Quantum Theory From a Non-Conventional Perspective

D Schuch

Institut für Theoretische Physik, J.W. Goethe-Universität Frankfurt am Main,
Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany

E-mail: `Schuch@em.uni-frankfurt.de`

Abstract. Quantum mechanics is essentially described in terms of complex quantities like wave functions. The interesting point is that phase and amplitude of the complex quantities are not independent of each other, but coupled by some kind of conservation law. This coupling exists in time-independent quantum mechanics as well as in in its time-dependent form. It can be traced back to a reformulation of quantum mechanics in terms of complex nonlinear Riccati equations, where the quadratic term in the latter equation explains the origin of the phase-amplitude coupling. Since realistic physical systems are always in contact with some kind of environment this aspect is also taken into account. It turns out that this dissipative effect can either affect the amplitude or the phase of the complex quantity describing the open quantum system. This suggests a relation between non-unitary transformations and gauge-transformations for these systems. A change of the amplitude also seems to be connected with a second quantum of action for "radial" changes , compared to $\hbar$ for "angular" changes, leading to an interpretation of Sommerfeld's constant.

1. Introduction

Looking at quantum theory in the conventional way one would say that it is a linear theory. Particularly in the Schrödinger picture, which is the one adopted in the following, the Schrödinger equation (SE) is a linear differential equation; therefore the superposition principle is valid which is in agreement with wave-particle duality. In the most common version it is also a Hamiltonian theory – although other formulations are possible like the one based on a Lagrangian, as in Feynman's path integral formulation. The operator corresponding to the Hamiltonian function determines the *energetics* of the system. It not only represents the energy of the system that can be determined via its mean value and corresponding eigenvalues but, as the Hamiltonian operator is usually hermitian, this also guarantees real eigenvalues. The energy of the quantum system, as in the classical case, is a conserved quantity (for time-independent (TI) potentials) or, at least, is not dissipated into heat. For certain systems the energy is also quantized, providing the spectrum of the system.

Moreover, the Hamiltonian also determines the *dynamics* of the system. The Hamiltonian enters the (imaginary) exponent of a time-evolution operator that transforms an initial state into a state at another time. As this transformation is unitary (only a "rotation" in Hilbert space with constant norm), this time-evolution is reversible, i.e., there is no direction of time.

While quantum theory is therefore a linear theory with reversible time-evolution and conserved energy, the macroscopic world we are observing daily is characterized by nonlinear

(NL) evolution processes. These are usually irreversible, thus having a direction of time, and (mechanical) energy is not conserved but transferred as heat into the environment of the system being considered.

A theory well-suited for describing the latter phenomena is *Nonlinear Dynamics*. Furthermore, this theory is usually scale invariant, i.e., independent of the size of the system; only relative changes matter. In this context, the Mandelbrot set and the aesthetic pictures of fractals [1] are also familiar. Interestingly enough, the customary spiral-type patterns of these fractals arise from angular changes that are accompanied by radial ones during the evolution process (in time or space). Manifestations of such processes are ubiquitous in nature; just look, for example, at the shell of a nautilus or the horn of a ram. The radial changes in particular determine the definition of a direction of time; as for an expanding system, a larger radius indicates a later state of the system than one with a smaller radius and vice versa for a contracting system. Since the time-evolution of quantum systems is usually described by a unitary transformation with constant norm, i.e., constant "length" of the state vector, this radial component is missing a priori in quantum theory. Or is it just hidden? Could this possibly be revealed using a different, maybe even NL, formulation of quantum theory? The following discussion is an attempt at providing some answers to these questions.

Such a formulation of quantum theory must be able to take into account aspects that are present in the theories mentioned previously, like nonlinearity, scale-invariance, radial changes, etc. In order to resolve this, one must first determine which properties of the established form of quantum theory are indispensable and cannot therefore be relinquished. As the name "quantum" theory already suggests, there are so-called quanta, i.e., fixed units of a physical quantity. These quantities can be expressed in terms of multiples (or sometimes fractions) of integers of these quanta. As Planck had shown, the essential quantity in nature that is quantized is *action*, quantized in terms of Plancks quantum h (or $\hbar = \frac{h}{2\pi}$).

Another essential ingredient of quantum theory is the occurrence of complex quantities –and not just as a mathematical tool to simplify calculations, but with intrinsic physical meaning. This was stressed by C.N. Yang in his talk "Square root of minus one, complex phase and Erwin Schrödinger" [2] presented 30 years ago at a conference in London celebrating the centenary of the birth of Schrödinger and also E. Wigner points this out in his paper "The unreasonable effectiveness of mathematics in the natural sciences" [3]. Remarkably, Schrödinger introduced his wave function $\Psi(\mathbf{r}, \mathbf{t})$ via the action S as $S = K \ln \Psi$ [4] where, later on, it turned out that $K = \frac{\hbar}{i}$ with $i = \sqrt{-1}$, so Schrödinger's action and also Ψ are complex quantities.

Which other essential property of quantum theory remains? The superposition principle was mentioned above as it seems to be necessary for the description of the wave-like properties of material systems. This is definitely in agreement with the linearity of the SE; but there are also NL differential equations that possess a kind of superposition principle. One of them is the NL Riccati equation. Can quantum theory therefore be rewritten in terms of NL Riccati equations? Or are these equations already somehow hidden in the conventional formulation of quantum theory? An initial answer is given in the next section where it is shown that the dynamics of a time-dependent (TD) quantum system can equally well be described by a complex NL Riccati equation. Besides other treatments, this equation can be linearized to a complex Newtonian equation. During their time-evolution in the complex plane, the amplitude and phase of the corresponding variable can display the simultaneous change of angle and radius mentioned previously. A similar treatment in terms of complex Riccati equations, or equations that can be derived directly thereof, is also possible in the TI case. In both cases this is connected with a conservation law that couples phase and amplitude of the complex quantity.

In Sec. 3 it is shown that the same systems considered in Sec. 2 can also be treated, including a linear velocity-dependent friction force. For this purpose, in the TD case, a NLSE with complex logarithmic nonlinearity is used that can be uniquely linked to other approaches (including

conventional ones). This leads to similar Riccati equations that can again be linearized to complex Newtonian equations. Integrating the dissipative friction force into the NL formalism developed in the non-dissipative case requires only minimal changes. A similar treatment is also possible in the TI case.

Comparison of the complex quadratic nonlinearity appearing in the Riccati equation and the problem of finding the so-called Pythagorean triples shows in Sec. 4 that the angular as well as the radial changes of the complex quantities in the Riccati version of quantum theory can always be quantized in principle. Searching for an example where quantization of the radial changes can be observed in nature leads to the quantum Hall effect (QHE) and the fractional quantum Hall effect (FQHE). The latter gives rise to the introduction of a second quantum of action that enables an interpretation of Sommerfeld's fine-structure constant. Sec. 5 summarizes the results and highlights future perspectives.

2. Complex nonlinear Riccati equations in time-dependent and time-independent quantum mechanics

In order to start the reformulation of quantum theory from save grounds, in the following systems are considered whose SEs have exact, even analytic solutions. In the TD case, these are Hamiltonians that are at most quadratic (or bilinear) in the variables of position and momentum, particularly in one dimension (although generalization to two or three dimensions usually does not cause a problem), in the TI case, in general three-dimensional problems are permitted.

Beginning with the TD case, the discussion focuses on the harmonic oscillator (HO) with potential $V = \frac{m}{2}\omega^2 x^2$ with constant frequency $\omega = \omega_0$, the parametric oscillator with TD frequency $\omega = \omega(t)$ and, in the limit $\omega \to 0$, the free motion, $V = 0$. For all three potentials the TDSE

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right\}\Psi(x,t) \tag{1}$$

has to be solved, what is possible using Gaussian wave packets (WPs) of the form

$$\Psi(x,t) = N(t)\exp\left\{ i\left[y(t)\tilde{x}^2 + \frac{\langle p \rangle}{\hbar}\tilde{x} + K(t) \right] \right\} \tag{2}$$

as solutions.

The variable $\tilde{x}$ in WP (2) is a shifted coordinate $\tilde{x} = x - \langle x \rangle = x - \eta(t)$, where the mean value $\langle x \rangle = \int_{-\infty}^{+\infty}\Psi^* x\Psi dx = \eta(t)$ corresponds to the classical trajectory, $\langle p \rangle = m\dot{\eta}$ represents the classical momentum and the coefficient of the quadratic term in the exponent, $y(t) = y_{\rm R}(t) + iy_{\rm I}(t)$, is a complex function of time and related to the WP width. The (possibly) time-dependent normalization factor $N(t)$ and the purely time-dependent function $K(t)$ in the exponent are not relevant for the dynamics of the WP maximum and width and therefore ignored in the folowing.

As a Gaussian function is totally determined by its maximum and width, the equations of motion that determine the dynamics of these properties are considered now. They are obtained by inserting WP (2) into the TDSE (1) and have the form

$$\ddot{\eta} + \omega^2\eta = 0, \tag{3}$$

and, multiplying $y(t)$ by the constant factor $\frac{2\hbar}{m}$ to obtain, just for formal reasons, the complex variable $\mathcal{C}(t) = \frac{2\hbar}{m}y(t)$,

$$\frac{d}{dt}\mathcal{C} + \mathcal{C}^2 + \omega^2 = 0, \tag{4}$$

the latter being the desired complex Riccati equation.

The Newtonian equation (3) simply conveys that the maximum of the WP, located at $x = \langle x \rangle = \eta(t)$, follows the classical trajectory. The equation for $\mathcal{C}(t)$ determines the time-dependence of the WP width that is related with the position uncertainty via $y_I = \frac{1}{4\langle \tilde{x}^2 \rangle}$ with $\langle \tilde{x}^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ being the mean square deviation of position.

The connection with the WP width and the corresponding equation of motion can be made even more transparent by introducing a new (real) variable α via $\mathcal{C}_I = \frac{1}{\alpha^2}$, where α can be expressed as $\alpha = \left(\frac{2m\langle \tilde{x}^2(t) \rangle}{\hbar} \right)^{1/2}$ and is directly proportional to the WP width. Inserting this definition of $\mathcal{C}_I$ into the Riccati equation (4), the real part $\mathcal{C}_R$ can be determined as $\mathcal{C}_R = \frac{\dot{\alpha}}{\alpha}$ and, inserting $\mathcal{C}_I$ and $\mathcal{C}_R$ in this form into the comple NL Riccati equation turns it into the NL real Ermakov equation (more details about this equation and a corresponding invariant can be found, e.g., in [5–7] and references cited therein, but are not relevant for the following)

$$\ddot{\alpha} + \omega^2 \alpha = \frac{1}{\alpha^3} \ . \tag{5}$$

As indicated in the Introduction, the Riccati equation can be linearized, in this case using the logarithmic derivative of a complex quantity $\lambda(t)$, $\mathcal{C} = \frac{\dot{\lambda}}{\lambda}$ (note, only the relative change of λ matters), to yield the complex Newtonian equation

$$\ddot{\lambda} + \omega^2 \lambda = 0, \tag{6}$$

First, a kind of geometric interpretation of the motion of $\lambda(t)$ in the complex plane shall be given. Expressed in Cartesian coordinates, λ can be written as $\lambda = u + iz$, turning Eq. (6) into two equivalent equations for u and z, respectively, without showing any connections between them. Writing λ in polar coordinates as $\lambda = \alpha e^{i\varphi}$ turns $\mathcal{C} = \frac{\dot{\lambda}}{\lambda}$ into

$$\mathcal{C} \ = \ \frac{\dot{\alpha}}{\alpha} + i \, \dot{\varphi} \ , \tag{7}$$

where the real part is already identical to $\mathcal{C}_R$, as defined above.

The quantity $\alpha(t)$ defined above in $\mathcal{C}_I$ as being proportional to the position uncertainty is identical to the absolute value of λ, as it can be shown that

$$\dot{\varphi} = \frac{1}{\alpha^2} \ . \tag{8}$$

This relation corresponds to the "conservation of angular momentum", but here for the motion in the complex plane (for further details, see [5–7])!

Relation (8) also shows that real and imaginary parts, or phase and amplitude, respectively, of the complex quantity $\lambda(t)$ are not independent of each other but uniquely coupled. This coupling is due to the quadratic nonlinearity in the Riccati equation.

An analogous situation can be found in the TI case, as shown by Reinisch [8]. For this purpose, Madelung's hydrodynamic formulation of quantum mechanics [9] is taken as starting point. Using the polar ansatz

$$\Psi(\mathbf{r}, t) = a(\mathbf{r}, t) \ \exp\left(\frac{i}{\hbar} S(\mathbf{r}, t) \right) \tag{9}$$

for the (complex) wave function $\Psi(\mathbf{r}, t)$ (where $a^2(\mathbf{r}, t) = \varrho(\mathbf{r}, t) = \Psi^* \Psi$), turns the linear SE (1)[1] into two coupled equations for the amplitude $a^2(\mathbf{r}, t)$ and the phase $S(\mathbf{r}, t)$; i.e., the continuity equation

$$\frac{\partial}{\partial t} a^2 \ + \ \frac{1}{m} \nabla(a^2 \, \nabla S) = 0, \tag{10}$$

---

[1] In the following $\frac{\partial^2}{\partial x^2}$ is replaced by the three-dimensional Laplace operator $\Delta = \nabla^2$ with $\nabla =$ nabla operator.

and the modified Hamilton–Jacobi-type equation

$$\frac{\partial}{\partial t}S + \frac{1}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\Delta a}{a} = 0. \tag{11}$$

Already here, the coupling of phase and amplitude can be seen clearly because the Hamilton–Jacobi equation for the phase[2] S contains a term (misleadingly called *"quantum potential"*, $V_{qu} = -\frac{\hbar^2}{2m}\frac{\Delta a}{a}$) depending on the amplitude $a(\mathbf{r}, t)$ and the continuity equation for the probability density $\varrho = a^2$ contains ∇S. In the following, it is shown that, also in the TI case, this coupling is not arbitrary but related to a conservation law originating from the quadratic nonlinearity of a complex Riccati equation.

For stationary states, the energy of the system is related to the action S via $\frac{\partial}{\partial t}S = -E = const$, and the density is TI, i.e., $\frac{\partial}{\partial t}a^2 = 0$. The continuity equation (10) then turns into

$$\nabla(a^2\nabla S) = 0 \tag{12}$$

and the modified Hamilton–Jacobi equation (11) into

$$-\frac{\hbar^2}{2m}\Delta a + (V - E)\, a = -\frac{1}{2m}(\nabla S)^2 a. \tag{13}$$

Equation (12) is definitely fulfilled for $\nabla S = 0$, turning Eq. (13) into the usual TISE for the real wave function $a = |\Psi|$ with position-independent phase S.

However, Eq. (12) can also be fulfilled for $\nabla S \neq 0$ if only the conservation law

$$\nabla S = \frac{C}{a^2} \tag{14}$$

is fulfilled with constant (or, at least, position-independent) C.

This expression now shows explicitly the coupling between phase and amplitude of the wave function a'nd is equivalent to Eq. (8) in the TD case. Inserting (4) into the rhs of Eq. (13) changes this to the Ermakov equation

$$\Delta a + \frac{2m}{\hbar^2}(E - V)\, a = (\frac{1}{\hbar}\nabla S)^2\, a = \left(\frac{C}{\hbar}\right)^2\frac{1}{a^3} \tag{15}$$

equivalent to Eq. (5) in the TD case. The corresponding *complex Riccati* equation equivalent to Eq. (4) in the TD case is given here by

$$\nabla\left(\frac{\nabla\Psi}{\Psi}\right) + \left(\frac{\nabla\Psi}{\Psi}\right)^2 + \frac{2m}{\hbar^2}(E - V) = 0 \tag{16}$$

where the following substitutions must be made

$$\frac{\partial}{\partial t} \leftrightarrow \nabla, \quad \left(\frac{2\hbar}{m}y\right) = \mathcal{C} = \frac{\dot{\lambda}}{\lambda} \leftrightarrow \frac{\nabla\Psi}{\Psi}, \quad \lambda = \alpha\, e^{i\varphi} \leftrightarrow \Psi = a e^{i\frac{S}{\hbar}}. \tag{17}$$

Note that whereas in the TD case the NL formulation is essentially possible for quadratic potentials, in the TI case the formulation in terms of a complex Riccati equation is possible for *any* potential $V(\mathbf{r})$.

Again, in linearizing the complex Riccati equation (16), only the logarithmic derivative $\frac{\nabla\Psi}{\Psi} = \nabla\ln\Psi$ is of importance; so the absolute value of Ψ is irrelevant. This might be a reason why the value is arbitrary and can also just be defined to be "one", thus leading to the probabilistic interpretation of the wave function[3].

---

[2] In the classical case, S is identical to the (real) action function of the system.

[3] The change of amplitude in the TI case is somehow "hidden" via normalization in order to fulfil this probabilistic interpretation. In the TD case, the change of the amplitude, e.g., caused by the spreading of a WP, is compensated by a TD normalization coefficient to guarantee normalizability at any time t, not only at the beginning $t = t_0$.

3. Dissipative systems with irreversible time-evolution

Realistic physical systems are always in contact with some kind of environment, leading to phenomena like irreversible time-evolution and dissipation of energy. So the question is how can this be taken into account in classical, particularly Hamiltonian (or Lagrangian) mechanics and then be transferred to a quantum mechanical description of those systems?

Without the boundary condition that necessitates a Hamiltonian theory in classical mechanics, a so-called phenomenological description is possible. Using the trajectory picture, this leads to the Langevin equation

$$m\ddot{x} \;=\; m\dot{v} \;=\; -\,m\gamma v \;-\; \frac{\partial}{\partial x}V \tag{18}$$

with the linear velocity dependent friction force $-m\gamma v$ (where γ is a friction coefficient), written here without a purely random TD stochastic force that generally vanishes on average. The same Brownian motion-type scenario can also be described in the picture of a (one-particle) distribution function ϱ by adding a diffusion term breaking the time-reversal symmetry. Particularly for a distribution function $\varrho(x,t)$ in position space, this leads to the Smoluchowski equation

$$\frac{\partial}{\partial t}\varrho_{\mathrm{cl}} + \frac{\partial}{\partial x}\left(\frac{F(x)}{m\gamma}\varrho_{\mathrm{cl}}\right) - \frac{k_{\mathrm{B}}\mathrm{T}}{m\gamma}\frac{\partial^2}{\partial x^2}\varrho_{\mathrm{cl}} \;=\; 0, \tag{19}$$

a Fokker–Planck-type equation where the diffusion coefficient obeys the Einstein relation $D = \frac{k_{\mathrm{B}}\mathrm{T}}{m\gamma}$ with k_{B} being Boltzmanns constant and T, temperature; $F(x)$ is an external force.

However, Eqs. (18) and (19) do not fit consistently into the Lagrange/Hamilton formalism that is invariant under canonical transformations and provides a basis for quantization.

There are numerous approaches in the literature for finding a Hamiltonian description of dissipative systems that can roughly be divided into three groups of methods:

1. the system-plus-reservoir approaches;

2. modifications of the classical Lagrange/Hamilton formalism with subsequent canonical qauntization;

3. modifications of the Hamiltonian operator, usually leading to NLSEs.

It has been shown [10, 11] that most of the approaches belonging to any of these three groups can be uniquely be linked to a particular NLSE with complex logarithmic nonlinearity that is free of the shortcomings of other related NLSEs and can uniquely be connected with the approaches of group two via non-unitary transformations on the quantum mechanical level (corresponding to non-canonical transformations on the classical level). As a connection between the canonical approaches of group two and the conventional system-plus-reservoir approach of group one is possible [12, 13], this NLSE can also be related to the conventional approaches.

Starting point for the derivation of the logarithmic NLSE is the irreversible Smoluchowski equation

$$\frac{\partial}{\partial t}\varrho \;+\; \frac{\partial}{\partial x}\left(\varrho\, v_-\right) \;-\; D\frac{\partial^2}{\partial x^2}\varrho \;=\; 0, \tag{20}$$

but now for the quantum mechanical probability density $\varrho(x,t) = \Psi^*(x,t)\Psi(x,t)$ and with the velocity field $v_-(x,t) = \frac{\hbar}{2m\mathrm{i}}\left(\frac{\frac{\partial}{\partial x}\Psi}{\Psi} - \frac{\frac{\partial}{\partial x}\Psi^*}{\Psi^*}\right)$. For the continuity equation, Madelung [14] and Mrowka [15] had shown that with a bilinear ansatz for the probability current density $j = \varrho\left(\frac{1}{m}\frac{\partial}{\partial x}S\right) = \varrho v_-$, this can be separated into the TDSE and its complex conjugate where the separation "constant" is proportional to the potential V. However, due to the diffusion term in (20) this is no longer possible in general. In order to achieve separability, an additional condition

must be imposed. One possibility is the choice

$$-D\frac{\frac{\partial^2}{\partial x^2}\varrho}{\varrho} \; = \; \gamma\left(\ln\varrho - \langle\ln\varrho\rangle\right), \tag{21}$$

because of $\ln\varrho = \ln\Psi + \ln\Psi^* = F_1(\Psi) + F_2(\Psi^*)$ the separation of terms containing either Ψ or Ψ^{ast} is possible. This leads to an additional *complex* logarithmic term in the SE [16],

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{NL}}(x,t) \; = \; \{H_{\mathrm{L}} \; + \; \gamma\frac{\hbar}{\mathrm{i}}\left(\ln\Psi_{\mathrm{NL}} - \langle\ln\Psi_{\mathrm{NL}}\rangle\right)\}\Psi_{\mathrm{NL}}(x,t) \; = \; \{H_{\mathrm{L}} + W_{\mathrm{lgNL}}\}\Psi_{\mathrm{NL}}(x,t), \tag{22}$$

where H_{L} is the usual linear Hamiltonian.

Also the logarithmic NLSE (22) possesses Gaussian WP solutions in the cases that were discussed for the TDSE without dissipation. Maximum and width of this Gaussian function are now determined by the modified equations of motion

$$\ddot{\eta} + \gamma\dot{\eta} + \omega^2\eta = 0 \tag{23}$$

for the maximum and

$$\frac{d}{dt}\mathcal{C}_{\mathrm{NL}} + \mathcal{C}_{\mathrm{NL}}^2 + \gamma\mathcal{C}_{\mathrm{NL}} + \omega^2 \; = \; 0 \tag{24}$$

for the width with an additional linear term depending on the friction coefficient γ.

The relation between the imaginary part of C_{NL} and the position uncertainty or Ermakov variable $\alpha(t)$ remains unchanged as in the non-dissipative case, $\mathcal{C}_{\mathrm{NL,I}} = \frac{\hbar}{2m\langle\tilde{x}^2(t)\rangle} = \frac{1}{\alpha_{\mathrm{NL}}^2}$, but the real part differs by a contribution from the friction coefficient, $\mathcal{C}_{\mathrm{NL,R}} = \frac{\dot{\alpha}_{\mathrm{NL}}}{\alpha_{\mathrm{NL}}} - \frac{\gamma}{2}$.

Inserting $C_{\mathrm{NL,I}}$ and $C_{\mathrm{NL,R}}$ into the Riccati equation (24) turns this into the Ermakov equation

$$\ddot{\alpha}_{\mathrm{NL}} + \left(\omega^2 - \frac{\gamma^2}{4}\right)\alpha_{\mathrm{NL}} = \frac{1}{\alpha_{\mathrm{NL}}^3}, \tag{25}$$

i.e., in comparison with the non-dissipative case, only ω^2 has been replaced by $\Omega = \left(\omega^2 - \frac{\gamma^2}{4}\right)$.

Riccati equation (24) can also be linearized, now using the ansatz $C_{\mathrm{NL}} = \frac{\dot{\tilde{\lambda}}}{\tilde{\lambda}} = \frac{\dot{\lambda}}{\lambda} - \frac{\gamma}{2}$ to yield the Newtonian equation with linear friction term

$$\ddot{\tilde{\lambda}} + \gamma\dot{\tilde{\lambda}} + \omega^2\tilde{\lambda} \; = \; 0 \;, \tag{26}$$

for the complex variable $\tilde{\lambda}(t) = \lambda\mathrm{e}^{-\gamma t/2} = \alpha_{\mathrm{NL}}\mathrm{e}^{-\gamma t/2 + \mathrm{i}\varphi}$. Inserting the polar form into Eq. (24) leads to the unchanged conservation law $\dot{\varphi} = \frac{1}{\alpha_{\mathrm{NL}}^2}$ as in the non-dissipative case and to the modified Ermakov equation (25).

Comparison of Eqs. (23), (24) and (26) with the corresponding Eqs. (3), (4) and (6) in the non-dissipative case shows the following modifications. In the linear second-order differential equations for η and λ, a linear term with first derivative has been added while in the first-order Riccati equation an additional term linear in C_{NL} appears. All the additional terms depend on the coefficient of the friction force.

Now the question is: is it possible to modify the corresponding equations in a similar way to include the effect of a dissipative environment into the TISE? The answer has been given in [9]. Here only a few details necessary for the following discussion will be cited.

The above-mentioned modifications in the TD case would mean the addition of a term $\Gamma\left(\frac{\nabla\Psi}{\Psi}\right)$ to Eq, (16) or a term $\Gamma\nabla\Psi$ to the TISE, respectively, where the coefficient Γ should somehow be related to the friction force but, in general, could be a complex function of $\mathbf{r}$ and t, $\Gamma(\mathbf{r}, t)$.

As Ψ is complex, a complex contribution from the imaginary part of the additional term $\Gamma\nabla\Psi$ to the continuity equation for $\varrho = \Psi^*\Psi$ of the form $\Gamma\,\nabla\varrho = \Gamma\,(\Psi^*\nabla\Psi + \Psi\nabla\Psi^*)$ should arise in the evolution equation for ϱ. The contribution from this term would enter the SE with an *imaginary* coefficient.

The additional term in the equation for ϱ, at least for cases with Gaussian WP solutions (i.e., $V = 0$ and HO) discussed in this paper, should have the same effect as the diffusion term or the $\ln\varrho$-term, i.e.,

$$-D\Delta_x\varrho \;=\; \gamma(\ln\varrho - \langle\ln\varrho\rangle)\varrho \;=\; \frac{\gamma}{2}\left(1 \;-\; \frac{\tilde{x}^2}{\langle\tilde{x}^2\rangle}\right)\varrho. \tag{27}$$

Comparison shows that

$$\nabla_x\left(\frac{\gamma}{2}\tilde{x}\varrho\right) \;=\; \frac{\gamma}{2}\left(1 \;-\; \frac{\tilde{x}^2}{\langle\tilde{x}^2\rangle}\right)\varrho \tag{28}$$

yields the desired result![4]

Taking into account this additional term, the modified continuity equation requires a modified action function $S' \;=\; S \;+\; m\frac{\gamma}{4}\tilde{x}^2 \;+\; f(t)$ in order to provide a conservation law of the form $\nabla S' \;=\; \frac{C}{a^2}$.

Separation of the continuity equation including the additional term (28) and achieving consistency of the modified Hamilton–Jacobi equation with the modified action function S' leads to a form of the dissipative term that is essentially identical to the one Hasse [17] and provides the full dissipative friction force. Further details can be found in [10].

4. Quantization of angular and radial aspects

We have seen so far that the changes of radial and angular components of a complex quantity are not independent of each other, if this complex quantity is somehow related to a complex Riccati equation due to the quadratic nonlinearity of this equation. This applies also, when the complex quantity is linearized via a logarithmic derivative. In the case of TD quantum mechanics, the complex quantity fulfilling the Riccati equation is $\mathcal{C} = \mathcal{R} + i\mathcal{I} = \frac{\dot{\alpha}}{\alpha} + i\dot{\varphi}$ where α is the amplitude and φ the phase angle of another complex quantity, $\lambda = \alpha e^{i\varphi}$ that was used to linearize the Riccati equation (4) to the linear complex Newtonian equation (6). How can this be connected with a quantization of radial and angular components?

Unlike modern physics that is looking for smallest building blocks of our material world (with largest accessible energy - and money), Plato's view of the world, expressed in his work "Timaios", was that it is build up in terms of right triangles that could be used form cube, tetrahedron, octahedron and icosahedron representing the four elements earth, fire, air and water of our material world - there is no word about the size of the triangles, i.e., scale invariance!

Certainly, in connection with right triangles one also thinks about Pythagoras and his law $a^2 + b^2 = c^2$. But there are special right triangles that can in a certain way be quantized (after Platon's "quantization", a kind of "second quantization").

These so-called Pythagorean triples are right triangles where all three sides can be expressed in terms of integers, e.g., 3, 4, 5 leading to $9 + 16 = 25$. Most people have difficulties to find even a second example for those triples, although infinitely many of them exist. The problem of finding them can easily be solved considering the square of a complex quantity as it occurs in our Riccati equations.

Assuming a complex quantity like $\mathcal{C} = \mathcal{R} + i\mathcal{I} = \frac{\dot{\alpha}}{\alpha} + i\dot{\varphi}$ represents a right triangle in the complex plane, made out of its real and imaginary parts and its absolute value. If $\mathcal{C}$ is squared, the result is again a complex quantity with real part $\mathcal{R}\{\mathcal{C}^2\} = \mathcal{R}^2 - \mathcal{I}^2$, imaginary part $\mathcal{I}\{\mathcal{C}^2\} = 2\,\mathcal{R}\mathcal{I}$ and absolute value $|\mathcal{C}^2| = \mathcal{R}^2 + \mathcal{I}^2$. It is straightforward to show that for

---

[4] In our one-dimensional case ∇_x means $\frac{\partial}{\partial x}$.

"quantized" $\mathcal{C}$, i.e., integer values of $\mathcal{R}$ and $\mathcal{I}$ (with $\mathcal{R} > \mathcal{I}$, all three sides of the right triangle formed by $\mathcal{C}^2$ are integers. Examples are: (a) $\mathcal{R} = 2$, $\mathcal{I} = 1$: $\mathcal{R}^2 - \mathcal{I}^2 = 3$, $2\,\mathcal{R}\mathcal{I} = 4$, $\mathcal{R}^2 + \mathcal{I}^2 = 5$ with $9 + 16 = 25$; (b) $\mathcal{R} = 3$, $\mathcal{I} = 2$: $\mathcal{R}^2 - \mathcal{I}^2 = 5$, $2\,\mathcal{R}\mathcal{I} = 12$, $\mathcal{R}^2 + \mathcal{I}^2 = 13$ with $25 + 144 = 169$.

In our case, $\mathcal{R} = \frac{\dot{\alpha}}{\alpha}$ and $\mathcal{I} = \dot{\varphi}$, so the question arises, where do we find a quantization of these quantities in physics? The quantization of $\dot{\varphi}$ can obviously be found considering the HO with constant frequency $\omega = \omega_0$. In this case, and for constant WP width, $\omega_0 = \dot{\varphi}$ is valid and the energy is quantized in terms of $\dot{\varphi}$ according to

$$E_n = \hbar \left(n + \frac{1}{2} \right) \dot{\varphi} \tag{29}$$

with n being an integer, i.e. there is a quantization in terms of $n\,\dot{\varphi}$.

In order to find a quantization of $\mathcal{R} = \frac{\dot{\alpha}}{\alpha}$, one has to look at the currents in the continuity equation. In the TD case for Gaussian WP solutions the velocity field v_-, according to the definition below Eq. (20), can be written as

$$v_- = \dot{\eta} \; + \; \frac{\dot{\alpha}}{\alpha} \tilde{x} \tag{30}$$

with the classical contribution $\dot{\eta} = \langle v \rangle$.

In the QHE, the current $j = \varrho\, v \propto \sigma E$ (here only one dimension shall be considered) is proportional to the electric field strength E and the conductivity σ with $\sigma = n\, c \left(\frac{e^2}{c\hbar} \right) = n\, c\, \alpha_{\text{Som}}$ where $n = $ integer, $c = $ velocity of light, $e = $ electic elementary charge and $\alpha_{\text{Som}} = \left(\frac{e^2}{c\hbar} \right) \approx \frac{1}{137}$ is Sommerfeld's fine structure constant (not to be confused with $\alpha(t)$, the WP width as defined in Sec. 2).

For fixed position of the system's centre of mass, $\dot{\eta} = 0$, the conductivity or current displays the following proportionality:

$$j \propto n \left(\frac{e^2}{c\hbar} \right) = n\, \alpha_{\text{Som}} \propto \frac{\dot{\alpha}}{\alpha}, \tag{31}$$

therefore, changes of j by integers n would correspond to integer changes of $\frac{\dot{\alpha}}{\alpha}$.

However, the continuity equation is valid for isolated systems without dissipation, whereas the QHE takes frictional effects into account. Looking at the dissipative scenario described by the Smoluchowski equation (20), the convective velocity field changes into $v_- = \dot{\eta} \; + \; \left(\frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2} \right) \tilde{x}$ and an additional contribution $v_D = -D \frac{\frac{\partial}{\partial x} \varrho}{\varrho} = \frac{\gamma}{2} \tilde{x}$ from the diffusion term shows up, leading to a total velocity field $v_{tot} = v_- + v_D = \frac{\dot{\alpha}}{\alpha} \tilde{x}$ that is again proportional to $\frac{\dot{\alpha}}{\alpha}$, so (for $\dot{\eta} = 0$) also in the dissipative situation that is present in the QHE, $\frac{\dot{\alpha}}{\alpha}$ is the quantity that is quantized. Particularly, $\frac{\dot{\alpha}}{\alpha}$ is proportional to $\frac{e^2}{c\hbar}$ that is dimensionless and is discussed in more detail below.

Apart from the QHE, there is also the FQHE observed where

$$j \propto \frac{n}{k} \left(\frac{e^2}{c\hbar} \right) \tag{32}$$

with n, k both being integers. There are different interpretations of this effect in the literature and not all possible combinations of n and k (being odd, even or both) have been observed. To my knowledge none of the existing theories is able to explain all observed combinations of n and k or even predict yet unobserved ones. Therefore, there might still be room for some unknown in the theory of the FQHE.

One interpretation that had been given by the author and colleagues [18] some time ago assumes that the FQHE is a manifestation of a phenomenon where two effects are involved,

each being quantized separately. Since it was already stated in the introduction that *action* is the quantity that is usually quantized in nature, we assumed that in the FQHE one of the effects is quantized in terms of $\hbar$ while the other is in terms of $\frac{e^2}{c}$, so the ratio of both is just Sommerfelds fine structure constant $\alpha_{\text{Som}} = \frac{e^2}{c\hbar}$.

Certainly, as shown above, the action $\hbar$ is well known in quantum mechanics; the action $\frac{e^2}{c}$ was already mentioned by Einstein [19], Schrödinger [20] and Eddington [21], assuming its importance but not knowing its physical relevance. Our interpretation of the FQHE is that this is the first observed physical effect that shows the relevance of a second quantum of action, $\frac{e^2}{c}$.

In [17], we called $\frac{e^2}{c}$ the "*quantum of electrostatic action*". Why this name? For two electrons interacting via Coulombs law, the electrostatic action involved is action = energy × time = $\frac{e^2}{r} \times t$. As the distance between the electrons, r, and their interaction time, t, can be related via the interaction velocity $v = \frac{r}{t}$, one can define the action = $\frac{e^2}{v}$. This action becomes minimum when the velocity attains its maximum value $v = c$ = velocity of light. So, the last Coulombic action is $\frac{e^2}{c}$.

But since nature is usually very economical, as expressed by the various extremal principles, e.g., in mechanics, optics, etc., why does it permit the luxury of two elementary quanta of action? Of course no definite answer shall be given at this point; but the following remarks, based on the afore-mentioned facts shall be allowed. Planck's constant (divided by 2π), $\hbar$, is well known, particularly in the quantum mechanics of oscillating systems where, e.g., the energy is quantized in terms of $\hbar\omega = \hbar\dot{\varphi}$, i.e., the frequency is related to some kind of angular velocity and thus to some kind of *angular* aspect. On the other hand, it has just been shown that $\frac{e^2}{c}$ is somehow related to a physical situation where a *radial* distance plays the important role.

As the radius and circumference of a circle cannot be expressed in terms of the same units ("*quanta*") since they are related via the irrational number π, maybe the "*radial*" *action* and the "*angular*" *action* also cannot be expressed in terms of the same fundamental units.

As seen from our discussion of integer values for $\frac{\dot{\alpha}}{\alpha}$ and $\dot{\varphi}$ related to $\mathcal{C} = \frac{\dot{\lambda}}{\lambda}$, also there the radial aspect is for integer values of $\frac{e^2}{c}$ connected to e = 2.718... whereas the angular aspect, $\dot{\varphi} = \frac{1}{\alpha^2}$, is connected to $\pi = 3.141...$. So, maybe $\frac{e^2}{c}$ and $\hbar$ just offer the possibility to quantize in a (complex?) two-dimensional (phase?) space radial changes as well as angular ones.

5. Conclusions and perspectives

Quantum theory is essentially based on complex quantities that posses an amplitude and a phase, like the wave function solving the linear SE. As has been shown above, this equation can also be considered as a linearized version of a complex Riccati equation. Due to the quadratic nonlinearity occurring in the Riccati equation, real- and imaginary-parts of the complex quantity are no longer independent of each other, but coupled (in our case by some kind of conservation law).

Considering the dissipative case the TDSE has shown that the effect of the the dissipative environment on the linearized complex quantity (i.e., $\lambda(t)$ fulfilling a complex Newtonian equation) is essentially an exponential decay of the amplitude of this quantity. In the TI case, this is by definition not possible, as the linearized quantity is the wave function whose amplitude is always normalized due to the probabilistic interpretation. Nevertheless a description of the same dissipative system was possible, but now requiring a change of the action function S into S' as defined below Eq. (28).

Therefore it seems that radial changes as they occur e.g. in irreversible decay processes can equally well be described by damping of an amplitude, i.e. radial changes, or changing of the phase of a complex quantity, i.e. angular changes. This would suggest that non-unitary transformations that appear to be necessary for the description of open quantum systems can

equally well be described by gauge-transformations changing only the phase of the wave function, but leaving the amplitude (and thus the interpretation of the wave function) untouched.

Attempts to link NLSEs with (NL) gauge trandformations have already been studied [22, 23], but there is still a wide field to explore.

Also the apparent occurrence of a second quantum of action that seems to be related to some radial changes that are usually not recognized as such in quantum theory needs further investigation.

References

[1] Peitgen H-O and Richter P H 1986 *The Beauty of Fractals* (Berlin: Springer) pp 9, 11 22 48, 84-86
[2] Yang C N 1987 *Schrödinger – Centenary Celebration of a Polymath* ed C W Kilmister (Cambridge: Cambridge University Press) p 53
[3] Wigner E 1960 *Commun. Pure Appl. Math.* **13** 1
[4] Schrödinger E 1926 *Ann. d. Phys.* **79** 361
[5] Schuch D 2008 *SIGMA* **4** 043
[6] Schuch D and Moshinsky M 2006 *Phys. Rev. A* **73** 062111
[7] Schuch D 2011 in: *Latin-American School of Physics XL ELAF; AIP Conf. Proceedings* **1334** 291
[8] Reinisch G 1994 *Physica* A **206** 229
[9] Madelung E 1926 *Z. Physik* **40** 322
[10] Schuch D 2012 *J. Phys. Conf. Ser.* **380** 012009
[11] Schuch D, Guerrero J, López-Ruiz F F and Aldaya V *Phys. Scr.* **90** 045209
[12] Yu L H and Sun C P 1994 *Phys. Rev. A* **49** 592
[13] Sun C P and Yu L H 1995 *Phys. Rev. A* **51** 1845
[14] Madelung E 1950 *Die Mathematischen Hilfsmittel des Physikers* (Berlin: Springer) p 432
[15] Mrowka B 1951 *Z. Phys.* **130** 164
[16] Schuch D, Chung K-M and Hartmann H 1983 *J. Math. Phys.* **24** 1652
[17] Hasse R W 1975 *J. Math. Phys.* **16** 2005
[18] Wadlinger R L P, Hunter G, Kostro L and Schuch D 1990 *Phys. Essays* **3** 194
[19] Einstein A 1909 *Z. Phys.* **6** 185
[20] Schrödinger E 1922 *Z. Phys.* **12** 23
[21] Eddington A S 1935 *New Pathways in Science* (Cambridge: Cambridge University Press) p 234
[22] Doebner H-D and Goldin G A 1996 *Phys. Rev. A* **54** 3764
[23] Schuch D 2000 in *Quantum Theory and Symmetries* (Singapore, World Scientific) p 152