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Analytical solution of subsonic gas flow past a circle

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Abstract. The Mach-Number series expansion of the potential function for the two dimensional flow of an inviscid, compressible, frozen gas past a circular cylinder are obtained to 29 terms. Analysis of these expansion allows the estimates for the critical Mach-Number M_*, (at which flow first becomes locally Sonic), and the radius of convergence of the series M_c for the maximum velocity. In this case of two-dimensional circular cylinder we obtain consistent results with 29 terms obtained by Van Dyke and Guttmann [19] for the case of diatomic gas with gamma equal to 7/5. For this case of two dimensions, we have been unable to determine the nature of singularities. We hope this work help to shed a light for resolving the 80-years old conterversary which says that whether the airfoil can posses a continuous range of shock free potential flow above the critical Mach-Number ($M_c > M_*$).

1. Introduction

This work is concerned with the two dimensional problem flow of an inviscid, compressible, perfect, frozen gas past a cylinder. The Jansen-Rayleigh approximation to subsonic flow is used. We hope to shed light on the transonic controversy, which may be phrased as: Can an airfoil have a continuous range of smooth shock-free potential flows above the critical Mach number?. Nocilla[11] have numerical results for flow past biconvex airfoils which suggest that shock -free flows can exist for up to 5 percentage. Generally numerical results of any kind like finite volume done by my student for circle case indicates range of shock -free flow above M_*. Inspired by original work of Rayleigh about 60 years ago G.I. Taylor raised the question whether an observed breakdown of flow near the critical Mach number occurs because irrotational flow ceases to be possible or occurs for some other reason connected with viscosity or other properties of the air not considered in the theory.

The goal of this analysis is to provide as complete a description as possible of this controversy. The analysis yields a solution for all values of Mach number from zero to one in a continuous fashion. We hope this work help to shed a light for resolving the 80-years old controversy which says that whether the airfoil can posses a continuous range of shock free potential flow above the critical Mach-Number ($M_c > M_*$). In the case of two-dimensional circular cylinder Van Dyke and Guttmann [19] showed that the radius of convergence for their Jansen-Rayleigh expansion exceeds the critical Mach-Number ($M_c > M_*$) by some 1.1 percentage. Bollman[2] uses series extension for flow of an inviscid, compressible, perfect gas along a sinusoidal wall in the transonic small-disturbance approximation. He generated 36 terms in the series expansion for maximum velocity and finds that the series is a Stieltjes kind. His claim was followed so that the series converge only up to the critical Mach-number.

As it has been illustrated in this example the singularity at Mach number equal to one corresponding to the transonic limit is not an isolated algebraic singularity. This method can be employed very successfully if care be exercised.

In this paper we will show that even in the context of compressible problems computer extension may not easily reveal the flow structure. More work and advanced analysis is required to deal with complications, such as zero radius of convergence, slowly converging type series or series have exponential type singularity. Supercomputers can handle these problems more effectively.

We consider here, that of steady flow of an inviscid, compressible, perfect gas past a circular cylinder without circulation. Following both Rayleigh[13] and Jansen[6], we have used the velocity potential as the dependent variable, rather than stream function, as the simplification of the surface boundary condition obtained by using the stream function as the dependent variable is brought at the price of a more complicated differential equation.

This equation for the velocity potential ϕ is given by Oswatitsch[12] in Cartesian co-ordinates, and re-expressing in the more appropriate cylindrical polar-coordinates ,we have:

$$\nabla^2\phi = M^2\left[\frac{\gamma-1}{2}\left(-1+\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta}\right)\nabla^2\phi\right]+$$

$$M^2\left[\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial r\partial r}+\frac{2}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial\theta\partial r}+\frac{1}{r^4}\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial\theta\partial\theta}-\frac{1}{r^3}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta}\right] \tag{1}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Where the angle θ , is measured from the horizontal axis. The corresponding boundary conditions expressing the normal velocity over the surface must be zero:

$$\frac{\partial\phi}{\partial r}=0 \quad \text{At } r=1$$
$$\phi = r\cos(\theta) \quad \text{At } r=a$$

2. Series derivation and computer extension

The equation for the velocity potential ϕ is just equation (1) in r,θ coordinate, by the requirement that ϕ to be an even function of θ, is enough to ensure the symmetric flow, free of circulation. By neglecting the right-hand side of equation (1), setting $M=0$, gives the basic incompressible solution. Enlightened separation of variables, trying $r^m\cos(m\theta)$ as suggested by the upstream condition, yields $m=1$ and $m=-1$. Hence the first approximation is found as:

$$\phi_1 = (r-\frac{1}{r})\cos(\theta)$$

Frankl and Keldysh[3] have proved that $\phi\left(M^2\right)$ is analytic in the neighbourhood of the origin, therefore one can systematically improve on this approximation by expanding in powers of M according to:

$$\phi = \phi_1 + M\phi_2 + M^2\phi_2 + \mathrm{L} = \sum_{n=0}^{\infty} \phi_n M^n \tag{2}$$

Where:

$$\phi_2 = (\frac{1}{12r^5} - \frac{1}{2r^3} + \frac{13}{12r})\cos(\theta) + (\frac{1}{12r^3} - \frac{1}{4r})\cos(3\theta)$$

$$\phi_3 = (\frac{28}{720r^9} - \frac{195}{720r^7} + \frac{414}{720r^5} - \frac{84}{72r^3} + \frac{1563}{720r})\cos(\theta)$$
$$+(\frac{-5}{360r^9} - \frac{3}{36r^7} + \frac{171}{360r^5} - \frac{1}{3r^3} - \frac{330}{360r})\cos(3\theta)$$
$$-(\frac{1}{10r^5} + \frac{5}{40r^3} + \frac{5}{40r})\cos(5\theta)$$

We have found 12 terms exactly by means of symbolic language, however for saving space only we report the result for the maximum velocity (U is free stream velocity) :

$$q_{\max} = (\frac{\partial\phi}{\partial\theta})_{r=1} d\theta$$

$$q_{\max} = 1 + \frac{7}{6}M^2 + \frac{281}{120}M^4 + \frac{9161}{1512}M^6 + \frac{115656949}{6350400}M^8 + \frac{10637691205579}{176033088000}M^{10} + \cdots = \sum_{n=0}^{\infty} q_n (M)^{2n-2} \tag{3}$$

Following the recipes given by Mansour[8] and Mansour[9], we have also obtained 50 terms of the series by writing a Fortran program in quadruple precision. The result for the local variable of maximum velocity is given in Table 1. The critical Mach number M_* is given by the solution of:

$$M_*^2 = \frac{2}{(\gamma+1)q_{\max}^2 - \gamma + 1}$$

The fundamental question we wish to resolve is whether (a) $(M_c > M_*)$ or (b) $(M_c = M_*)$.

3. Analysis of series and Discussion

Pade approximants has been used in original forms to enable us to increase the range of applicability of the series as has been used in the works of Mansour [8] and Mansour[9]. This method does not necessarily require any information about the radius of convergence. The Pade approximants provide an approximation that is invariant under an Euler transformation of the independent variables. The theory of Pade approximants has been used extensively in Mansour [9]. Briefly stated, the Pade approximant is the ratio P(K)/Q(K) of polynomials P and Q of degree m and n, respectively, that, when expanded, agrees with the given series through terms of degree m+n, and normalized by P(Q)=1. Such rational fractions are known to have remarkable properties of analytic continuation. The coefficients of the power series must be known to degree m+n. By equating like power of g(x) and P(K)/Q(K), the linear system of m+n+1 equation must be solved to obtain the coefficients in the functional form P(K)/Q(K).).Pade approximation of orders [1/1], [2/2] and [3/3] for $q_{\max}$ series are respectively:

pade[1/1]:

$$\frac{-2+\dfrac{299M}{105}}{(1-\dfrac{281M}{140})}$$

pade[2/2]:

$$\frac{-2+\left(\dfrac{22264113M}{2876314}\right)-\dfrac{5601925387M^2}{1208051880}}{1-\left(\dfrac{38429719M}{8628942}\right)+\dfrac{13574887007M^2}{3624155640}}$$

pade[3/3]:

$$\frac{\left(-2+\dfrac{838806830007803891009 11M}{656026242337163518821 0}+\dfrac{1327659800594433582005 09(M)^2}{6262068676854742679655}+\dfrac{216379848216780444042732289M^3}{2893075728706891118000 6100}\right)}{\left(1-\dfrac{2288358062367849087178 9M}{3280131211685817594105}+\dfrac{4959394809598266363664843(M)^2}{36737469570881157053976 0}-\dfrac{1198994215831331655567749 57(M)^3}{1851568466372410315520390 4}\right)}$$

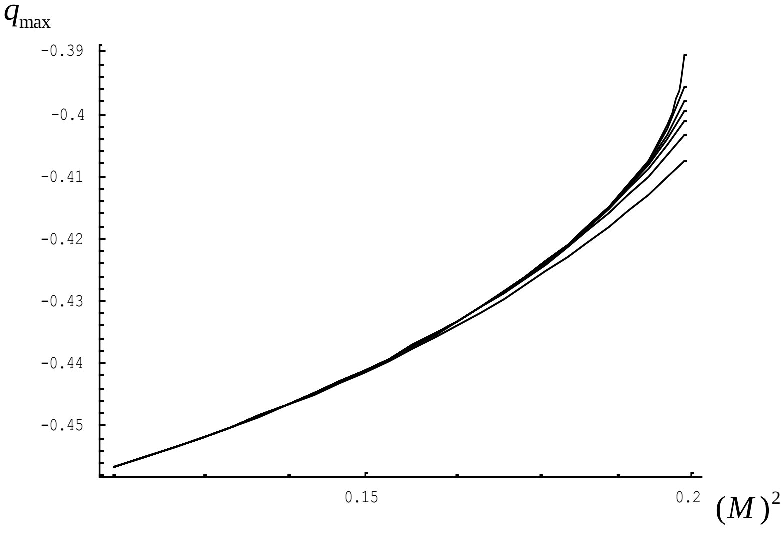

Figure 1. Plots of [8/8], [7/7], [6/6],[5/5], [4/4], [3/3]and [2/2] of the Pade approximants for q_{max} series versus $(M)^2$

When we form the ratios [8/8], [7/7], [6/6],[5/5], [4/4], [3/3]and [2/2of the Pade approximants, It can be shown, they agreed up to the value $(M)^2$=.17. This conclusion is confirmed as is plotted in Figure 1. This means that we are successful to do analytic continuation for Mach number up to .4

Table 1 . Coefficients of the series for maximum speed (q_{max})

n	q_n	n	q_n
1	2.00000000E+00	18	1.6242596E+10
2	1.16666667E+00	19	7.4261616E+10
3	2.3416666+00	20	3.4195181E+11
4	6.0588624E+00	21	1.5847778E+12
5	1.8212545E+01	22	7.3879922E+12
6	6.0430066E+01	23	3.4627706E+13
7	2.1495893E+02	24	1.6310689E+14
8	8.0496612E+03	25	7.7180403E+14
9	3.1354777E+03	26	3.6676008E+15
10	1.2598968E+04	27	1.7497171E+16
11	5.1915440E+05	28	8.3781619 E+16
12	2.1841806E+05	29	4.0255125 E+17
13	9.3512994E+06	30	
14	4.0637934E+07	31	
15	1.7889006E+07	32	
16	7.9639262E+08	33	
17	3.5807695E+09	34	

Note: ***E*** is a scientific abbreviation of multiplication by, 10 powered by the following number.

Previous works on the problem for the case of diatomic gas, the first attempt was made by O. Janzen [6] and by Lord Rayleigh [13]. Both independently obtained ϕ_1 and ϕ_2 , and Lord Rayleigh expressed the view that $(M_c = M_*)$. Later on Taylor and Sharman [17] and some years later Sauer [14] supported Rayleigh 's view for the cylinder, but pointed out there is some doubt. However

Shapiro stated that $(M_c = M_*)$. For both the cylinder and the sphere cases as Lighthill [7] expressed the view which is possible that $(M_c > M_*)$. Conclusions of Lighthill[7] presumably relied on the work of Imai[5] and Simasaki[16]. Imai[5] extended the series by obtaining the ϕ_3 term, and in 1941 the ϕ_4 term. Simasaki[16] published the next two terms ϕ_5 and ϕ_6. He estimated $M_* = 0.40$ and $M_c = 0.50$ for $\gamma = 1.405$ approximately. Of course that was a remarkable hand conclusion, but still very inaccurate. Later on by help of computer Hoffman [4] found 7 terms and gave for $\gamma = \frac{7}{5}$ a value of $M_* = 0.3983 + 0.0002$, which compares well with numerical integration technique done by Melnick and Ives[10]. Hoffman [4] finally concluded that $(M_c > M_*)$ by 5.5 percentage . Vandyke and Guttmann [19] obtained 29 terms and they gave for $\gamma = \frac{7}{5}$ the estimates $M_* = 0.39823780$ and $M_c = 0.402667605$, so that M_c is larger than M_* by 1.11percentage. We have found 50 terms and agreed well the estimate given by work of Vandyke. Since we have not been unable to found the singularity even 20 more terms can not help. Of course our analysis is based on work by Barber [1] and his transformation is known for losing lots of accuracy, only 20 coefficients by this way can be used. In brief as has been seen in works of Hoffman [4] and Vandyke and Guttmann [19] by having 7 terms and 29 terms, the gap between M_c and M_* has decreased from 5.5 to 1.1 percentages respectively. The author [20] has discussed some aspect of these contradictory conclusions about the appearance of shocks in transonic flow. Question of whether 29 terms obtained in this work with different gamma namely equal one can reduce even further this gap or not?. The answer is, there are some difference between different diatomic gas previously studied and frozen gas we report here, however at this stage we have found no better than 1.1 percentages.

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