Black hole geometrothermodynamics

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Abstract.
We review the main aspects of geometrothermodynamics which is a geometric formalism to describe thermodynamic systems, taking into account the invariance of classical thermodynamics with respect to Legendre transformations. We focus on the particular case of black holes, and present a Riemannian metric which describes the corresponding space of equilibrium states. We show that this metric can be used to describe the stability properties and phase transition structure of black holes in different gravity theories.

1. Introduction

The main goal of geometrothermodynamics (GTD) is to construct a formalism that allows us to investigate the properties of thermodynamic systems by using the geometric properties of the corresponding equilibrium spaces [1]. In fact, the idea is that to any given thermodynamic system we can associate an equilibrium space whose points represent possible states at which the system can remain at equilibrium. This means that at any given moment of time the system can be represented geometrically as existing on a particular point of the equilibrium space.

To develop such a geometric formalism, we use as a guiding principle the already existing geometric representation of classical fields. Consider, for instance, the gravitational field. According to general relativity, a gravitational field can be represented by a 4-dimensional Riemannian manifold, the spacetime, whose metric satisfies the Einstein equations. Two geometric concepts are very important in this construction. First, the Riemannian curvature is known to represent the gravitational interaction. This is to say, if no gravitational field is presented, it is expected that the Riemannian curvature vanishes. This is true in general relativity since from the lack of curvature it follows the flat spacetime of special relativity. The second important concept is that of curvature singularities. Intuitively, we expect that singularities indicate the limits of applicability of the theory because they correspond to an infinite gravitational interaction which cannot be treated correctly in general relativity. In fact, the presence of singularities is a major problem in general relativity that is expected to be completely solved only in the framework of a theory of quantum gravity, which is not yet developed.

Other classical fields permit also a geometric representation in terms of principal fiber bundles. In the case of gauge fields, one can take the base space as the flat Minkowski spacetime and the standard fiber as the gauge symmetry. The corresponding curvature of the principal fiber bundle turns out to correspond to the gauge field strength, and the curvature singularities again

indicate the breakdown of the underlying classical gauge field theory. In this way, we see that all the classical field interactions existing in Nature permit a representation in terms of concepts of differential geometry [2]. In all the above examples of classical interactions, the explicit construction of the theory is based upon the use of the diffeomorphism and gauge invariance for the gravitational and gauge fields, respectively. These are very important symmetries of the theory that have a very deep physical significance.

Now we ask the question whether it is possible to construct a geometric formalism for the theory of classical equilibrium thermodynamics [3] such that the curvature represents the thermodynamic interaction and the singularities indicate the breakdown of the classical theory. GTD is an attempt to construct such a geometric formalism. An important ingredient of GTD is the Legendre invariance which will play a role similar to that of physical symmetries in classical fields. As mentioned above, the idea is to equip the equilibrium space with a geometric structure which allows one to define the curvature of that space. This can be done in several ways. The formalism of thermodynamic geometry,for instance, consists in introducing different Hessian metrics, computed from different thermodynamic potentials, into the equilibrium space [4, 5, 6]. The interesting feature of using Hessian metrics in the equilibrium space is that their components acquire a very interesting physical interpretation in fluctuation theory. In fact, small fluctuations of the thermodynamic potential can be decomposed into moments, and it turns out that the second moment of the fluctuation is essentially determined by the components of the corresponding Hessian metric. Nevertheless, some difficulties have been found when applying Hessian metrics to certain thermodynamic systems. It turns out that different Hessian metrics predict different properties for the same system. In other words, the properties of a thermodynamic system, as represented in thermodynamic geometry, depend on the choice of thermodynamic potential. This is not consistent with the results of classical thermodynamics [3]. The formalism of GTD avoids these difficulties by using only geometric objects which are invariant with respect to Legendre transformations.

In this work, we review the main aspects of the GTD formalism and apply it to characterize the thermodynamic properties of black holes from a geometric point of view. In Sec. 2, we discuss the properties of Legendre transformations when interpreted as transformations between the coordinates of an abstract space, the phase space, which is explicitly constructed in Sec. 3. We show that the phase space can be endowed with a contact structure and a Riemannian structure, both being invariant with respect to Legendre transformations. The definition of the equilibrium manifold is given in Sec. 4 as a submanifold of the phase manifold which inherits its invariance properties. Then, in Sec. 5, we investigate explicitly the geometrothermodynamic properties of black holes in Einstein's gravity. Finally, in Sec. 6, we discuss our results and some tasks of future research. We use throughout geometric units with $c = G = \hbar = k_B = 1$.

2. Legendre invariance

In classical equilibrium thermodynamics, to describe a system with n degrees of freedom, it is necessary to specify n extensive variables E^a ($a = 1, ..., n$), their intensive duals I^a and a thermodynamic potential Φ. Let us for the moment suppose that all these variables are independent from each other, and denote them by $Z^A = \{\Phi, E^a, I^a\}$. We can therefore use this set as coordinates of an abstract space. If, in addition, we suppose that this space is differentiable, we can use the standard procedures of differential geometry in order to introduce other coordinates by means of diffeomorphisms. In particular, we can consider Legendre transformations as coordinate transformations in this abstract space and investigate their properties from the point of view of differential geometry. This is one of the main assumptions of GTD, and we will see that it leads to consistent results. In addition, this opens the possibility to consider general diffeomorphisms that act on the geometric structure of GTD.

Let us now define a total Legendre transformation[1] by means of [7]

$$Z^A \rightarrow \tilde{Z}^A = \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\} \tag{1}$$

with

$$\Phi = \tilde{\Phi} - \tilde{E}^a \tilde{I}_a \ , \quad E^a = -\tilde{I}^a, \quad I^a = \tilde{E}^a \ . \tag{2}$$

This representation is equivalent to the standard one which is used in the equilibrium space. However, it allows us to consider a Legendre transformation as a coordinate transformation determined by the matrix

$$\frac{\partial Z^A}{\partial \tilde{Z}^B} = \begin{pmatrix} 1 & -\tilde{Z}^{n+1} & -\tilde{Z}^{n+2} & ... & -\tilde{Z}^{2n} & -\tilde{Z}^1 & -\tilde{Z}^2 & ... & -\tilde{Z}^n \\ 0 & 0 & 0 & ... & 0 & -1 & 0 & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & -1 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 & 0 & 0 & ... & 0 \end{pmatrix} \ , \tag{3}$$

where the indices A and B represent rows and columns, respectively. The determinant of this matrix is equal to one. Therefore, one can compute the inverse transformation which is the condition for the above transformation to be considered as a diffeomorphism.

As we see from the above expressions, a Legendre transformation interchanges the role of the extensive and intensive variables. In this sense, it resembles the role of canonical transformations of classical mechanics in which the generalized coordinates can become momenta by means of a canonical transformation.

Our goal is to construct geometric structures which are invariant with respect to Legendre transformations. This is similar to the Lorentz invariance of special relativity. Indeed, consider, for instance, the $2-$dimensional Minkowski metric in Cartesian coordinates

$$ds^2 = -dt^2 + dx^2 \ . \tag{4}$$

If we apply a special Lorentz transformation

$$(t, x) \rightarrow (\tilde{t}, \tilde{x}) \tag{5}$$

given by the relationships

$$t = \frac{1}{\sqrt{1 - v^2}} \left(\tilde{t} + v\tilde{x} \right) \ , \quad x = \frac{1}{\sqrt{1 - v^2}} \left(\tilde{x} + v\tilde{t} \right) \ , \tag{6}$$

where v is a constant, the line element (4) transforms into

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 \ , \tag{7}$$

which has the same functional dependence as the original line element.

Let us now consider the abstract space with coordinates Z^A. As an example, we check if the flat metric

$$ds^2 = \delta_{AB} dZ^A dZ^B \tag{8}$$

[1] Partial Legendre transformations can be defined analogously. However, in this work, we limit ourselves to the study of total transformations.

is Legendre invariant. For concreteness, consider the case $n = 1$, i.e.,

$$ds^2 = d\Phi^2 + (dE^1)^2 + (dI^1)^2 \ . \tag{9}$$

Applying the transformation (2), it is straightforward to see that the flat metric transforms into

$$ds^2 = d\tilde{\Phi}^2 - 2d\tilde{\Phi}(\tilde{E}^1 d\tilde{I}^1 + \tilde{I}^1 d\tilde{E}^1) + 2\tilde{E}^1 \tilde{I}^1 d\tilde{E}^1 d\tilde{I}^1 + (d\tilde{E}^1)^2 + (d\tilde{I}^1)^2 \ , \tag{10}$$

whose functional dependence is clearly different. We conclude that a Legendre invariant metric is necessarily curved.

In general, it is not easy to find Legendre invariant quantities because the set of Legendre transformations do not form a group. Moreover, in the case of subsets which form a group, one can show that not all them can be obtained from the generators of infinitesimal Legendre transformations [8]. The search for Legendre invariant geometric quantities must therefore be conducted in an empiric way in which the functional invariance should be represented explicitly, giving rise to a set of algebraic equations to be solved by using the standard methods of linear algebra. This is the method we will use in the following sections.

3. Phase manifold

One the main geometric components of GTD is the phase manifold which is necessary in order to represent Legendre transformations as diffeomorphisms. The triad $(\mathcal{T}, \Theta, G)$ is called phase manifold if the following conditions are satisfied.

(i) $\mathcal{T}$ is a $(2n + 1)-$dimensional differential manifold, a local patch of which is characterized by the set of coordinates $Z^A = \{\Phi, E^a, I^a\}$.

(ii) Θ is a contact $1-$form defined on the cotangent manifold $T^*_{\mathcal{T}}$, i.e., it satisfies the condition $\Theta \wedge (d\Theta)^n \neq 0$, meaning that $\mathcal{T}$ is maximally non-integrable. According to Darboux theorem, locally it is possible to express the contact $1-$form as

$$\Theta = d\Phi - I_a dE^a \ . \tag{11}$$

(iii) G is a Legendre invariant Riemannian metric on $\mathcal{T}$, i.e., $G = G_{AB}dZ^A dZ^B$.

In general, the triad $(\mathcal{T}, \Theta, G)$ represents a Riemannian contact manifold. The contact form is Legendre invariant in the sense that under a Legendre transformation $Z^A \rightarrow \tilde{Z}^A$, it behaves as

$$\Theta \rightarrow \tilde{\Theta} = d\tilde{\Phi} - \tilde{I}_a d\tilde{E}^a \ . \tag{12}$$

In GTD, we also demand that the metric G be Legendre invariant. As mentioned in the previous section, it is difficult to generate Legendre invariant objects due to the non-standard properties of the set of Legendre transformations. Therefore, in order to construct a Legendre invariant metric, we proceed as follows. We start from a general metric $G = G_{AB}dZ^A dZ^B$ and transform the components G_{AB} with the transformation matrix (3). For the resulting components $\tilde{G}_{AB}$ we demand functional invariance as explained above. This leads to a system of algebraic equations that can be solved to obtain specific Legendre invariant metrics [1]. Among the particular solutions we have found so far [9, 10], the metric

$$G^{II} = \Theta^2 + (\xi_{ab}I^a E^b)(\eta_{cd}dI^c dE^d) \ , \tag{13}$$

has been shown to be applicable in the case of thermodynamic systems with second-order phase transitions, like black holes. Here $\eta_{cd} = \text{diag}(-1, \cdots, 1)$ and ξ_{ab} is a diagonal constant matrix.

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It is important to note that to the above matrix structure it is still possible to add quadratic terms [1, 8]

$$\delta_{ab}(dE^a dE^b + dI^a dI^b) \, , \tag{14}$$

non-conformal terms

$$\delta_{ab}E^b I^a dE^a dI^b \, , \tag{15}$$

and mixed conformal terms

$$\Omega \, \epsilon_{ab} dE^a dI^b \, , \tag{16}$$

where $\delta_{ab} = \mathrm{diag}(1, \cdots, 1)$, $\epsilon_a{}^b = 1, -1, 0$ for $a > b$, $a < b$, $a = 0$, respectively, and Ω is an arbitrary Legendre invariant function of the coordinates E^a and I^a. However, as we will see below, none of them satisfies the physical condition that the thermodynamic curvature of the equilibrium manifold vanishes when thermodynamic interaction is lacking. In this sense, the metric (13) is essentially unique. In fact, the contact form Θ can still be multiplied by an arbitrary non-zero function of the coordinates Z^A which, however, does not affect the geometric structure of the physical equilibrium manifold. Moreover, the second term of the metric (13) is determined modulo a multiplicative constant which can be set equal to one, without loss of generality.

We thus see that the phase manifold of GTD is a Legendre invariant, Riemannian, contact manifold [11]. The above procedure shows that the construction of the phase manifold is essentially unique. Indeed, once the coordinates of $\mathcal{T}$ are fixed, the contact form is given in a canonical way, modulo a non-zero conformal function which does not affect the physical results. The metric structure G is also essentially unique as explained above. The constant diagonal matrix ξ_{ab} which enters the metric (13) contains, in principle, n arbitrary constants which are not fixed by the Legendre invariance condition. However, they can be fixed by imposing an additional physical condition which is related to the Euler identity as we will see in the following sections.

4. Equilibrium manifold

In the previous section, we introduced the concept of phase manifold as an auxiliary space to handle Legendre transformations as diffeomorphisms. On the other hand, we mentioned that thermodynamic geometry uses only the equilibrium space to describe thermodynamic systems from a geometric perspective, without considering the Legendre invariance of classical equilibrium thermodynamics, i.e., without considering the corresponding phase space. In this section, we will show that in fact it is possible to consider the equilibrium space of any thermodynamic system as subspace of the phase manifold.

In GTD, an equilibrium manifold is a Riemannian manifold $(\mathcal{E}, g)$ defined by the following conditions:

(i) $\mathcal{E} \in \mathcal{T}$ with $\dim(\mathcal{E}) = n$.

(ii) There exists a smooth embedding map $\varphi : \mathcal{E} \to \mathcal{T}$ such that

$$\varphi^*(\Theta) = 0 \, , \tag{17}$$

where φ^* is the pullback of φ.

(iii) There exists a canonically induced metric g determined by

$$g = \varphi^*(G) \, . \tag{18}$$

We see that the definition of an equilibrium manifold is closely related with the existence of the smooth map φ. It is therefore important to establish the conditions under which the map φ can exist. To this end, consider the explicit form of the map φ in a particular coordinate system. Let us consider E^a as the coordinates of $\mathcal{E}$. Then, the map $\varphi : \mathcal{E} \to \mathcal{T}$ implies that

$$\varphi : \{Z^A\} \mapsto \{\Phi(E^a), E^a, I^a(E^a)\} \ , \tag{19}$$

i.e, all the coordinates of $\mathcal{T}$ should now depend on the coordinates of $\mathcal{E}$. Consider the function $\Phi(E^a)$. As mentioned in the previous section, Φ was introduced as a thermodynamic potential and E^a are the extensive variables. Then, $\Phi(E^a)$ represents a fundamental thermodynamic equation and Φ can be either the entropy or the internal energy of the system [3]. We conclude that the equilibrium manifold can be defined only if the fundamental equation is known. This is in accordance with the results of classical thermodynamics in which all the properties of a system can be obtained from the explicit form of the fundamental equation. Now, we have in GTD a similar situation. We need to know the fundamental equation $\Phi(E^a)$, as part of the smooth map φ, in order to construct the equilibrium manifold. This implies that for a given thermodynamic system, given by a unique fundamental equation $\Phi(E^a)$, there exists a unique equilibrium manifold $\mathcal{E}$.

Consider now condition (17) which in coordinates Z^A implies that

$$d\Phi = I_a dE^a \ , \quad I_a = \frac{\partial \Phi}{\partial E^a} \ . \tag{20}$$

This implies that $I_a = I_a(E^a)$, as demanded by the map (19), and that I_a is the intensive variable dual to E^a as in classical thermodynamics. On the other hand, we can now interpret the condition $\varphi^*(\Theta) = 0$ as the first law of thermodynamics.

The condition (18) determines the metric of the equilibrium manifold which in the case of the Legendre invariant metric (13) can be expressed as

$$g^{II} = \varphi^*(G^{II}) = \left(\xi_a{}^b E^a \Phi_{,b} \right) \left(\eta_a^b \Phi_{,bc} \, dE^a dE^c \right) \ , \tag{21}$$

where $\eta_a^b = \mathrm{diag}(-1, 1, \cdots, 1)$, $\xi_a{}^b = \xi_{ac}\delta^{bc}$, and $\Phi_{,a} = \frac{\partial \Phi}{\partial E^a}$. We see that all the components of the metric g^{II} can be calculated explicitly once the fundamental equation $\Phi(E^a)$ is given. The conformal term can be put proportional to the potential Φ by using Euler's identity. Indeed, if Φ is a homogeneous function of degree β, i.e., $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$, Euler's identity reads $E^a \Phi_{,a} = \beta\Phi$. If Φ is instead a generalized homogeneous function [18], i.e., $\Phi(\lambda^{\alpha_a} E^a) = \lambda^{\alpha_\Phi}\Phi(E^a)$, then $\alpha_a E^a \Phi_{,a} = \alpha_\Phi \Phi$. Therefore, we see that it is always possible to choose the components of the diagonal matrix $\xi_a{}^b$ proportional to α_a so that the conformal factor becomes proportional to Φ, i.e.,

$$g^{II} = \Phi \left(\eta_a^b \Phi_{,bc} \, dE^a dE^c \right) \ . \tag{22}$$

We conclude that the equilibrium manifold as defined in GTD contains in a canonical manner the information about the first law of thermodynamics and the fundamental equation of the thermodynamical system under consideration. The second law of thermodynamics should be contained in the fundamental equation as a consequence of the condition that the corresponding system is physical. The third law implies that there exists a point in the equilibrium manifold which cannot be reached by a finite number of quasi-static processes. This implies geometrically that the point of minimum entropy should not be part of the equilibrium manifold. In the case of the ideal gas, this implies that the topology of the equilibrium manifold is not trivial [12].

In the following diagram, we summarize the geometric structure of GTD.

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ \varphi\ } & \mathcal{T} \\
\downarrow & & \downarrow \\
T_{\mathcal{E}} & \xrightarrow{\ \varphi_*\ } & T_{\mathcal{T}} \\
\downarrow & & \downarrow \\
T_{\mathcal{E}}^* & \xleftarrow{\ \varphi^*\ } & T_{\mathcal{T}}^* \\
g & \xleftarrow{\ \varphi^*\ } & G
\end{array}
$$

Due to the differentiability properties of the phase manifold $\mathcal{T}$, it is possible to introduce the corresponding tangent $T_{\mathcal{T}}$ and cotangent space $T_{\mathcal{T}}^*$. Since the embedding map φ is smooth, it is also possible to define in a canonical way the pushforward φ_* and pullback φ^* that act on the tangent and cotangent spaces, respectively. Finally, once the metric G is given on $\mathcal{T}$, the embedding map induces a canonical metric g on $\mathcal{E}$.

It is important to mention that if we insert the fundamental equation for an ideal gas

$$
S = \frac{3}{2} \ln U + \ln V \ , \tag{23}
$$

where U is the internal energy, V is the volume and $N = 1$ is the molar number, the curvature of the metric (22) vanishes, implying that no thermodynamic interaction exists, as expected from classical thermodynamics. However, if we add anyone of the Legendre invariant terms (14), (15) or (16), we obtain a non-flat equilibrium manifold. Because of this inconsistency, those additional terms are not considered in GTD.

5. Black holes

Since the discovery that the area of a black hole horizon satisfies a series of rules that resemble the laws of thermodynamics [13, 14, 15], black holes have been intensively studied by using the standard methods of classical thermodynamics [16]. It is in this sense that black holes can be considered as thermodynamic systems. However, this is still a controversial approach in particular because a definite microscopic model for black holes is still missing [17].

The main postulate of black hole thermodynamics is that the entropy S is proportional to the horizon area A as

$$
S = \frac{1}{4} A \ , \tag{24}
$$

which can be considered as the fundamental equation of black holes. This implies that the entropy is a function of the extensive variables $S = S(E^a)$ which correspond to the physical parameters of the black hole.

Consider now the most general black hole in Einstein's theory whose spacetime is completely determined by the Kerr metric [19]

$$
\begin{aligned}
ds^2 = \ & -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\varphi \\
& + \frac{(r^2 + a^2)^2 - a^2 \sin^2\theta\,\Delta}{\Sigma} \sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \ ,
\end{aligned} \tag{25}
$$

$$\Sigma = r^2 + a^2 \cos^2\theta \, , \quad \Delta = (r - r_+)(r - r_-) \, , \quad r_\pm = M \pm \sqrt{M^2 - a^2} \, , \tag{26}$$

where M is the total mass of the black hole and $a = J/M$ is the specific angular momentum. It is then straighforward to compute the area of the exterior horizon to obtain

$$S = 2\pi(M^2 + \sqrt{M^4 - J^2}) \, . \tag{27}$$

To construct the GTD of the Kerr black hole, we first have to identify the thermodynamic variables. Since $S = \Phi$ is the thermodynamic potential, then $E^a = (M, J)$. Therefore, the metric of the equilibrium manifold (22) becomes

$$g^{K} = S\left(-S_{MM}dM^2 + S_{JJ}dJ^2\right) \, , \tag{28}$$

where the subscript denotes partial derivative. Now, the corresponding scalar curvature can be shown to be of the form

$$R^{K} = \frac{N^{K}}{S_{MM}S_{JJ}} \, , \tag{29}$$

where N^{K} is a function of M and J which is in general non-zero. The fact that the curvature of the the equilibrium manifold is non-vanishing indicates that the Kerr black hole is a system with thermodynamic interaction. Moreover, we see that curvature singularities appear when $S_{MM} = 0$ and $S_{JJ} = 0$ which, according to the interpretation of the thermodynamic curvature, implies the existence of phase transitions. On the other hand, if we interpret black holes as classical thermodynamic systems, it is possible to explore the behavior of the response functions in order to find the phase transition structure. Indeed, according to the Ehrenfest scheme [3], a zero or a discontinuity in the $n-$th derivative of a thermodynamic potential indicates the presence of an $n-$th phase transition. We conclude that the thermodynamic curvature of the equilibrium manifold of the Kerr black hole predicts the existence of second order phase transitions. The location of the transition points in the equilibrium space has been derived explicitly in [20].

We now consider the Einstein-Maxwell theory. The most general black hole in this case is described by the Kerr-Newman metric which is again given by Eq.(25) with different values for the horizon radii, namely,

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \, , \tag{30}$$

where Q is the electric charge of the black hole. Then, the fundamental equation is given by

$$S = \pi\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2}\right) \, , \tag{31}$$

indicating that the coordinates of the equilibrium manifold are $E^a = (M, J, Q)$. The GTD of the Kerr-Newman black hole is then determined by the thermodynamic metric

$$g^{KN} = S\left(-S_{MM}dM^2 + S_{JJ}dJ^2 + S_{QQ}dQ^2 + 2S_{JQ}dJdQ\right) \, . \tag{32}$$

We notice that in this case there is some ambiguity in the choice of the coordinate which should be associated with the minus sign of the matrix $\eta_a^b = \mathrm{diag}(-1, 1, 1)$. Since there are three different possible choices, it turns out that the right choice corresponds to the coordinate M. The reason is that the minus sign should be associated with the coordinates which determine the thermodynamic potential of the fundamental equation, i.e, the entropy or the internal energy. This shows that although in GTD it is possible to use any thermodynamic potential which is obtained from the entropy or the internal energy by means of a total Legendre transformation, the canonical variables associated with the fundamental equation are in certain sense privileged.

A straightforward computation of the curvature tensor of the metric (32) indicates that in general several components are different from zero. This means that the Kerr-Newman black hole is characterized by thermodynamic interaction. For the study of the curvature singularities, we have to investigate scalars formed by the components of the curvature tensor. The Ricci scalar $R = g^{ab}g^{cd}R_{acbd}$ is the simplest scalar available, but in the $3-$dimensional case under consideration, there are other independent scalars, in particular, those which are quadratic in the curvature tensor, namely, the Kretschmann scalar $R_{abcd}R^{abcd}$, the Euler scalar $*R_{abcd}R^{abcd}$, and the Pontrjagin scalar $*R_{abcd} * R^{abcd}$, where $*R_{abcd}$ is the dual curvature tensor [21]. A computation of all these scalars shows that all of them yield the same singularities, which are equivalent to the singularities of the Ricci scalar, i.e.,

$$R^{KN} = \frac{N^{KN}}{S_{MM}(S_{JJ}S_{QQ} - S_{JQ}^2)} \, , \tag{33}$$

where N^{KN} is a non-zero function of the coordinates M, J and Q. We see that there two types of singularities for

$$S_{MM} = 0 \, , \quad \text{and} \quad S_{JJ}S_{QQ} - S_{JQ}^2 = 0 \, . \tag{34}$$

The first type of singularities corresponds again to a second order phase transition as can be corroborated by calculating the corresponding response function [20]. The second type of singularities is connected with

$$S_{JJ}S_{QQ} - S_{JQ}^2 \geq 0 \, , \tag{35}$$

which corresponds to the convexity and stability conditions for any thermodynamic system with J and Q as thermodynamic degrees of freedom [3]. It follows that the second type of singularities implies that during the phase transition the black hole passes from an unstable state $(S_{JJ}S_{QQ} - S_{JQ}^2 < 0)$ to a stable state $(S_{JJ}S_{QQ} - S_{JQ}^2 > 0)$.

These results show that in fact the curvature of the equilibrium manifold contains the thermodynamic information of black holes.

6. Final remarks

In this work, we reviewed the main geometric structures which are necessary in GTD to develop a formalism which is independent with respect to Legendre transformations. The formalism involves concepts of contact geometry, Riemannian geometry and classical equilibrium thermodynamics. Here, we focused of the applications of GTD in the context of black hole thermodynamics.

First, we introduce the concept of phase manifold in order to handle Legendre transformations as coordinate transformations. Since the phase manifold is necessarily odd dimensional, it is possible to introduce in a canonical way a contact structure which generates the first law of thermodynamics. We endow the phase manifold with a Riemannian metric which is invariant with respect to total Legendre transformations. The auxiliary phase manifold contains only geometric objects which are Legendre invariant and so it represents the main ingredient of GTD.

The equilibrium manifold is defined as contained in the phase manifold and its properties are determined by means of an embedding map whose definition contains a fundamental thermodynamic equation. This means that in principle there exists a unique equilibrium manifold for each thermodynamic system. The goal of GTD is to describe the thermodynamic properties of a given system in terms of the geometric properties of the corresponding equilibrium manifold. In particular, we associate curvature with thermodynamic interaction and singularities with phase transitions.

We present as particular example the GTD of black holes and analyze the most general black holes in Einstein and Einstein-Maxwell theories. In the case of the Kerr and Kerr-Newman

black holes, the curvature of the corresponding equilibrium manifolds turns out to be non-zero, pointing to the existence of thermodynamic interaction in these black holes. In addition, the curvature singularities of the equilibrium manifolds are shown to correspond to divergences of the response functions, as defined in classical thermodynamics. Moreover, the thermodynamic curvature also contains information about the stability properties of black holes. Indeed, we found that during a phase transition the black hole passes from an unstable state to a stable state.

Finally, we mention that black hole GTD has been also applied to a large number of black hole configurations in different gravity theories like Einstein-Born-Infeld, Einstein-Yang-Mills, etc. (see Ref. [10] and references cited therein). So far, black hole GTD has shown to lead to consistent results in all the cases in which it has been applied. Therefore, GTD represents an alternative method to study thermodynamic properties of black holes.

A particular contribution of GTD to the physics of black holes is the definition of thermodynamic interaction in terms of curvature. This is certainly an invariant method to explore internal properties of black holes. Indeed, in classical black hole thermodynamics there is no method to define interaction. This is only possible at the level of statistical physics in which the presence of a potential term in the Hamiltonian function indicates the existence of thermodynamic interaction. In black hole physics, however, no definite Hamiltonian is known that could yield from statistical physics the fundamental equation in the corresponding thermodynamic limit. We therefore believe that by investigating the behavior of the thermodynamic curvature, it would be possible to extract information about the interaction inside a black hole.

Acknowledgements

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