IOP Conf. Series: Journal of Physics: Conf. Series **839** (2017) 012011 doi:10.1088/1742-6596/839/1/012011

Branched Hamiltonians for a Class of Velocity Dependent Potentials

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Abstract. Hamiltonians that are multivalued functions of momenta are of topical interest since they correspond to the Lagrangians containing higher-degree time derivatives. Incidentally, such classes of branched Hamiltonians lead to certain not too well understood ambiguities in the procedure of the quantization. Within this framework we pick up a model which samples the latter ambiguities and which, simultaneously, turns out to be amenable to a transparent analytic and perturbative treatment.

1. Introduction

Models of classical systems with branched structures [1], in either coordinate (x) space or in its momentum (p) counterpart, have of late been a subject of active theoretical enquiry [2–9]. The key idea is that classical Lagrangians possessing time derivatives in excess of quadratic powers inevitably lead to p becoming a multi-valued function of velocity (v) thereby yielding a multivalued class of Hamiltonian systems.

Branched Hamiltonians in the classical context, and their quantized forms, have been recently discussed by Shapere and Wilczek [2]. Following it, Curtright and Zachos [3] analyzed certain representative models for a classical Lagrangian described by a pair of convex, smoothly tied functions of v. The underlying v turns out to be a double-valued function of p. Proceeding to the quantum domain shows that the double-valued Hamiltonians thus obtained have the inherent feature of being expressible in a supersymmetric form in the p space. Subsequently, a class of nonlinear systems whose Hamiltonians exhibit branching was explored by Bagchi et al [4] who also considered the possibility of quantization for some specific cases of the underlying coupling parameter.

In this paper we present a class of velocity-dependent Lagrangians which define a canonical momentum that yields exactly a pair of velocity variables in fractional terms. As a consequence, the corresponding Hamiltonians develop a branching character. An interesting aspect of our scheme is that it is well-suited for a perturbative treatment.

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2. Branched Hamiltonians: A brief review

Let us briefly review the example of a branched system that was put forward in [3]. It was noted that a typical classical model of branched Hamiltonians results from a non-conventional form of the Lagrangian say, for the one given by

$$L = (v-1)^{\frac{2k-1}{2k+1}} - V(x) \tag{1}$$

where the traditional kinetic-energy term features the replacement of a typical quadratic form by a fractional function of "velocity" v while the function $V(x)$ stands for a convenient local interaction potential. The fractional powers of the difference $v-1$ was invoked to make plausible connections to known phenomenology such as the supersymmetric pairing. In detail, the $(2k+1)-$st root was required to be real and positive or negative for $v > 1$ or $v < 1$, respectively. Correspondingly, the quantity v turned out to be a double-valued function of p.

Working out the standard steps leads to the following two branches:

$$H_\pm = p \pm \frac{1}{4k-2}\left(\frac{1}{\sqrt{p}}\right)^{2k-1} + V(x)\,. \tag{2}$$

Note that the $k=1$ case speaks of the canonical supersymmetric structure [10] for the difference $H_\pm - V(x)$ namely, $p \pm \frac{1}{2\sqrt{p}}$, but in the momentum space if viewed as a quantum mechanical system. The spectral and boundary condition linkages of these Hamiltonians are not difficult to set up.

3. A velocity dependent potential

Against the above background we consider setting up of an extended Lagrangian model having a velocity dependent potential $U(x,v)$ that gives rise to a branched Hamiltonian under Legendre transformation:

$$L(x,v) = C(v-1)^{\frac{2k+1}{2k-1}} - U(x,v) \quad \text{where} \quad C = \frac{2k-1}{2k+1}\left(\frac{1}{4}\right)^{\frac{2}{2k+1}} \tag{3}$$

where we assume $U(x,v)$ to be given in a separable form $U(x,v) = f(v) + V(x)$, $f(v)$ and $V(x)$ are certain functions of v and x respectively.

Using the standard definition of the canonical momentum, we find that it is given by

$$p = \left(\frac{1}{4}\right)^{\frac{2}{2k+1}} (v-1)^{\frac{2}{2k-1}} - f'(v)\,. \tag{4}$$

The above equation is too complicated to put down the multivalued nature of velocity in a tractable closed form.

If we try to determine the associated branches of the Hamiltonian corresponding to this Lagrangian (3), $H_\pm$ emerge in a mixed form involving the momentum p, the function $f(v)$ and its derivative.

$$H_\pm = p \pm \frac{1}{4}\left[p + f'(v)\right]^{-\frac{2k-1}{2}} \left(\frac{2k+1}{2k-1} - p\left[p+f'(v)\right]^{-1}\right) + U(x,v)\,. \tag{5}$$

Since a Hamiltonian has to be a function of the coordinate and the corresponding canonical momentum, $H_\pm$ as derived above is of little use.

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We note that the case $k = 1$ is particularly interesting to understand the spectral properties of $L(x, v)$. Explicitly, the Lagrangian assumes the simple but a general form

$$L = 3\left(\frac{1}{4}\right)^{\frac{2}{3}} (v-1)^{\frac{1}{3}} - f(v) - V(x)\,. \tag{6}$$

A sample choice for $f(v)$ could be

$$f(v) = \lambda v + 3\delta(v-1)^{\frac{1}{3}} \tag{7}$$

with $\lambda\,(\geq 0)$ and $\delta\left(< 4^{-\frac{2}{3}}\right)$ being suitable real constants. Observe that the presence of δ rescales the kinetic energy coefficient which now enjoys a parametric representation.

The above form of f facilitates determination of the canonical momentum p in a closed form as given by

$$p = \mu(v-1)^{-\frac{2}{3}} - \lambda \tag{8}$$

where $\mu = 4^{-\frac{2}{3}} - \delta > 0$. On inversion, we find a pair of relations for the velocity depending on p :

$$v_{\pm}(p) = 1 \mp \mu^{\frac{3}{2}}(p+\lambda)^{-\frac{3}{2}}\,. \tag{9}$$

As a consequence, we run into two branches of the Hamiltonian which we put down in the form

$$H_{\pm} - V(x) = (p+\lambda) \pm \frac{2\gamma}{\sqrt{p+\lambda}} \tag{10}$$

For the ease of notation, note that we have replaced $\mu^{3/2}$ with γ. In the special case where $\lambda = 0$

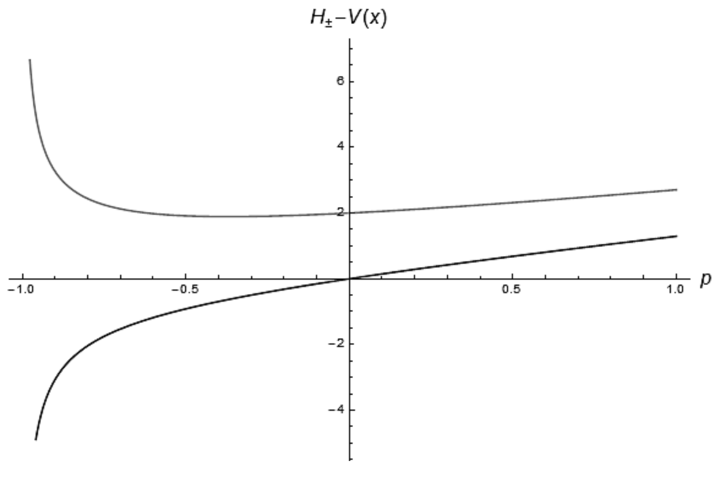

Figure 1. When $\lambda = 1$ and $\gamma = \frac{1}{2}$, $H_{\pm} - V(x)$ branches correspond to the upper and lower curves respectively.

and $\gamma = \frac{1}{4}$, we recover the Hamiltonian derived in [3]. However, the presence of the parameter γ in (10) is nontrivial as our following treatment of perturbative analysis will show. In Figure 1, we have given a graphical illustration (for $\lambda = 1$ and $\gamma = \frac{1}{2}$) of the behavior of the two branches of the Hamiltonian against some typical values of the momentum variable. As in the $\lambda = 0$ case of [3] here also we encounter a cusp asymptotically with regard to p for a fixed γ.

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4. Lowest excitations and the Fourier transform

After one decides to consider just small excitations of our quantum system over a local or global minimum (x_0) of a generic analytic potential $V(x)$, one may put the origin of the coordinate axis to this minimum, $x \to y = x - x_0$, and write down the Taylor series

$$V(x) = V(x_0) + (x - x_0)V'(x_0) + \frac{1}{2}(x - x_0)^2 V''(x_0) + \dots \tag{11}$$

Recall that $V'(x_0) = 0$ and the zero of the energy scale can be shifted in such a manner that $V(x_0) = 0$. Finally, the series is truncated after the first non-trivial term yielding, in *ad hoc* units,

$$V(x_0 + y) = y^2. \tag{12}$$

After a Fourier transform to the momentum space, we get a transformed quantum form of the Hamiltonian guided by the second-order differential operator,

$$H = -\frac{d^2}{dp^2} + W(p) \tag{13}$$

containing a one-parametric family of pseudo-potentials

$$W(p) = p + \frac{2\gamma}{\sqrt{p}}\,. \tag{14}$$

Here, the original subscript $_\pm$ entering Eq. (10) may be perceived as equivalent to an optional switch between positive coupling-type parameter $\gamma > 0$ and its negative alternative $\gamma < 0$. Besides such a freedom of the sign of the dynamical characteristic, the consequent quantum-theory interpretation of the model requires also a few nontrivial mathematical addenda. The form Eq. (14) matches with Eq. (10) for $\lambda = 0$ which will now be our point of inquiry.

First of all, the most natural tentative candidate

$$H\phi_n(p) = E_n\phi_n(p)\,, \qquad p \in (-\infty, \infty) \tag{15}$$

for the quantum Schrödinger equation living on the whole real line of momenta (i.e., with $\phi_n(p) \in L^2(\mathbb{R})$) is characterized by the asymptotically linear *decrease* of the pseudo-potential (14) along the left half-line. Hence, the negative half-axis of momenta p must be excluded, *a priori*, as unphysical. In other words, the acceptable wave functions $\phi_n(p)$ should vanish, identically, whenever $p \in (-\infty, 0)$. The consistent quantization of our model must be based on the modified, half-line version of Eq. (15), viz., on Schrödinger equation

$$H\phi_n(p) = E_n\phi_n(p)\,, \qquad p \in (0, \infty) \tag{16}$$

such that (cf. also [2] and [3])

$$\phi_n(p) \in L^2(\mathbb{R}^+)\,. \tag{17}$$

Still, the discussion is not yet complete. Due care must be also paid to the fact that the inverse-square-root singularity of $W(p)$ in the origin is "weak" (see, e.g., Ref. [11] for a detailed explanation of the rigorous, "extension theory" mathematical contents of this concept). In the language of physics, such a comment means that the information about possible bound states and physics represented by Eq. (16) with constraint (17) is incomplete.

In the rest of this paper (i.e., in sections 5 and 6) we shall, therefore, describe the two alternative versions of the completion of the missing, phenomenology-representing information.

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5. Eligible "missing" boundary conditions at small γ and $p = 0$

As we emphasized above, the existence of the usual discrete spectrum of bound states can only be guaranteed via an *additional* physical boundary condition at $p = 0$. Although, from the point of view of pure mathematics, the choice of such a condition is flexible and more or less arbitrary, the necessary suppression of this unwanted freedom can rely upon several forms of the physics-based intuition.

Let us split the problem into two subcategories. In a simpler scenario we shall assume that the central core is repulsive and strong (i.e., that our parameter is positive and large, $\gamma \gg 1$). This possibility will be discussed in the next section 6. For the present, let us admit that the (real) value of γ is arbitrary and that the regular nature of our ordinary differential Schrödinger equation near $p = 0$ implies that the integrability condition (17) itself still does not impose any constraint upon the energy E [11]. A fully explicit and constructive demonstration of such an observation may be based on the routine reduction of (16) to its simplified, leading-order form

$$-\sqrt{p}\,\frac{d^2}{dp^2}\psi(p) + 2\gamma\,\psi(p) = 0\,. \tag{18}$$

Being valid at the very small (though still positive) values of $p \ll 1$ this equation is exactly solvable in terms of Bessel functions [12]. Thus, one may choose either $\gamma > 0$ or $\gamma < 0$.

After some algebra we obtain the respective two-parametric families of the general solutions which depend on two parameters $C_{1,2}$ or $D_{1,2}$ and which remain energy-dependent. At small p they behave, respectively, as follows,

$$\psi\,(p) = C_1\,\sqrt{p}\,I_{2/3}\left(\frac{4\sqrt{2}}{3}\sqrt{\gamma}\,p^{\frac{3}{4}}\right) + C_2\,\sqrt{p}\,K_{2/3}\left(\frac{4\sqrt{2}}{3}\sqrt{\gamma}\,p^{\frac{3}{4}}\right) \tag{19}$$

and

$$\psi\,(p) = D_1\,\sqrt{p}\,J_{2/3}\left(\frac{4\sqrt{2}}{3}\sqrt{-\gamma}\,p^{\frac{3}{4}}\right) + D_2\,\sqrt{p}\,Y_{2/3}\left(\frac{4\sqrt{2}}{3}\sqrt{-\gamma}\,p^{\frac{3}{4}}\right) \tag{20}$$

On this purely analytic background, one of the most natural resolutions of the paradox of the ambiguity of the physical boundary conditions at $p = 0$ may be based on the brute-force choice of the parameters $C_{1,2}$ or $D_{1,2}$ in these formulae.

Finally, let us emphasize that intuitively by far the most plausible requirement of the absence of the jump in the wave functions at $p = 0$, i.e., the Dirichlet boundary condition

$$\lim_{p\to 0}\psi\,(p) = 0 \tag{21}$$

would remove the latter ambiguity of quantization in the most natural manner. The resulting pair of the requirements

$$C_2 = 0, \quad D_2 = 0 \tag{22}$$

may be then recommended as easily derived from the well known approximate formulae for the Bessel functions near the origin [12].

6. Perturbation-theory analysis at large $\gamma \gg 1$

In a purely formal spirit one could complement the above-recommended Dirichlet boundary condition (21) by its Neumann vanishing-derivative analogue

$$\lim_{p\to 0}\psi'\,(p) = 0 \tag{23}$$

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or, more generally, by a suitable Robin boundary condition. In this context it is worth adding that with a systematic strengthening of the repulsive version of the barrier (i.e., with the growth of the positive coupling constant γ) the specification of the additional boundary conditions at $p = 0$ becomes less and less relevant because the two alternative energy levels will degenerate in the limit $\gamma \to \infty$.

The most immediate explanation of this phenomenon may be provided by perturbation theory. In the dynamical regime, when the parameter is large, a perturbative approach seems to be particularly well suited. With $\gamma \gg 1$, we look at the absolute minimum of the potential $W(p)$ which occurs at p_0, say. This value is, incidentally, unique

$$p_0 = \gamma^{\frac{2}{3}} \gg 1 \tag{24}$$

With the construction of a Taylor series in its vicinity,

$$W(p) = W(p_0) + (p - p_0)W'(x_0) + \frac{1}{2}(p - p_0)^2 W''(p_0) + \dots \tag{25}$$

we observe that the first term, which is given by

$$W(p_o) = 3\gamma^{\frac{2}{3}} \tag{26}$$

in very large in this scenario. In contrast, all of the further Taylor coefficients remain very small and asymptotically negligible,

$$W''(p_o) = \frac{3}{2}\gamma^{-\frac{2}{3}}, \quad W'''(p_o) = -\frac{15}{4}\gamma^{-\frac{4}{3}} \quad \dots \tag{27}$$

Clearly then, with $\gamma \gg 1$, H can be expressed as

$$H = -\frac{d^2}{dp^2} + 3\gamma^{\frac{2}{3}} + \frac{3}{4}\gamma^{-\frac{2}{3}}(p - p_0)^2 + \mathcal{O}(\gamma^{-\frac{4}{3}}(p - p_0)^3)\,. \tag{28}$$

After one re-scales the axis $p = \rho q$, equation (16) acquires the modified form

$$\tilde{H}\tilde{\phi}_n(q) = E_n \rho^2 \tilde{\phi}_n(q) \tag{29}$$

where,

$$\tilde{H} = -\frac{d^2}{dq^2} + 3\rho^2\gamma^{\frac{2}{3}} + \frac{3}{4}\rho^4\gamma^{-\frac{2}{3}}(q - q_0)^2 + \mathcal{O}(\rho^5\gamma^{-\frac{4}{3}}(q - q_0)^3)\,. \tag{30}$$

One may now set

$$\rho = \left(\frac{4}{3}\right)^{\frac{1}{4}} \gamma^{\frac{1}{6}} \tag{31}$$

yielding the very weakly perturbed harmonic-oscillator Hamiltonian

$$\tilde{H} = -\frac{d^2}{dq^2} + (q - q_0)^2 + \gamma\sqrt{12} + \mathcal{O}(\rho^5\gamma^{-\frac{4}{3}}(q - q_0)^3)\,. \tag{32}$$

In full analogy to many models with similar structure (cf., the study [13] containing further references), the exact solvability of the model in the leading-order harmonic-oscillator approximation proves sufficient because in the domain of large $\gamma \gg 1$ the contribution of the anharmonic corrections becomes negligible.

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7. Summary

To summarize, we looked at the particular example of a non-conventional form of a velocity-dependent Lagrangian that leads to a double-valued structure of the associated Hamiltonian for some specific choice of the underlying coupling parameter. We showed that our scheme allows for a perturbative analysis by constructing a Taylor series near the vicinity of the absolute minimum of the potential.

Acknowledgments
For one of us (BB), it is a pleasure to thank Prof. Sara Cruz y Cruz and Prof. Oscar Rosas-Ortiz for warm hospitality during Quantum Fest 2016 held at UPIITA-IPN, Mexico City. MZ acknowledges the short-stay hospitality by Shiv Nadar University and the support by the GAČR Grant Nr. 16-22945S. We also all thank Prof. Bhabani Prasad Mandal for fruitful discussions.

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