

Influence of Torsional Excitation on Dynamic Responses of Rotors with a Breathing Slant Crack

Zhiwen Lu, Chunrong Hua, Dawei Dong, Bing Yan, Kang Fan

School of Mechanical Engineering, Southwest Jiaotong University, Chengdu 610031

Email: hcrong@swjtu.cn

Abstract. Focusing on a rotor-bearing system with a breathing slant crack in the power transmission machine, influence of torsional excitations on the coupled nonlinear responses of the system is studied in this work. The slant crack element stiffness matrix is derived based on energy principal and the crack breathing phenomenon is simulated by the Crack Closure Line Position (CCLP) model; and the time-varying coupled dynamic equation of a rotor with a slant breathing crack considering the eccentricity of static unbalance is established using the finite element method and is solved by the NEWMARK method; then the influences of static torque and periodic torsional excitations on rotor dynamic responses in transverse and torsional directions are discussed. Results show that with the increment of static torque, cracks will become open gradually and the nonlinearity degree of rotors will increase firstly and then decrease. For periodic torsional excitation, the torsional excitation frequency and its rotating frequency combination can be found in transverse vibration response, and the larger is the amplitude of excitation, the larger are the combinational frequency components. Then a crack monitoring method for power transmission machines can be suggested by monitoring the coupled response characteristics and their variation from transverse responses of rotors before and after the loads change.

1. Introduction

Rotors are the critical parts of power transmission machines, the shaft easily produces fatigue crack and induces structural bearing failure under long-term complicated and severe operating condition. The slant crack is the most common form of crack owing to the torque transmitting characteristic of power transmitting rotor, and the crack will appear open and close breathing phenomenon under the rotor gravity, eccentricity, and external excitation when the rotor is rotating. Therefore, it is significant for crack detection to investigate the influence of torsional excitation on the dynamic characteristic of the power transmission rotor with slant breathing crack.

Though much research has been carried out on the cracked rotor dynamics ^[1-4], most of which aim at rotors with transverse cracks and assume the gravity dominant, yet the slant crack is seldom studied at present. However, the cracks in rotors are often slant cracks due to transmitted torque. Moreover, the static torque applied on rotors will affect the dynamic characteristics of rotor systems, which has not been paid enough attention at present.

Based on the above problems, the coupled vibration characteristic of a rotor-bearing system with a slant breathing crack under torsional excitation including periodic and static torsional excitation is investigated in this paper.

A model that reflects the essential behaviour of a crack is vitally important. There are many methods



to model a crack. Nonlinear 3D finite element method was adopted to model a breathing crack in a rotor in [5-6], which may be the most accurate model, but the computation workload is very heavy. In [7], rigid finite element method was put forward to model a cracked rotor, which also had good accuracy to model a breathing crack. By considering gravity dominant, rotation dependent breathing model such as switching model ^[8], cosine model ^[9], truncated Fourier series model ^[10] etc. are promoted for breathing crack modelling. Ref. [3] reviewed the crack model in rotors based on strain energy release rate (SERR), and showed that the method put forward by Darpe et al. ^[11, 12] could model a breathing crack in rotors more accurately, and it had the advantages of allowing general excitations without assuming that gravitational force is dominant, and the behaviour of the breathing crack was response dependent instead of being rotation dependent.

As gravity dominant is not always the case for rotors of power transmission equipment, it is unreasonable to simulate the crack model of these rotors with rotation dependent breathing model. Considering synthetically the model accuracy and computation complexity, the Crack Closure Line Position (CCLP) model proposed by Darpe et al. is adopted in this paper, the slant crack element stiffness matrix is derived based on energy principal, and then the time-varying coupled dynamic equation of a rotor with a slant breathing crack considering the rotor eccentricity is established; the NEWMARK method is used to solve the motion equations in time domain, and the influences of static torque and periodic torsional excitations on rotor dynamic responses in transverse and torsional directions are discussed. The research will provide theoretical basis for crack detection in rotors of power transmission equipment.

2. Stiffness matrix of a shaft with a breathing slant crack

2.1. The slant crack element stiffness derivation based on energy principle

Fig.1 shows a slant cracked shaft element of length l and radius R . P_1 – P_{12} are the loads acting on the 12 degrees-of-freedom of the two nodes in the element coordinate system $x - y - z$. The local coordinate system $x' - y' - z'$ is defined on the flat crack face to describe the crack cross-section. θ_c is the crack angle between the crack face and the shaft centreline (formed by the negative z' axis turning to the negative x axis in the counter-clockwise direction), x_L is the location of the crack centre in the element coordinate system. CCL (crack closure line) is an imaginary line that separates the open and closed parts of the crack which will be used to simulate the breathing of crack. The hatched area corresponds to the open area of the crack.

Based on Castigliano theorem ^[13], the flexibility coefficient of a crack element can be represented as:

$$g_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} (U^0 + U^c) = \frac{\partial^2 U^0}{\partial P_i \partial P_j} + \frac{\partial^2 U^c}{\partial P_i \partial P_j} = g_{ij}^0 + g_{ij}^c \quad (1)$$

where u_i is the generalized displacement of the i th freedom, P_i is the node force in the i th freedom, U is the total strain energy, U^0 is the strain energy of the un-cracked element; U^c is the additional strain energy induced from crack, g_{ij} and g_{ij}^0 are flexibility coefficients of the cracked and un-cracked element respectively; g_{ij}^c is the additional flexibility coefficient caused by crack.

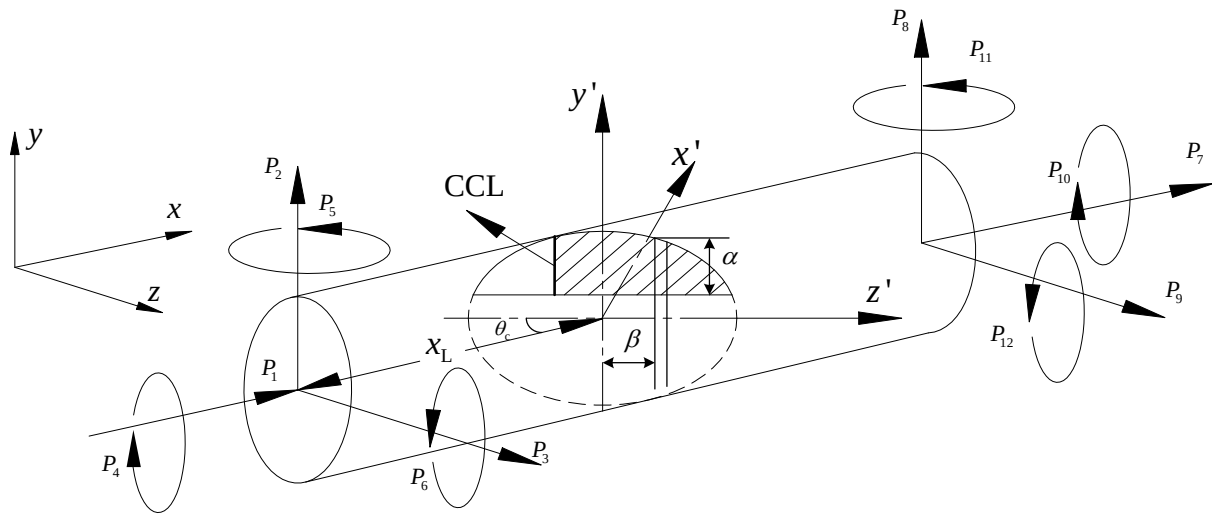


Fig. 1 Element with slant crack and diagram of the crack area

The strain energy of the un-cracked shaft element in Fig.1 is:

$$U^0 = \frac{1}{2} \int \left[\frac{P_1^2}{EA} + \frac{\alpha_s P_2^2}{GA} + \frac{\alpha_s P_3^2}{GA} + \frac{P_4^2}{EI_0} + \frac{(P_3 x_L + P_5)^2}{EI_y} + \frac{(P_6 - P_2 x_L)^2}{EI_z} \right] dx \quad (2)$$

where E is elastic modulus, G is shear modulus, A is cross section of the shaft; α_s is shear coefficient, $\alpha_s = (1+\nu)/(1.305+1.273\nu)$; I_0 is polar moment of inertia; I_x, I_y are area moment inertias.

According to the strain energy release rate theory^[3], the additional strain energy of the shaft resulted from crack is:

$$U^c = \frac{1}{E'} \iint_{A^c} \left[\left(\sum_{i=1}^6 K_{Ii} \right)^2 + \left(\sum_{i=1}^6 K_{IIi} \right)^2 + (1+\nu) \left(\sum_{i=1}^6 K_{IIIi} \right)^2 \right] dA^c \quad (3)$$

where $E' = E / (1-\nu^2)$, ν is Poisson's ratio; K_{Ii} , K_{IIi} , K_{IIIi} are the opening, sliding and tearing mode SIFs respectively; A^c is the open area of crack.

Substituting equation (2) and (3) into equation (1), and flexibility matrix of the cracked and un-cracked shaft element $(\mathbf{G}^0)_{6 \times 6}$ and $(\mathbf{G}^c)_{6 \times 6}$ can be derived, and the expression of each element of the matrix can be obtained from [11].

Then the total flexibility matrix of the cracked element is:

$$\mathbf{G}_{ce} = \mathbf{G}^0 + \mathbf{G}^c \quad (4)$$

Considering static equilibrium of the shaft element in Fig.1:

$$\{P_1, P_2, \dots, P_{12}\}^T = \mathbf{T} \{P_1, P_2, \dots, P_6\}^T \quad (5)$$

where

$$\mathbf{T}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & l \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -l & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (6)$$

According to the definition of flexibility

$$\mathbf{T}^T \{u_1, u_2, \dots, u_{12}\}^T = \mathbf{G}_{ce} \{P_1, P_2, \dots, P_6\}^T \quad (7)$$

From equation (5) and (7), one can get:

$$\{P_1, P_2, \dots, P_{12}\}^T = \mathbf{T}(\mathbf{G}_{ce})^{-1} \mathbf{T}^T \{u_1, u_2, \dots, u_{12}\}^T \quad (8)$$

So the stiffness matrices of the cracked and un-cracked shaft element are:

$$\mathbf{K}_{ce} = \mathbf{T}(\mathbf{G}_{ce})^{-1} \mathbf{T}^T \quad (9)$$

$$\mathbf{K}_{uce} = \mathbf{T}(\mathbf{G}^0)^{-1} \mathbf{T}^T \quad (10)$$

2.2. The breathing slant crack model

The crack will represent open and close breathing state when the rotor is rotating, so the equation (9) will change with the crack state. Using the breathing crack model based on the Crack Closure Line Position (CCLP) ^[11], the crack front is subdivided into m part, K_I can be calculated from equation (11), which determines the position of the crack closure line, the positive K_I represents the corresponding position of the crack front is open and vice versa.

$$K_I = \sum_{i=1}^6 K_{I_i} \quad (11)$$

The SIFs of the slant cracked element under node forces from each freedom in Fig.1 are:

$$\begin{cases} K_{I1} = P_1 / (\pi R^2) \sin^2 \theta_c \sqrt{\pi \alpha} F_1; K_{I2} = 0; K_{I3} = k P_3 / (\pi R^2) \sin 2\theta_c \sqrt{\pi \alpha} F_1 \\ K_{I4} = 2 P_4 / (\pi R^4) \sqrt{R^2 - \beta^2 \sin^2 \theta_c} \sin 2\theta_c \sqrt{\pi \alpha} F_2 \\ K_{I5} = 4(P_5 + P_3(x_L + \beta \cos \theta_c)) / (\pi R^4) \beta \sin^3 \theta_c \sqrt{\pi \alpha} F_1 \\ K_{I6} = 4(P_2(x_L + \beta \cos \theta_c) - P_6) / (\pi R^4) \sqrt{R^2 - \beta^2 \sin^2 \theta_c} \sin^2 \theta_c \sqrt{\pi \alpha} F_2 \end{cases} \quad (12)$$

where, θ_c is crack angle, F_1 and F_2 are empirical coefficients of SIF, and the definition of other parameters are shown in Fig.1. From equation (12), it can be seen that the torsional excitation will affect the K_{I4} , then influence the open and close state of crack and thus affect the system responses.

3. The motion equation of the cracked rotor and its time domain solution

For the slant crack rotor-bearing system in Fig. 2(a), supposing the crack only influence the local stiffness, and the system motion equation will be derived once the breathing slant crack element is integrated into the rotor-bearing system based on the finite element method.

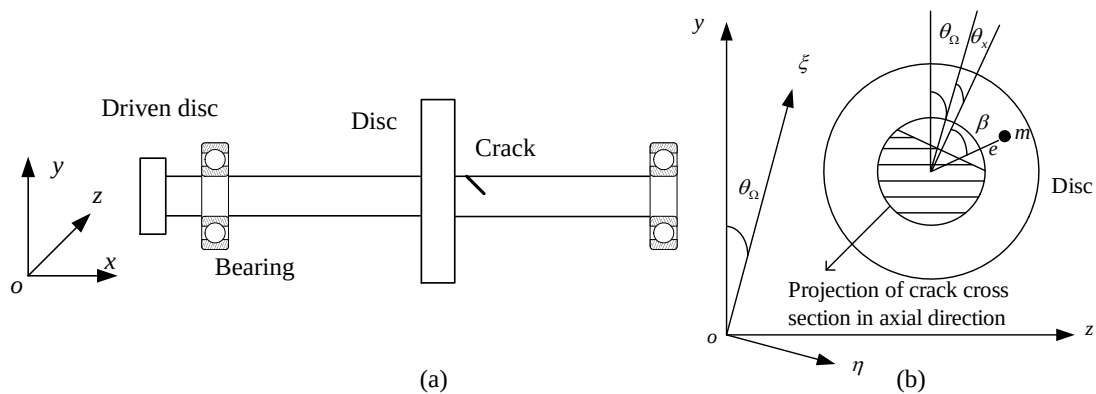


Fig. 2 (a) Diagram of slant cracked rotor-bearing system in power transmission (b) Rotating and stationary coordinates of the rotor

Setting the displacement vector of node i is $\mathbf{q}_i$, and 3 translational and 3 rotational degrees of freedoms are considered in each node,

$$\mathbf{q}_i = \{x_i, y_i, z_i, \theta_{xi}, \theta_{yi}, \theta_{zi}\}^T \quad (13)$$

Therefore, the equation of motion of the cracked rotor in global static coordinate system can be represented as:

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \Omega\mathbf{D}_g)\dot{\mathbf{q}} + \mathbf{K}(t)\mathbf{q} = \mathbf{F}_e + \mathbf{F}_g + \mathbf{F}_{ex} \quad (14)$$

$$\mathbf{q} = \{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_i, \dots, \mathbf{q}_n\}^T \quad (15)$$

where, $\mathbf{M}$ is system mass matrix; $\mathbf{K}$ is system stiffness matrix (the stiffness matrix of the cracked rotor are composed from un-cracked and cracked element stiffness matrix according the node position, as the stiffness matrix of breathing crack element is response-dependent nonlinear, so does the system stiffness matrix). $\mathbf{D}$ is system damping matrix, considering Rayleigh damping, $\mathbf{D} = a\mathbf{M} + b\mathbf{K}$; Ω is rotating speed of the rotor; $\mathbf{D}_g$ is Gyroscopic matrix; $\mathbf{F}_g$ is gravity vector; $\mathbf{F}_e$ is eccentric excitation; $\mathbf{F}_{ex}$ is external excitation; $\mathbf{q}$ is generalized displacement vector.

The gravity vector of the disc in node i :

$$\mathbf{F}_{gi} = \{0, -mg, 0, 0, 0, 0\}^T \quad (16)$$

Eccentric excitation of the rotating disc is ^[14]:

$$\mathbf{F}_{ei} = \{F_{exi}, F_{eyi}, F_{ezi}, M_{exi}, M_{eyi}, M_{ezi}\}^T \quad (17)$$

$$\begin{cases} F_{exi} = 0; M_{eyi} = 0; M_{ezi} = 0 \\ F_{eyi} = me[(\Omega + \theta_{xi})^2 \cos(\Omega t + \theta_{xi} + \beta) + \theta_{xi} \sin(\Omega t + \theta_{xi} + \beta)] \\ F_{ezi} = me[(\Omega + \theta_{xi})^2 \sin(\Omega t + \theta_{xi} + \beta) - \theta_{xi} \cos(\Omega t + \theta_{xi} + \beta)] \\ M_{exi} = me[y \sin(\Omega t + \theta_{xi} + \beta) - (z + g) \cos(\Omega t + \theta_{xi} + \beta)] \end{cases} \quad (18)$$

where, $\theta_{\Omega}(t)$ is the relative angular displacement between the rotating and stationary coordinate system; $\theta_x(t)$ is the torsional angular displacement fluctuation of the rotor; β is unbalanced orientation angle. These parameters are shown in Fig. 2(b).

From equation (18), one can see that the eccentric excitation of rotating disc is bending and torsional coupling which will result in more complicated rotor responses.

In this paper the torsional excitation is mainly considered as external excitation, including the static load torque T_s applied into the torsional freedom of the output node and the periodic torsional excitation T_p applied into the torsional freedom of the rotating disc node.

As the breathing state of a crack is decided by the nodal forces in the cracked element, and the rotor responses are related to the eccentric excitation, therefore the motion equation of the cracked rotor has time-variable and nonlinear characteristic. NEWMARK method ^[15] is adopted to solve the equations in time-domain, and updating the system stiffness matrix, damping matrix and coupling excitation in each time step, and the next time step starts as the last time step converges.

4. The influence of torsional excitation on the dynamic response of rotor

The main influence paths of torsional excitation on dynamic responses of a cracked rotor are: the static torque produces static torsional deformation influencing the opening mode SIF then the crack breathing state, leading to variation of system stiffness and responses; the periodic torsional excitation changes system inputs on one hand, and affects the crack state on the other hand, resulting in changes of the dynamic responses. Furthermore, the above factors act on a nonlinear cracked rotor with stiffness coupling which will make the vibration responses in each direction more complicated.

Parameters of the rotor in Fig. 2 (a) are in Tab.1. The rotor is divided into 15 two-node Timoshenko beam elements. Each node includes 3 translational and 3 rotational degrees-of-freedom. The two discs are considered rigid bodies and simplified as three translational and three rotational inertias, which are added to the mass matrix elements at the corresponding degrees-of-freedom. Gyroscopic effect of the two discs is also included. The ball bearings are simplified as isotropic linear springs and dampers, one of which constrains the axial degree-of-freedom. The torsional degree-of-freedom in the driven disc of the rotor is also constrained and the connection misalignment is neglected.

The slant crack is in the 8th element, the ratio of crack depth to diameter of the rotating shaft is 0.2, the crack angle θ_c is 45° , the eccentricity is 10% of the maximum static deformation of the rotor, the unbalance orientation angle is 0° . The rotating speed is set as 1/10 of the critic speed to investigate the coupled vibration responses of the rotor at subcritical speed.

Tab. 1 Parameters of the rotor

Parameter	Value(units)	Parameter	Value(units)
Shaft length	0.75m	Young's modulus	2.1×10^{11} Pa
Shaft diameter	0.016m	Poisson's ratio	0.3
Disc mass	2.17Kg	Rotating speed	162r/min
Polar moment of inertia of the disc	7×10^{-3} Kg.m ²	Rayleigh damping coefficient (a)	1.3186
Area moment of inertia of the disc	3.5×10^{-3} Kg.m ²	Rayleigh damping coefficient (b)	1.905×10^{-4}
Driven disc mass	0.96Kg	Bearing stiffness	2.5×10^5 N/m
Polar moment of inertia of the driven disc	1.2×10^{-3} Kg.m ²	Bearing damping	100Kg/s
Area moment of inertia of the driven disc	6.33×10^{-4} Kg.m ²	First bending frequency	26.9Hz
Maximum static deformation of rotor	2.66×10^{-4} m	Second bending frequency	64.1Hz
Gravitational acceleration	9.8 m/s^2	First torsional frequency	68.3Hz
Density of rotor	7.8×10^3 kg/m ³	First axial frequency	1018.7Hz

The NEWMARK method is adopted to solve the system in time domain, and the initial condition is the initial static deformation of the un-cracked rotor, the NEWMARK constant is set as 0.25 and 0.5, and the convergence precision is 10^{-11} , the sampling frequency is **1000Hz**.

4.1. The influence of the static torque

When the rotor has no crack, the transmitted static torque does not affect the dynamic response of the rotor system, as the rotor appears crack, the static torque will change the extension direction of the crack, and change the stress intensity factor of the front crack, which will influence the respiratory state of the crack.

In order to analyze the influence of static torque, vibration responses of each direction of the rotor are compared, when only the static torque and gravity force are applied into the rotor. The value of the static torque is provided by the ratio between the static torque and gravity $r_{S/G}$.

Fig.3 is the torsional and vertical vibration responses of the rotor as $r_{S/G}$ are 0, 0.18 and 0.3. It shows that the transmitted static torque not only affects the static torsional deformation, but also influences the torsional and transverse vibrations of the rotor system. As to the torsional vibration, the static torsional deformation increases with the transmitted static torque, and doubling frequency components appear in the spectra, which increase first and decrease and then diminish with increasing the static torsional torque. As to the vertical vibration, with the increase of the static torque, the amplitude of fundamental frequency and double frequency components increase gradually, the triple frequency and more high frequency components increase first and decrease and then diminish. The influence of torsional excitation on transverse vibration is similar to that on vertical vibration.

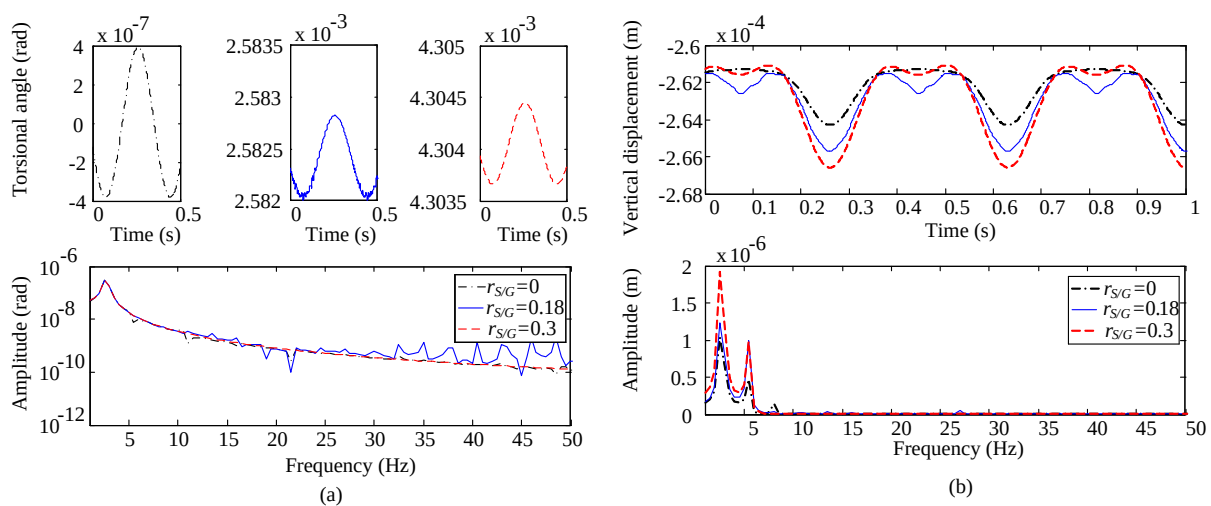


Fig.3 Vibration responses of the rotor with $r_{S/G} = 0, 0.18, 0.3$ (a) Torsional response; (b) Vertical response

In order to explain the influence of static torque on each direction vibration of the rotor, the variation of crack element stiffness is recorded as $r_{S/G}$ are 0, 0.1, 0.18 and 0.3 (see Fig.4). It shows that, with the increase of the static torque, the crack is more largely in open state, and when the static torque is not large enough, the crack will experience close to open completely, but the developing speed of the process will become faster, so the nonlinearity will be increased, leading to the increase of the higher

order components. However, when the static torque increases continually, the crack will experience partially open to full open instead of full close to full open, so the crack is developing to an open crack, making the system stiffness variation smaller and smaller, and resulting in reduction in the nonlinearity of the cracked rotor system and decrease of the higher order frequency components. As the static torque exceeds some value, the crack is in full open state, the nonlinear system is developed to a linear system, as the stiffness of the cracked rotor is asymmetric, just the fundamental frequency and double frequency components are remained as well as other higher frequency components being vanished.

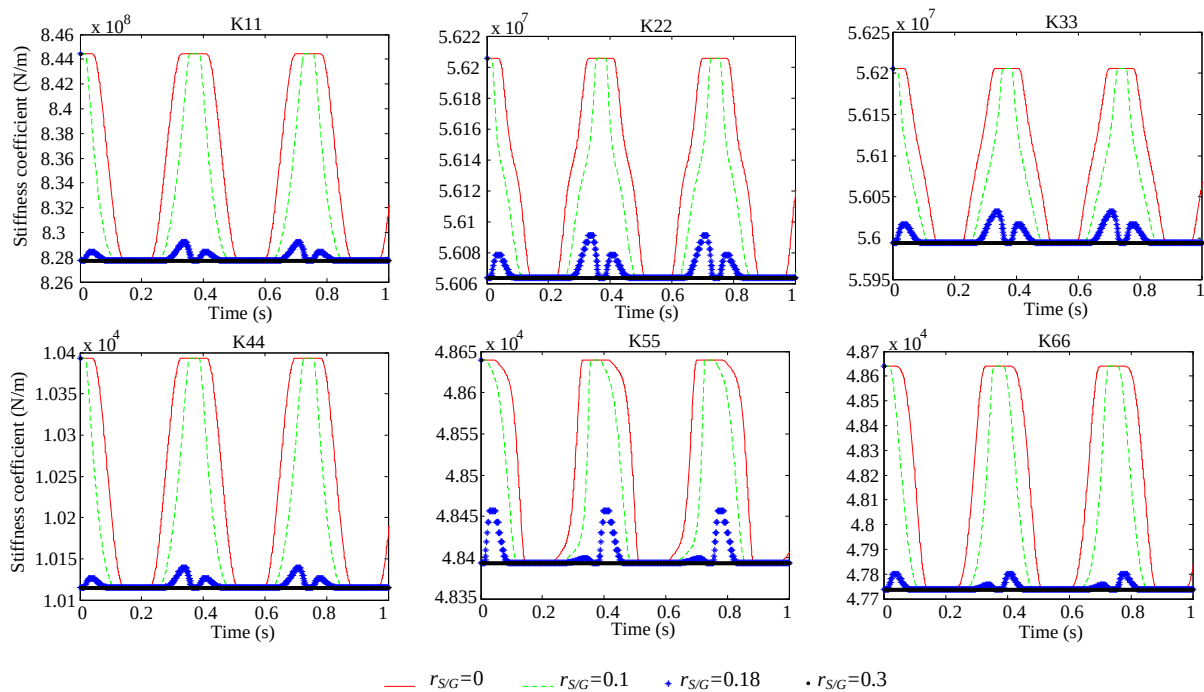


Fig.4 Main stiffness coefficients of crack element with $r_{S/G} = 0, 0.1, 0.18, 0.3$

4.2. The influence of periodic torsional excitation

The power transmission rotor may transmit periodic torque besides the static torque. Supposing the rotating disc is under periodic torsional excitation in different magnitudes, the exciting frequency (ω_T) of which is the first bending frequency (ω_b). The amplitude of the periodic torsional excitation is given by the ratio between torsional excitation to the rotor gravity $r_{P/G}$.

Fig.5 is the vibration responses in each direction of the slant cracked rotor as $r_{P/G} = 5\%, 10\%, 15\%$. It shows that, torsional vibration presents torsional single frequency response, and frequency components $\Omega, 2\Omega$, torsional exciting frequency ω_T and its combined frequencies $\omega_T \pm \Omega$ appear in vertical vibration response, indicating there are bending and torsional vibration coupling, and the amplitude of various frequencies increase with excitation amplitude; frequency components $\Omega, 2\Omega$ and the combined torsional frequencies $\omega_T \pm \Omega, \omega_T \pm 2\Omega$ appear in the transverse vibration response, the amplitude of various frequencies also increase with the excitation amplitude. The frequency components and change rule of the system response are in accordance with Ref. [11].

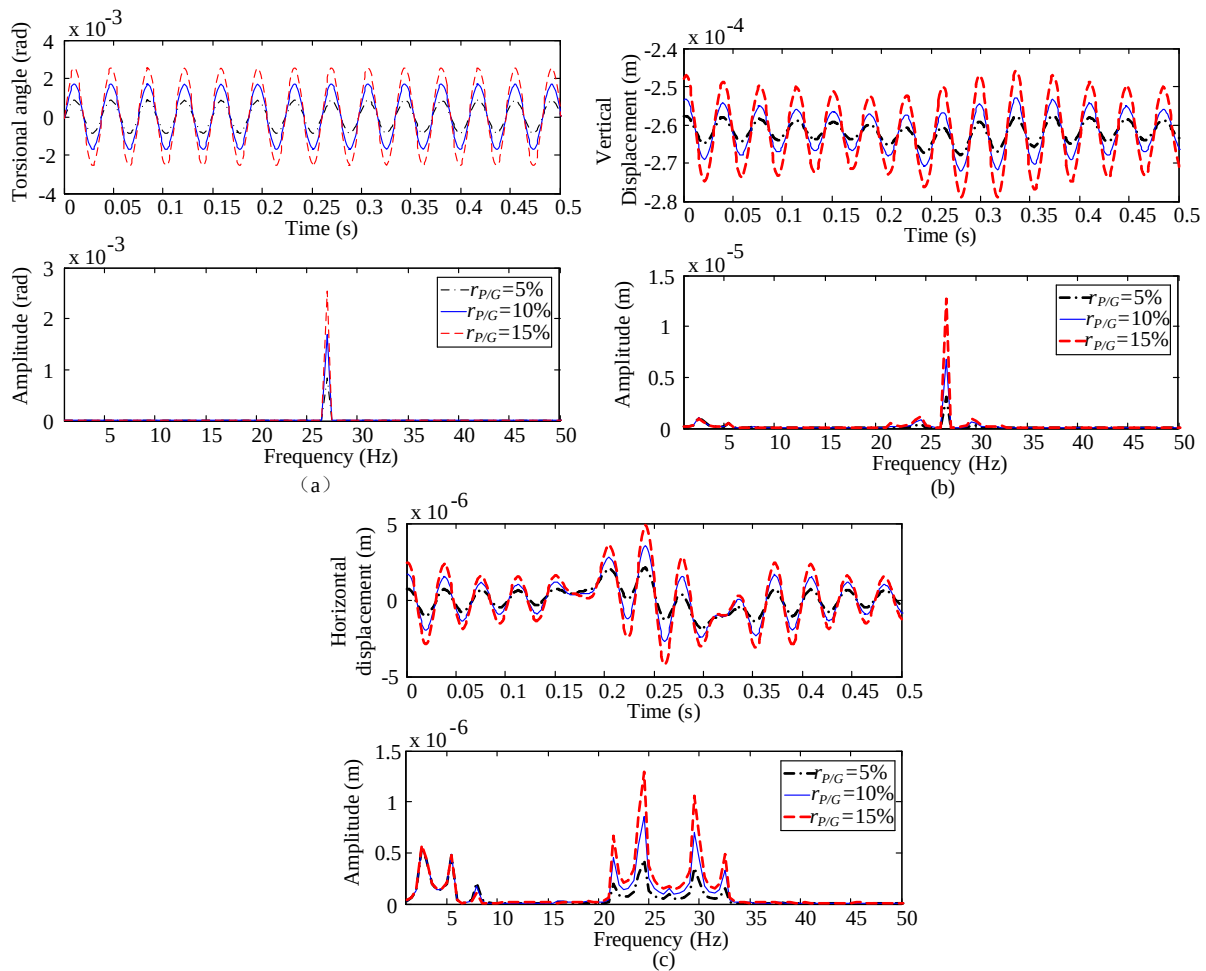


Fig.5 Vibration responses of the rotor with $r_{p/G}=5\%$ 、 10% 、 15% (a)Torsional response; (b) Vertical response; (c) Horizontal response

As to the torsional vibration, the torsional vibration excited by periodic torsional excitation is far greater than that by coupling, so the frequency components by coupling is submerged, and the torsional vibration shows single frequency response. As to the transverse vibration, the larger is the amplitude of the periodic torsional excitation, the greater is the influence on fluctuation of the crack stiffness. Though the crack is in the unchanged breathing rule from full open to full close, more local fluctuation in partial close state is added from open to close state of the crack, therefore, the greater is the nonlinearity of the rotor system, resulting in greater higher order frequency components and combined frequency components.

4.3. The synthesized effect

The static torque, periodic torsional excitation and eccentricity always simultaneously exist in power transmission machines, so their synthesized effect is investigated. Supposing the amplitude of the periodic torsional excitation is 10% of the transmitted static torque. Fig.6 shows the vibration responses of the rotor under synthesized action with $r_{s/G}=0.1, 0.15, 0.3$.

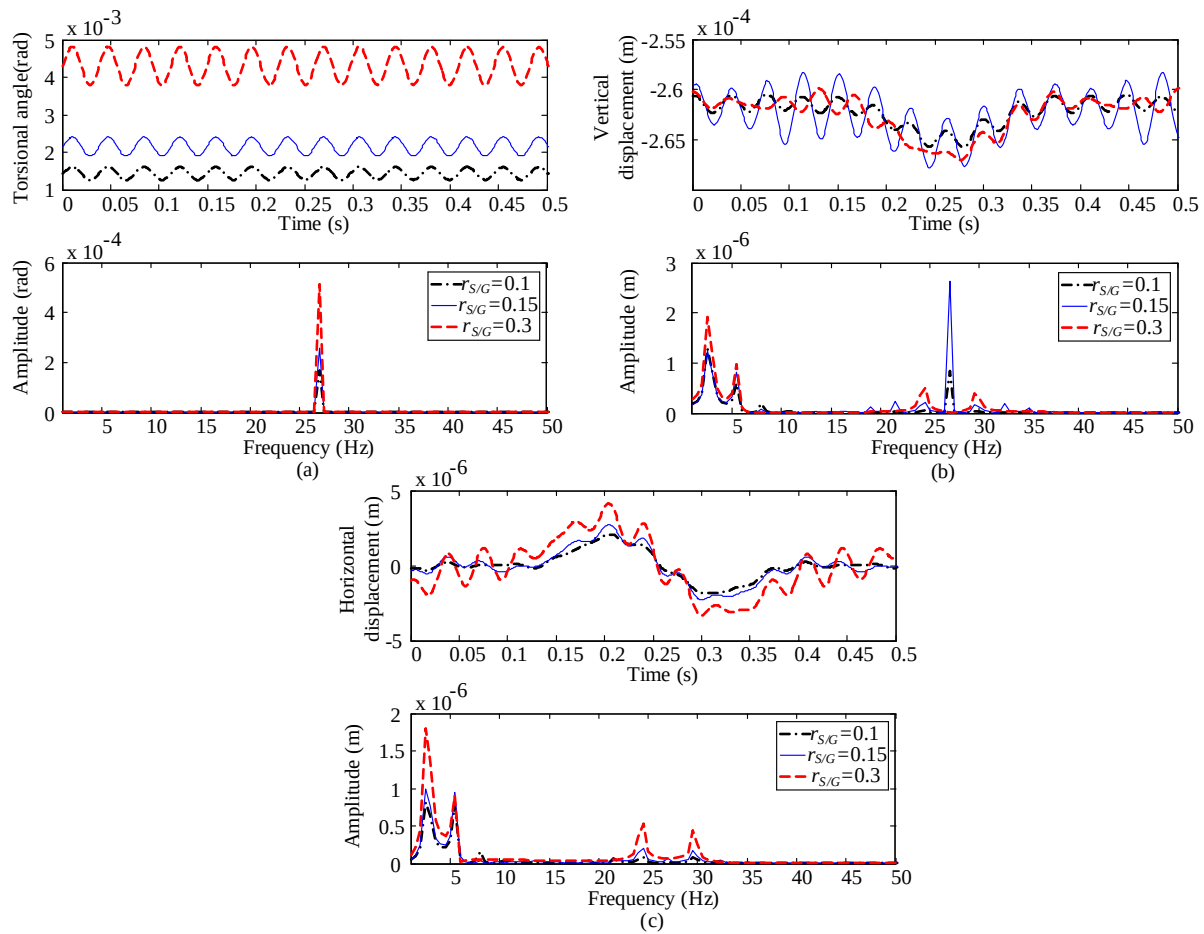


Fig.6 Vibration responses of the rotor with $r_{S/G} = 0.1, 0.15, 0.3$ under comprehensive factors (a) Torsional response; (b) Vertical response; (c) Horizontal response

It can be seen from Fig.6 that, the torsional vibration mainly represents the frequency component of periodic torsional excitation, and the amplitude of which increases with the static torque while the double frequency components is submerged. The frequency component Ω and combined frequency components $\omega_T \pm \Omega$ in vertical and transverse responses all increase with the static torque; the frequency components $2\Omega, 3\Omega$ and ω_T increase first and then decrease with the static torque. However the frequency component ω_T in vertical vibration increase with the periodic torsional excitation when there is no static torque (see Fig. 5), which indicates that the static torque changes the breathing state of the crack, leading to variation of the nonlinearity and coupling behaviour of the rotor.

Hence, as the transmitted static torque will make the crack open and change the nonlinearity of the system, the crack detection can be carried out by loading and unloading the power transmission machines first, and then testing the vibration characteristics before and after load change, furthermore, the load should be used as an important monitoring parameter in condition monitoring on rotor systems.

5. Conclusions

By studying the influence of the static torque and periodic torsional excitation on the transverse and

torsional dynamic response of an unbalanced rotor with breathing slant crack, conclusions are can be drawn as follows:

- (1) Static torque has great effect on vibration responses of each direction of breathing slant cracked rotor. Static torque makes the breathing crack become open crack, and the nonlinearity of the cracked rotor increase first and then decrease with the static torque. Thus the influence of static torque should be considered in crack detection.
- (2) Owing to the coupling of bending and torsional vibration of slant cracked rotor, the torsional excitation frequency and the combined frequency components of rotating frequency will appear in transverse vibration response of the rotor under the action of periodic torsional excitation, and the higher the exciting amplitude is, the larger the combined frequency components are.
- (3) As the transmitted static torque will make the crack in normally open state and change the nonlinearity of the system, the crack detection can be carried out by loading and unloading the power transmission, and comparing the vibration characteristics before and after load change; furthermore, the load should be used as an important monitoring parameter in condition monitoring on rotor systems.

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