

Monoid spaces and linearized gravity

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Abstract. We propose an alternative representation for linear quantum gravity. It is based on the use of a structure that bears some resemblance to the Abelian loop representation used in electromagnetism but with the difference that the space of extended object on which waves functions take values has a structure of commutative monoid instead of Abelian group. The generator of duality of the theory is realized in this representation and a geometrical interpretation is discussed.

The introduction of loop representation has opened a new avenue for quantization of gauge theories such as electromagnetism [1, 2], Yang-Mills [3] and General Relativity [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. This representation allows an elegant way of solve the constraints of the theories as well as obtain certain geometrical information about the space of extended objects on which the wave functions take values. For example, it is well known that the Abelian loop representation in electromagnetism allows to solve the Gauss constraint immediately [1, 2]. Also, because the metric independence of the generator of duality, their realization in this representation leads to a knot invariant, more precisely, the Gauss linking number between Abelian loops [2]. Within the same spirit, the loop representation has been used to quantize linearized gravity [14, 15]. For example in reference [15] it was obtained a representation in terms of skein of loops. Although it could be found a realization for the canonical algebra and constraints of the theory were solved in the reduced phase space, a geometrical interpretation of skein representation keeps unclear.

In this paper we develop an alternative representation for linearized quantum gravity. As we shall see, the wave functions in this representation does not depend on a list of three tangled Abelian loops as in reference [15] but in only one extended object. However, the price which has to be paid is that the space of this extended objects does not form an Abelian group but a commutative monoid space.

We start by considering a set of parametrized curves γ on R^3 given by $\vec{z}_\gamma = \vec{z}(l)$, with l the arc length parameter, and define

$$I^{ab}[\vec{x}, \gamma] := \int_{\gamma} dl u_{T_\gamma}^a u_{T_\gamma}^b \delta^3(\vec{z}_\gamma - \vec{x}), \quad (1)$$



with $\hat{u}_{T_\gamma} := d\vec{z}_\gamma/dl$ the tangent vector to γ and $a, b = 1, 2, 3$. It is easy to check that (1) is symmetric and independent of the curve orientation. It is worth mentioning that for an arbitrary parameter τ we have

$$I^{ab}[\vec{x}, \Gamma] = \int_{\tau_1}^{\tau_2} d\tau \frac{dz^a(\tau)}{d\tau} \frac{dz^b(\tau)}{d\tau} \left| \frac{d\vec{z}(\tau)}{d\tau} \right|^{-1} \delta^3(\vec{x} - \vec{z}(\tau)), \quad (2)$$

with $\tau_1 < \tau_2$ independent of the orientation of γ .

Let \mathcal{M} be the space which elements Γ are the union of disjoint curves, i.e, $\Gamma = \gamma_1 \cup \dots \cup \gamma_n$ with arbitrary n . From definition (1), it follows that

$$I^{ab}[\vec{x}, \Gamma] = I^{ab}[\vec{x}, \gamma_1 \cup \dots \cup \gamma_n] = \sum_{i=1}^n I^{ab}[\vec{x}, \gamma_i]. \quad (3)$$

Now, we shall say that two curves Γ and Γ' belong to the same equivalence class if $I^{ab}[\vec{x}, \Gamma] = I^{ab}[\vec{x}, \Gamma']$ and hereafter, they will represent the same element or *path*. We will denote the space formed by such a paths by \mathfrak{M} .

For elements in \mathfrak{M} we define the product \circ among two paths Γ_1 and Γ_2 as follows

$$I^{ab}[\vec{x}, \Gamma_1 \circ \Gamma_2] := I^{ab}[\vec{x}, \Gamma_1 \cup \Gamma_2] = I^{ab}[\vec{x}, \Gamma_1] + I^{ab}[\vec{x}, \Gamma_2]. \quad (4)$$

As can be seen from the above definition, this product is commutative. In the other hand, for an element $\Gamma_e \in \mathfrak{M}$ such as $I^{ab}[\vec{x}, \Gamma_e] = 0$ (the null path) occurs that

$$I^{ab}[\vec{x}, \Gamma \circ \Gamma_e] = I^{ab}[\vec{x}, \Gamma]. \quad (5)$$

Hence Γ_e is the neutral element for the multiplication defined in (4). In summary, the elements of \mathfrak{M} along with (4) form a commutative monoid, i.e, a semi-group with an identity.

Let us now consider square integrable functions (at least formally) $\Psi : \mathfrak{M} \rightarrow C$. We define the path derivative operator $\delta_{cd}(\vec{y}, \hat{v})$ as

$$v^c v^d \delta_{cd}(\vec{y}, \hat{v}) \Psi[\Gamma] := \lim_{L \rightarrow 0} \frac{1}{L} (\Psi[\Gamma] \circ \Gamma[\vec{y}, \hat{v}, L] - \Psi[\Gamma]). \quad (6)$$

As can be seen, this derivative compute the change of $\Psi[\Gamma]$ when a path $\Gamma[\vec{y}, \hat{v}, L]$ of infinitesimal length L starting at \vec{y} in the \hat{v} direction is appended to the path Γ . Now with the above definition, is straightforward to prove that the path derivative for $I^{ab}[\vec{x}, \Gamma]$ is given by

$$\delta_{cd}(\vec{y}, \hat{v}) I^{ab}[\vec{x}, \Gamma] = \frac{1}{2} (\delta_c^a \delta_d^b + \delta_d^c \delta_c^b) \delta^3(\vec{x} - \vec{y}). \quad (7)$$

With this tools at hand we are ready to obtain a monoidal representation for linearized gravity. Before doing that, let us recall the most relevant aspects of linearized gravity at Hamiltonian level. In the first place the canonical variables are the spatial components of the metric perturbation field h_{ab} and this conjugated momentum p^{ab} with Poisson algebra

$$\{h_{ab}(\vec{x}), p^{cd}(\vec{y})\} = \frac{1}{2} (\delta_a^c \delta_b^d + \delta_b^c \delta_a^d) \delta^3(\vec{x} - \vec{y}). \quad (8)$$

Also, there are two first class constraints given by

$$\begin{aligned}\partial_a p^{ab} &\approx 0 \\ \partial^a \partial^b h_{ab} - \nabla^2 h &\approx 0,\end{aligned}\tag{9}$$

where h is the trace of h_{ab} . As has been discussed in reference [15], the constraints (9) lead to conclude that observables of the theory must be the transverse and traceless part of the canonical pair, in the same way that Gauss constraint in electromagnetism leads to the observable in that case is the transverse part of the electric field.

In order to obtain the quantum theory for linearized gravity, we shall proceed to implement the Dirac quantization program as described below. The canonical variables are promoted to operators on certain Hilbert space

$$\begin{aligned}h_{ab} &\rightarrow \hat{h}_{ab} \\ p^{ab} &\rightarrow \hat{p}^{ab},\end{aligned}\tag{10}$$

with commutator given by

$$[\hat{h}_{ab}(\vec{x}), \hat{p}^{cd}(\vec{y})] \rightarrow i\{h_{ab}(\vec{x}), p^{cd}(\vec{y})\} = \frac{i}{2}(\delta_a^c \delta_b^d + \delta_b^c \delta_a^d) \delta^3(\vec{x} - \vec{y}).\tag{11}$$

The physical states are those annihilated by the first class constraints

$$\begin{aligned}\partial_a \hat{p}^{ab} |\Psi\rangle &= 0 \\ (\partial^a \partial^b \hat{h}_{ab} - \nabla^2 \hat{h}) |\Psi\rangle &= 0,\end{aligned}\tag{12}$$

and their time evolution is given by the Schrödinger equation.

Now, to obtain the quantum theory for linearized gravity we must represent the operators (10) acting on $\Psi[\Gamma]$ in such a way that they fulfill the canonical algebra (11) and solve the first class constraints (12). With the purpose of carrying out the quantization previously described, let us consider the follow realization

$$\begin{aligned}\hat{h}_{ab} \Psi[\Gamma] &\rightarrow i\delta_{ab}(\vec{x}, \hat{v}) \Psi[\Gamma] \\ \hat{p}^{ab} \Psi[\Gamma] &\rightarrow I^{ab}[\vec{x}, \Gamma] \Psi[\Gamma].\end{aligned}\tag{13}$$

After a straightforward calculation involving derivative (7) it can be shown that this prescription fulfill the canonical algebra (11). Regrettably, unlike electromagnetism case in the Abelian loop representation, the first class constraints can not be realized in a direct way. However, we can consider the quantization of the theory in the reduce phase space, i.e, taking the transverse and traceless part of the canonical variables as described in reference ([15]). Let \hat{h}_{ab}^{TT} and \hat{p}^{abTT} be the transverse and traceless part of the dynamical field and their conjugate moment respectively and $P_{ab/cd} = P_{ac}P_{db} - \frac{1}{2}P_{ab}P_{cd}$ be the transverse and traceless part projector (P_{ab} is the usual transverse projector of electromagnetism [15, 16, 17]). The transverse and traceless parts of our original operators are given by

$$\begin{aligned}\hat{h}_{ab}^{(TT)} &= P_{ab/cd} \hat{h}_{cd} \\ \hat{p}_{ab}^{(TT)} &= P_{ab/cd} \hat{p}^{cd}.\end{aligned}\tag{14}$$

It worth mentioning that considering (14) as our basics quantum operators, constraints (9) are “strong” equalities.

At this stage we can ask if our results (13) admits an easy geometrical interpretation as in electromagnetism case. In fact, Abelian loops are interpreted as quantum Faraday’s lines which are closed in absent of sources and this solve automatically the Gauss constraint. In the other hand, the quantum operator associated to the electric field is interpreted as a measure of the density of electric flux. However, in the monoidal representations that we found such an interpretation is unclear. In the first place, we have not found a solution for the constraints in terms of conditions on the paths space but we have restricted to reduced phase space. Secondly, the observable associated with a kind of density of flux of Faraday’s lines is p^{abTT} which have non-local terms that make difficult to extract some geometrical information.

In spite of the former observations, let us consider the monoid representation for the generator of duality [15, 17]

$$\hat{G} = \int d^3x \left(-\frac{1}{4} \hat{h}_{ab}^{(TT)} \varepsilon^{acd} \partial_c \hat{h}_d^{(TT)b} + \hat{p}_{ab}^{(TT)} \nabla^{-2} \varepsilon^{acd} \partial_c \hat{p}_d^{(TT)b} \right). \quad (15)$$

As can be seen, this generator is similar to obtained in the electromagnetic case (see for example [2]) in the sense that it depends only on the observables of the theory combined in a form that “looks like” Chern-Simons terms. But these similarities are only formals because terms in (15) are metric dependents and for this reason we are not expecting that their realization leads to knot invariants. However, some interesting information can be obtained as we shall see in what follow. Performing the representation (13) and the projector (14) in (15) we obtain

$$\begin{aligned} \hat{G} = & \int d^3x \delta_{ab}(\vec{x}, \hat{v}) \varepsilon^{acd} \partial_c \delta_d^b(\vec{x}, \hat{v}) + \int d^3x \partial^b \delta_{ab}(\vec{x}, \hat{v}) \varepsilon^{acd} \partial_c \nabla^{-2} \partial^e \delta_{de}(\vec{x}, \hat{v}) \\ & + \int d^3x I^{ab}[\vec{x}, \Gamma] \varepsilon_{acd} \nabla^{-2} \partial^e I^d{}_b[\vec{x}, \Gamma] + \int d^3x \partial_b I^{ab}[\vec{x}, \Gamma] \varepsilon_{acd} \partial^c \nabla^{-4} \partial_e I^{de}[\vec{x}, \Gamma]. \end{aligned} \quad (16)$$

To obtain an explicit form of the generator in terms of the paths Γ we must to use definitions (1) and (6). However in the same way as electromagnetic case the “electric part” of the generator have an interesting interpretation in this representation. In our case it corresponds to the last two terms. For this reason let us consider only the “electric terms”. Also, in order to simplify some calculus and to facilitate the geometrical interpretation, we shall assume that each term has support on different closed paths. Taking into account the former considerations, the third term, that we call \hat{G}_3 , can be expressed as

$$\hat{G}_3 = -\frac{1}{4\pi} \oint_{\Gamma_1} dl_{\Gamma_2} \oint_{\Gamma_2} dl_{\Gamma_1} (\hat{u}_{T_{\Gamma_1}} \cdot \hat{u}_{T_{\Gamma_2}}) \int_{\gamma}^{\vec{z}_{\Gamma_1}} dl_{\gamma} (\hat{u}_{T_{\Gamma_1}} \times \hat{u}_{T_{\Gamma_2}}) \cdot \hat{u}_{T_{\gamma}} \delta^3(\vec{z}_{\Gamma_2} - \vec{w}_{\gamma}), \quad (17)$$

where the last integral is performed on an auxiliar open straight curve γ from $-\infty$ to \vec{y} . Please, note that this term comes from replace

$$-\frac{1}{4\pi} \frac{(\vec{x} - \vec{y})^a}{|\vec{x} - \vec{y}|^3} = \int_{\gamma}^{\vec{y}} dz^a \delta^3(\vec{x} - \vec{y}) + \varepsilon^{abc} \partial_b f_c(|\vec{x} - \vec{y}|), \quad (18)$$

where $f_c(|\vec{x} - \vec{y}|)$ is an arbitrary function and as in electromagnetism case [2], the curl does not contribute to the generator. It can be seen that (17) is a kind of Gauss linking number. The

only difference is that it has the term $\hat{u}_{T_{\Gamma_1}} \cdot \hat{u}_{T_{\Gamma_2}}$ which is a measure of the angle between tangent vector of the closed paths. Note that despite of that, \hat{G}_3 will have non-vanish contribution when closed path Γ_1 intersect the surface closed by Γ_2 . However, instead of obtain ± 1 every time this occurs as in electromagnetism [2], our result is angle dependent, or in other words, is metric dependent as expected. Another way to interpret \hat{G}_3 is the following. Note that it will contribute as long as tangent vectors $\hat{u}_{T_{\Gamma_1}}, \hat{u}_{T_{\Gamma_2}}, \hat{u}_{T_{\gamma}}$ form a trihedron but also the dihedral angle between planes intersecting in the line generated by $\hat{u}_{T_{\gamma}}$ must be necessarily different from 0 and $\pi/2$.

In the other hand the fourth term, that we call \hat{G}_4 , is given by

$$\hat{G}_4 = - \left(\frac{1}{4\pi} \right)^2 \oint_{\Gamma_1} dl_{\Gamma_1} \kappa_{\Gamma_1} \oint_{\Gamma_2} dl_{\Gamma_2} \int_{\gamma_1}^{\vec{z}_{\Gamma_1}} dl_{\gamma_1} (\hat{u}_{N_{\Gamma_1}} \times \hat{u}_{T_{\Gamma_2}}) \cdot \hat{u}_{T_{\gamma_1}} \int_{\gamma_2}^{\vec{w}_{\gamma_1}} dl_{\gamma_2} \hat{u}_{T_{\gamma_2}} \cdot \hat{u}_{T_{\Gamma_2}} \delta^3(\vec{w}_{\gamma_2} - \vec{z}_{\Gamma_2}), \quad (19)$$

where $\hat{u}_{N_{\Gamma_1}}$ is the normal vector to Γ_1 and κ_{Γ_1} is their curvature. The interpretation for this term is less direct than the former. However if we consider the auxiliary curves γ_1 and γ_2 be parallel, we can say that this term will contribute as long as $(\hat{u}_{N_{\Gamma_1}}, \hat{u}_{T_{\Gamma_2}}$ and $\hat{u}_{T_{\gamma_1}})$ form a trihedron and also the dihedral angle between planes intersecting in the line generated by $\hat{u}_{N_{\Gamma_1}}$ must be different from 0 and $\pi/2$.

It seems possible to extend the above formulation to treat with another models. For example, linearized massive gravity could also be formulated in the monoidal representation following the scheme of quantization presented here. We want to make it clear that the only advantages of the monoidal representation presented here in comparison with the well known loop representation in the context of linearized gravity, is that it allows to represent symmetric objects (namely the canonical pair (h_{ab}, p^{ab})) with just one kind of extended objects instead of three. We found that this feature allows us to carry out a kind of “geometric interpretation” of interesting objects like the generator of duality of the theory. If the monoidal representation allows to obtain some hints that can be further applied or generalized to full gravity or if can provide more insights respect to the standard loop formulation is under investigation.

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