

# A Time-Dependent Anharmonic Oscillator

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**Abstract.** Supersymmetric Quantum Mechanics is commonly used to generate time independent Hamiltonians with a desired spectrum. This technique can be generalized to construct time dependent potentials. In this work, the harmonic oscillator and a coherent state are taken to perform a generalized SUSY transformation in order to obtain a time dependent anharmonic oscillator.

## 1. Introduction

In 1984 B. Mielnik [1] constructed with a novel technique a potential with the spectrum of the quantum harmonic oscillator known as the Abraham-Moses potential [2]. Later this technique, called Supersymmetric Quantum Mechanics (SUSY), was studied in more detail and generalized in order to modify the spectrum of Hamiltonians, [3–6]. Another generalization of SUSY where the starting point is a time dependent Schrödinger equation with a potential that could depend on time is considered in [7] and [8]. In this article the SUSY technique is used to generate a time dependent anharmonic oscillator departing from the harmonic oscillator. Solutions for the new Schrödinger equation can also be obtained. The general theory is presented in Sec. 2, introducing first the simplest case known as 1-SUSY and later an iteration known as second order confluent SUSY transformation. In Sec. 3 the confluent technique is used to obtain the time-dependent anharmonic oscillator. Conclusions are presented in the last section.

## 2. Time-dependent supersymmetry transformation

As in the time-independent case, a supersymmetry transformation can be done relating two Schrödinger operator through an intertwining operator. In Sec. 2.1 we introduce a 1-SUSY transformation using a first order differential operator. In Sec. 2.2 we iterate this technique to generate new solvable potentials, this iteration will have a different restriction giving to this second order transformation more freedom.

### 2.1. Time-dependent 1-SUSY transformation

The time-dependent Schrödinger equation is given by

$$i\partial_t\psi + \partial_{xx}\psi - V_0\psi = 0, \quad x \in (x_\ell, x_r), \quad t \in (-\infty, \infty), \quad (1)$$

where the potential  $V_0 = V_0(x, t)$  is a real known function;  $x_\ell$  and  $x_r$  are the left and right endpoints of the domain of the potential. In order to generate a new exactly solvable system



consider the following intertwining relationship between two Schrödinger operators

$$S_1 L_1 = L_1 S_0, \quad (2)$$

where the Schrödinger operators  $S_0, S_1$ , and the intertwining operator  $L_1$  are defined as

$$S_j = i\partial_t + \partial_{xx} - V_j; \quad j = 0, 1; \quad L_1 = A_1 \left( -\partial_x + \frac{u_x}{u} \right), \quad (3)$$

where  $A_1 = A_1(t)$  is in principle a complex valued function but in this article it will be considered only the case when  $A_1$  is real-valued, the function  $u = u(x, t)$  is called transformation or seed function and  $u_x = \partial_x u(x, t)$ . A comparison of the expansions of the left and right hand sides of the intertwining relationship (2) after some simplifications lead us to the couple of equations

$$V_1 = V_0 + i \frac{A_{1t}}{A_1} - 2\partial_{xx} \ln u, \quad i\partial_{xt} \ln u + \partial_x \left( \frac{u_{xx}}{u} - V_0 \right) = 0, \quad (4)$$

where  $A_{1t} = \partial_t A_1(t)$  and  $u_{xx} = \partial_{xx} u(x, t)$ ; integration of the last equation with respect to  $x$  simplifies the equation  $u$  needs to fulfill to

$$i\partial_t u + u_{xx} - V_0 u = 0, \quad (5)$$

as in the time-independent SUSY (see [6]), the transformation function  $u(x, t)$  satisfies the Schrödinger equation, but now the time-dependent version; moreover, no separability of  $u(x, t)$  is assumed. Asking  $V_1$  to be a real potential, i.e.  $\text{Im}(V_1) = 0$ , and using (4), a second condition the transformation function needs to fulfill is

$$\partial_{xxx} \ln \left( \frac{u}{u^*} \right) = 0. \quad (6)$$

Since we are restricting ourselves to the case where  $A_1$  is a real valued function, it can be obtained as

$$A_1 = \exp \left\{ -i \int^t \partial_{xx} \ln \left[ \frac{u(x, s)}{u^*(x, s)} \right] ds \right\} = \exp \left\{ 2 \int^t \text{Im} [\partial_{xx} \ln u(x, s)] ds \right\}, \quad (7)$$

where  $\text{Im}(\cdot)$  stands for the imaginary part. The second expression shows explicitly that  $A_1$  is a real function. Substituting this last expression in (4) a simplified expression for  $V_1$  can be obtained

$$V_1 = V_0 - 2\partial_{xx} \ln |u|. \quad (8)$$

Note that in order to obtain a potential  $V_1$  with no more singularities than the ones already existing in  $V_0$ , it is necessary to use a transformation function  $u$  without zeros in the domain  $(x_\ell, x_r)$  at any time.

The solutions  $\phi$  of the new Schrödinger equation

$$i\partial_t \phi + \partial_{xx} \phi - V_1 \phi = 0, \quad x \in (x_\ell, x_r), \quad t \in (-\infty, \infty), \quad (9)$$

can be obtained by applying the operator  $L_1$  onto the solutions  $\psi$  of the original equation,  $\phi = L_1 \psi$ .

It is also important to notice that the adjoint equation of (2) give us a new intertwining relationship

$$L_1^\dagger S_1 = S_0 L_1^\dagger, \quad \text{where} \quad L_1^\dagger = A_1 \left( \partial_x + \frac{u_x^*}{u^*} \right), \quad (10)$$

and, as can be seen from (10), the function annihilated by  $L_1^\dagger$  could also be solution of  $S_1\phi = 0$ . To construct this solution  $\phi$  the first step is to solve the first order differential equation  $L_1^\dagger\phi = 0$  and then to ask the condition  $S_1\phi = 0$ , the final form of this extra solution is

$$\phi = \frac{1}{A_1 u^*}. \quad (11)$$

Thus, solutions of the equation  $S_1\phi = 0$  can be obtained with the operator  $L_1$  as  $\phi = L_1\psi$  or as given by (11).

Summarizing, given a Schrödinger equation with the set of solutions  $\psi$  satisfying the boundary conditions (physical solutions) and a transformation function  $u(x, t)$  fulfilling (5), (6) and  $u(x, t) \neq 0$  for all  $x \in (x_\ell, x_r)$  and  $t \in (-\infty, \infty)$  a new time-dependent potential  $V_1(x, t)$  given by (8) can be generated and the solutions of the corresponding Schrödinger equation (9) will have the form  $\phi = L_1\psi$ , and an extra solution can be found using (11), this is the so called missing state.

## 2.2. Second-order confluent supersymmetry algorithm

The time-independent confluent SUSY transformation introduced in [9–12] can be seen as an iteration of a 1-SUSY transformation. In the same way, an iteration can be performed for the time-dependent situation [8]. Let us consider the intertwining relation (2) and also

$$S_2 L_2 = L_2 S_1, \quad \text{where} \quad S_j = i\partial_t + \partial_{xx} - V_j, \quad j = 1, 2, \quad L_2 = A_2 \left( -\partial_x + \frac{v_x}{v} \right), \quad (12)$$

the intermediate transformation function  $v(x, t)$  satisfies  $S_1 v = 0$ . If we use the function (11) as transformation function turns out that  $V_2 = V_0$ , i.e. we obtain the original Schrödinger operator as a result of the second transformation, so we need a more general solution  $v$

$$v = \frac{1}{A_1 u^*} \left( \omega + \int^x |u(s, t)|^2 ds \right), \quad (13)$$

where  $\omega$  is considered in this manuscript as a real constant. Now, the new potential is given by

$$V_2 = V_1 - 2\partial_{xx} \ln |v| = V_0 - 2\partial_{xx} \ln \left| \omega + \int^x |u(s, t)|^2 ds \right|. \quad (14)$$

An important remark is that every value of the constant  $\omega$  defines a potential  $V_2$ , for different values different physics are modeled.

In order to produce regular potential the condition changed, we need a transformation function  $u(x, t)$  such that

$$\omega + \int^x |u(s, t)|^2 ds \neq 0, \quad (15)$$

which means that any normalized function will satisfy this condition, but also some non-square-integrable functions are going to fulfill this condition. The reality condition for  $V_2$  in terms of  $u$  can be expressed as

$$\partial_{xxx} \ln \left( \frac{v}{v^*} \right) = 0 \quad \Rightarrow \quad \partial_{xxx} \ln \left( \frac{u}{u^*} \right) = 0. \quad (16)$$

Finally, the function  $A_2(t)$  appearing in the definition of the operator  $L_2$  in (12) is

$$A_2 = \exp \left\{ -i \int^t \partial_{xx} \ln \left[ \frac{v(x, s)}{v^*(x, s)} \right] ds \right\} = \exp \left\{ -i \int^t \partial_{xx} \ln \left[ \frac{u(x, s)}{u^*(x, s)} \right] ds \right\}. \quad (17)$$

The assumption  $A_2 = A_1$  will only be considered in this manuscript, (compare eqs. (7) and (17)).

Solutions  $\phi$  of the equation  $S_2\phi = 0$  where  $S_2$  is given by (12) with  $V_2$  as in (14) can be obtained from solutions  $\psi$  of the original equation as

$$\phi = L_2 L_1 \psi = A_1^2 \{ \partial_{xx} - [\partial_x \ln(uv)] \partial_x + [(\partial_x \ln v)(\partial_x \ln u) - \partial_{xx} \ln u] \} \psi \quad (18)$$

and a missing state as

$$\phi = \frac{1}{A_1 v^*} = \frac{u(x, t)}{\omega + \int^x |u(s, t)|^2 ds}. \quad (19)$$

With this iteration  $u$  still needs to satisfy (5) and (6) but a transformation function with zeros can be used as long as the constant  $\omega$  is such that (15) is satisfied.

### 3. A time-dependent anharmonic oscillator

The time-dependent Schrödinger equation for the harmonic oscillator is given by [13, 14]

$$\left[ i\hbar \partial_t + \frac{\hbar^2}{2m} \partial_{xx} - \frac{1}{2} m \tilde{\omega}^2 x^2 \right] \psi = 0. \quad (20)$$

In this example units where  $\hbar = 1$ ,  $m = 1/2$  and  $\tilde{\omega} = 2$  will be considered, then the Schrödinger equation is

$$(i\partial_t + \partial_{xx} - x^2) \psi = 0. \quad (21)$$

Our initial potential  $V_0 = x^2$  is a time-independent potential. Stationary solutions are well known and are given by

$$\psi_n(x, t) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} H_n(x) \exp\left(-\frac{x^2}{2} - iE_n t\right), \quad \text{where } E_n = 2n + 1, \quad n = 0, 1, 2, \dots \quad (22)$$

and  $H_n(x)$  are Hermite polynomials [15].

An interesting set of solutions of this equation is given by the so called coherent states, these states are Gaussian packets that maintain their shape with time and they are characterized by a complex number  $z = r \exp(i\theta)$  [16]. One of this coherent states can be used as transformation function  $u$  in order to obtain SUSY partners of the harmonic oscillator, let  $u$  be given by

$$u(x, t) = \frac{1}{\pi^{1/4}} \exp \left\{ -i t - \frac{[x - r \cos(2t)]^2}{2} + i \frac{r^2 \cos(2t) \sin(2t)}{2} - i r x \sin(2t) \right\} \quad (23)$$

where for simplicity a parameter  $z = r$  with  $r \in \mathbb{R}$  is used. This election of the phase of  $z$  does not affect the generality of the results since they are periodic states,  $t = 0$  is selected when  $\theta = 0$ . This transformation function satisfies the reality condition (6) and it is never and nowhere equal to zero. If we try to use  $u$  as given by (23) to generate new potentials with a first-order transformation we will obtain

$$V_1 = x^2 - \partial_{xx} \ln |u|^2 = V_0 + 2, \quad (24)$$

i.e., after the first order transformation we obtain a displaced version of the harmonic oscillator, this property is known in the time-independent SUSY as shape invariance and it allows to

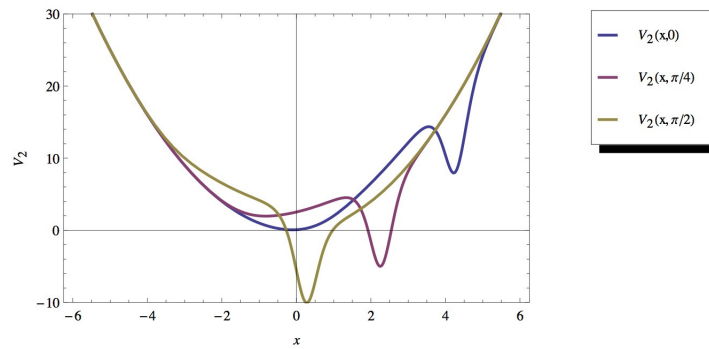
generate the solutions of the Schrödinger equation for the original potential with algebraic methods [3].

To obtain new potentials using the transformation function  $u$  we can employ the second-order confluent SUSY algorithm developed in section 2. Using the equation (14) the time-dependent SUSY partners of the harmonic oscillator are

$$V_2 = x^2 - \partial_{xx} \ln \left| \omega + \frac{1}{\pi^{1/2}} \int_{-\infty}^x \exp \left\{ -[s - r \cos(2t)]^2 \right\} ds \right|^2 \quad (25)$$

Note that when  $t = (2m+1)\pi/4$  with  $m = 0, 1, 2, \dots$  the generated potential  $V_2$  correspond to a Abraham-Moses potential [2], rediscovered and obtained with a first-order SUSY transformation by Mielnik [1]. It is also important to notice that to avoid singularities in  $V_2$  the parameter  $\omega$  must be in the interval  $(-\infty, -1] \cup [0, \infty)$ .

In Fig. 1 the time-dependent anharmonic potential  $V_2(x, t)$  given by (25) with the parameters  $\omega = -1.001$  and  $r = 2$  is plotted at three different times,  $t_0 = 0$ ,  $t_1 = \pi/4$ ,  $t_2 = \pi/2$ . To complete a cycle note that  $V_2(x, 3\pi/2) = V_2(t_1)$ . The perturbation or local minimum that can be seen in  $V_2(x, t_0)$  moves to the left until the position shown in  $V_2(x, t_2)$  and then travels back to its initial position in the same time.



**Figure 1.** New potential  $V_2(x, t)$  at three different times, the parameters are  $\omega = -1.001$  and  $r = 2$ .

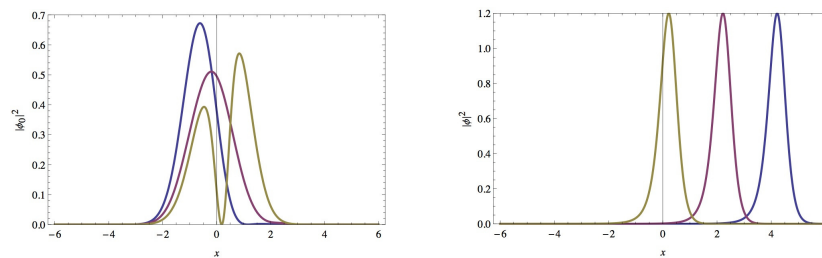
To obtain solutions  $\phi$  of the new Schrödinger equation  $S_2\phi = 0$  where the potential  $V_2$  involved is given by (25) equation (18) can be used, and also (19). First step is to calculate  $A_1$  through (7), with the selection of  $u$  in this example we can fix  $A_1 = 1$ . The function  $v$  is then constructed with (13). Any solution of the Harmonic Oscillator can be mapped, for example the ground state  $\psi_0$  will be mapped to a function  $\phi_0$

$$\phi_0 = \frac{1}{\pi^{3/4}} r \exp \left( -\frac{x^2}{2} - 3it \right) \left\{ \frac{2 \exp(-[x - r \cos(2t)]^2)}{\text{erf}(x - r \cos(2t)) + 2\omega + 1} + \sqrt{\pi} (2x - r \exp(2it)) \right\}, \quad (26)$$

where,  $\text{erf}(\cdot)$  is the error function [15]. On the left of figure 2 it can be seen the plot of the probabilities densities  $|\phi_0(x, t)|^2$  at  $t_0$  (blue curve),  $t_1$  (purple curve) and  $t_2$  (yellow curve). The missing state can be obtained using (19):

$$\phi = \frac{\exp \left\{ -it - \frac{[x - r \cos(2t)]^2}{2} + i \frac{r^2 \cos(2t) \sin(2t)}{2} - i r x \sin(2t) \right\}}{\pi^{1/4} \left\{ \frac{1}{2} [\text{erf}(x - r \cos(2t)) + 1] + \omega \right\}}. \quad (27)$$

On the right of Fig. 2 the probability density of the missing state (27) at  $t_0$  (blue curve),  $t_1$  (purple curve) and  $t_2$  (yellow curve) are shown.



**Figure 2.** Probabilities densities of  $\phi_0(x, t)$  (left) and the missing state  $\phi(x, t)$  (right) at  $t = 0$  (blue curve),  $t = \pi/4$  (purple curve) and  $t = \pi/2$  (yellow curve). The parameters are  $\omega = -1.001$  and  $r = 2$ .

#### 4. Conclusions

A time-dependent anharmonic oscillator was constructed using a generalized supersymmetry transformation, in particular it was shown that application of the simplest version of SUSY is not enough to generate a new potential from the traditional harmonic oscillator when a coherent state is used as transformation function. An iteration was needed to generate this family of potentials, we called this two-step procedure a confluent time-dependent SUSY transformation. Solutions of the corresponding time-dependent Schrödinger equation can be generated using any solution of the harmonic oscillator and the constructed operators  $L_1$  and  $L_2$ , furthermore, a missing state was also found.

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