributed and pointwise displacements are obtained through refined finite element computation. Three sets of static measurements for 11, 51 and 101 measurement points are studied to see the effect of the number of measurement points on damage identification. To measure the error, an error index ERI is introduced as

$$\text{ERI} = \max_{x \in X} \left| k^{id}(x) - k^{ex}(x) \right| \tag{16}$$

with k^{id} and k^{ex} denoting the identified and exact stiffness. Eventually, the piecewise constant stiffness is identified for all the three sets of measurements as displayed in **Figure 5**. It is found that the increase of number of measurement points from 11 to 51 makes ERI decrease substantially and the stiffness is

better identified; however, when the number continues to rise to 101, ERI increases, being even larger than that under 11 measurement points. To explain the unexpected result under 101 measurement points, the bending moment shown in **Figure 5**(d) is examined. The largest error is found to occur in the vicinity of zero bending moment where $M \approx 0$ and hence $w'' \approx 0$; that is to say, the nearby elements hardly deform and therefore, the perturbation and deviation in **Figure 5**(c) is reasonable due to the limited identifiability of the proposed approach.

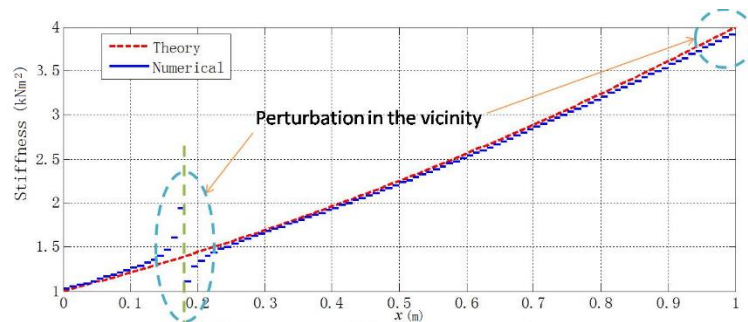**Figure 5.** Damage identification in propped cantilever beam under right-end concentrated moment

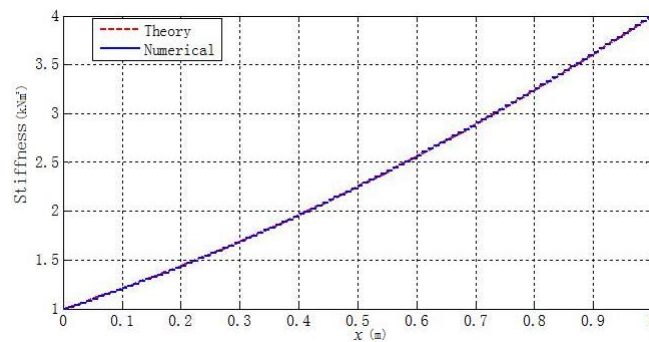**Figure 6.** Stiffness identification in propped cantilever beam under concentrated force at *x*=0.3

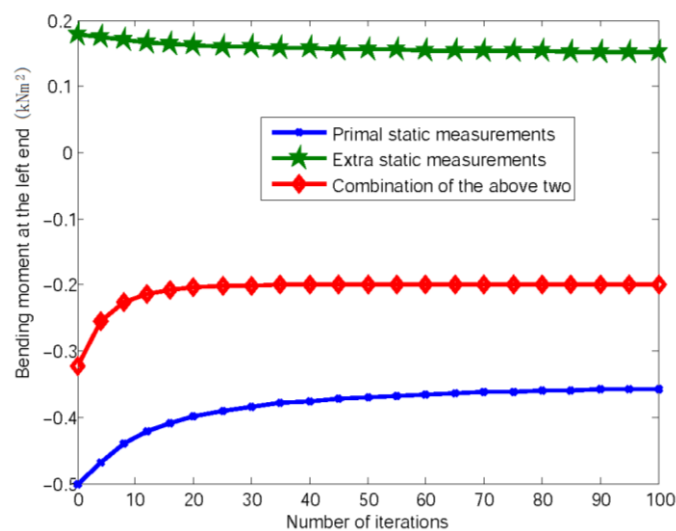**Figure 7.** Damage identification in propped cantilever beam using two sets of static measurements

Figure 8. Convergence of bending moment at left end of propped cantilever beam

To identify the stiffness of the beam with 101 measurement points accurately, the strategy presented in Section 2.3 is adopted. Another set of static measurements is required. The load is selected as a concentrated force $F = 1$kN at $x = 0.3$m. Under the single extra set of static measurements, the stiffness can be identified similarly and the bending moment is also obtained as exhibited in **Figure 6**. As expected, the identified stiffness perturbs in the vicinity of the zero bending moment. Fortunately, it is found that the location of the zero bending moment is different from that in the previous case (see **Figure 5**(d)) and it is deduced that after combination of the extra measurements with the previous measurements, the stiffness can be well identified. Then, the identified results are shown in **Figure 7**. Evidently, the stiffness is perfectly identified.

In addition, to show the convergence of the proposed two-step substitution algorithm, the bending moment at the left end of the beam $M(x = 0)$ is observed at each iteration step for all the three cases: the single primal set of static measurements, the single extra set of static measurements and combination of the former two. Detailed results are displayed in **Figure 8**. It is seen that the bending moment converges after 80 iterations for all the three cases, verifying the convergence of the two-step substitution algorithm. Furthermore, the combination case becomes convergent after only 20 iterations, being obviously faster than other two individual cases. This is reasonable since the introduction of an extra set of static measurements can enhance identifiability and stability of the algorithm.

4. Conclusions

A new approach based on the minimum constitutive relation error (CRE) principle has been proposed for damage identification in Bernoulli-Euler beams. A two-step substitution algorithm has been established to fulfil the practical implementation and its convergence has been proved. The flaw of the proposed approach for the stiffness in zero bending moment regions can be remedied by introducing another set of static measurements. Numerical tests have been carried out to verify the approach. The sound performance of the proposed approach in the following aspects has been observed:

- It is well applicable to both statically determinate and indeterminate beam structures;
- The proposed remedy for identifiability limits performs well;
- It can well identify single damage as well as multiple damage, under concentrated or uniformly distributed testing loads;
- Damage can be well identified under a noise up to 5%, verifying the robustness of the approach.
- The two-step substitution algorithm converges well in practice.

Therefore, it is believed that the proposed approach can serve as an effective tool for practical structural damage identification.

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