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A hybrid method for improving the robustness of WENO scheme

Ding Z H[1], Wu S P[1, 2], Liu J[*3]

[1]School of Aeronautic Science and Engineering, Beijing University of Aeronautics and Astronautics, 37 Xueyuan Road, Beijing 100191, China
[2]Ministry-of-Education Key Laboratory of Fluid Mechanics, Beijing University of Aeronautics and Astronautics, China
[3]Sino-French Engineer School, Beijing University of Aeronautics and Astronautics, 37 Xueyuan Road, Beijing 100191, China

Email: lantingjixulj@126.com

Abstract. In this paper, a hybrid scheme based on fifth-order finite difference WENO scheme is proposed for solving spurious oscillation in strong shock-capturing. In this scheme, we consider smoothness indicator as a measurement of gradient and construct a function of indicator, using which the new hybrid can switch between WENO scheme and Lax-Friedrichs (LxF) scheme. Numerical experiments demonstrate that the hybrid scheme dominates the accuracy of the solution in smooth region and non-oscillation can be promised around discontinuity. The new hybrid scheme could essentially preserve positive density and positive pressure. The idea of constructing the hybrid can be easily extended to other high resolution schemes (TVD, ROE etc.) coupled with WENO scheme. These ideas will be illustrated in further study.

1. Introduction

Exact solution of scalar conservation law

$$\begin{aligned} u_t + \nabla \cdot F(u) = 0 \\ u(\boldsymbol{x},0)=u_0(\boldsymbol{x}) \end{aligned} \tag{1.1}$$

has an important property that it satisfies a strict maximum principle, namely: $u(\boldsymbol{x},t) \in \left[\min_x u_0(\boldsymbol{x}), \max_x u_0(\boldsymbol{x})\right]$ [1], the same property is desired to be satisfied in numerical simulation.

Weighted essentially non-oscillatory (WENO) schemes, which are based on essentially non-oscillatory (ENO) schemes, perform well in high accuracy solution but still have some problems in shock-capturing. [2, 3, 4, 5] develop advanced smoothness indicators of WENO scheme since Jiang and Shu proposed classical WENO-JS [6], among which WENO-M and WENO-Z can measure smoothness indicator in the vicinity of critical points, and assign substantially larger weights to discontinuous stencils [3], which improves the robustness of smoothness indicator. Due to the Gibbs' phenomenon, high order Lagrangian interpolation always occur oscillation. Numerical experiment demonstrate that classical WENO-JS shows mild oscillation while Flux Vector Splitting (FVS) WENO-

Z is sensitive to discontinuity and performs severe oscillation which might lead to negative density and pressure during calculation in some complex problem.

The spurious oscillation of fifth-order WENO scheme comes from following reasons: (a) Odd rank truncation error term ($\partial^{(2n-1)}u/\partial x^{(2n-1)},(n=1,2,......)$) in WENO scheme, because the dispersion always comes from $(2\mathrm{n}-1)^{\mathrm{th}}$ -derivative and the dissipation from $(2\mathrm{n})^{\mathrm{th}}$ -derivative $(n=1,2,......)$, detail analysis of the problem in [7]; (b) The solution of unbalanced weights in advanced WENO schemes ($\mathrm{WENO-M, WENO-Z, WENO-NS, WENO-}\eta$) may do not strictly satisfy maximum-principle in scalar conservation law, which lead to severe oscillation when using characteristic splitting.

In this paper, we further deal with the spurious oscillation in the vicinity of discontinuity. There are mainly three techniques for solving the problem: (a) Add higher order term to WENO scheme. Then counteract the fifth-derivative ($\partial^{(5)}u/\partial x^{(5)}$) of the truncation error, then the main term of truncation error is a sixth-derivative which mainly perform dissipation; (b) Direct limitation: solution $u(\boldsymbol{x},t)$ that does not satisfy maximum-principle will be forced into $\left[\min_x u_0(\boldsymbol{x}),\max_x u_0(\boldsymbol{x})\right]$, this technique perform well in some mildly oscillatory solution; (c) As will be discussed in following part, a hybrid of WENO and low accuracy scheme that satisfies maximum principle.

It is well-known that Lax-Friedrichs (LxF) schemes solving equation (1.1) strictly satisfies the maximum principle [1]. Both explicit and implicit 1^{st}-order LxF scheme keep positive density and internal energy [8]. And some Positive-Preserving WENO scheme is a combination of WENO and LxF and a good behaviour of such schemes are demonstrated [9]. Thus we use LxF scheme to construct a Hybrid based on discontinuity recognition.

In following sections, we review the WENO-Z scheme and LxF scheme, then we propose the hybrid scheme. Finally, numerical experiments can demonstrate the robustness of the scheme.

2. Review of WENO and LxF

In this section, we will describe the construction of the two schemes applied on one dimension Euler system:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}}{\partial x}=0 \tag{2.1}$$

Where

$$\boldsymbol{U}=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix},\boldsymbol{F}=\begin{pmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{pmatrix} \tag{2.2}$$

Spatial-discrete: on the cell $\boldsymbol{I}_j=[x_j-\Delta x/2,x_j+\Delta x/2]$, we have:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\boldsymbol{H}_{j+1/2}-\boldsymbol{H}_{j-1/2}}{\Delta x}=0 \tag{2.3}$$

Numerical flux $\boldsymbol{F}_{j+1/2}$ is used for approximating flux $\boldsymbol{H}_{j+1/2}$.

2.1. FVS WENO-Z scheme

On stencils $S=\{\boldsymbol{I}_{j-2},\boldsymbol{I}_{j-1},\boldsymbol{I}_j,\boldsymbol{I}_{j+1},\boldsymbol{I}_{j+2}\},\boldsymbol{F}_{j+1/2}$ of WENO-Z scheme can be obtained by:

$$\boldsymbol{F}_{j+1/2}=\boldsymbol{F}^{+}_{j+1/2}+\boldsymbol{F}^{-}_{j+1/2} \tag{2.4}$$

Hereinafter, only the positive part $\boldsymbol{F}^{+}_{j+1/2}$ is described for brevity, the formulas for the negative part of the split flux are symmetric.

$$\boldsymbol{F}_{j+1/2}^{+} = R_{j+1/2} \sum_{k=0}^{2} \omega_k \overline{\boldsymbol{F}}_k^{+} \tag{2.5}$$

In the right hand term of (2.5), $\overline{\boldsymbol{F}}_k^{+}$ can be obtained on sub stencil $S_k = \{I_{j+k-2}, I_{j+k-1}, I_{j+k}\}$, namely:

$$\overline{\boldsymbol{F}}_k^{+} = \Lambda_{j+1/2}^{+} L_{j+1/2} \sum_{l=0}^{2} \boldsymbol{c}_{k,l} \boldsymbol{U}_{j+k+l-2} \tag{2.6}$$

where $\Lambda_{j+1/2}^{+}, L_{j+1/2}, R_{j+1/2}$ is eigenvalues, left eigenvectors and right eigenvectors of Jacobi matric $A_{j+1/2} = \left(\partial F / \partial x\right)_{j+1/2}$ respectively, which can be obtained by Roe-average.

$$\omega_k = \frac{\alpha_k}{\alpha_0 + \alpha_1 + \alpha_2}, \quad \alpha_k = \frac{C_k}{IS_k}. \tag{2.7}$$

Where C_k is a constant named ideal weight and denominator IS_k :

$$IS_k = \frac{IS_k^{JS} + \grave{o}}{IS_k^{JS} + \grave{o} + \tau_5} \tag{2.8}$$

Where IS_k^{JS} is smoothness indicator of classical WENO-JS [6], the role of $\grave{o}$ is avoid denominator being zero.
And τ_5 in fifth-order WENO-Z scheme:

$$\tau_5 = | IS_2^{JS} - IS_0^{JS} | \tag{2.9}$$

2.2. LxF scheme

$\boldsymbol{F}_{j+1/2}$ of LxF scheme can be directly obtained by:

$$\boldsymbol{F}_{j+1/2}^{LF} = \frac{1}{2}\left(\boldsymbol{F}_j + \boldsymbol{F}_{j+1} - (|u| + c)_{\max j} (\boldsymbol{U}_{j+1} - \boldsymbol{U}_j)\right) \tag{2.10}$$

3. Construction of hybrid scheme

The basic idea of the hybrid is distinguishing discontinuity from flow region and constructing a function θ to construct modified numerical flux:

$$\boldsymbol{F}_{j+1/2}^{m} = \theta\left(\boldsymbol{F}_{j+1/2}^{LF} - \boldsymbol{F}_{j+1/2}\right) + \boldsymbol{F}_{j+1/2} \tag{3.1}$$

Where θ can vary from 0-1, and θ can be a function of measurement of discontinuity.

3.1. A choice for recognizing discontinuity

In this section, we devise a measurement for recognizing discontinuity, which is the variable for function θ.

Generally, smoothness indicators (IS) play an important role in constructing weights of WENO scheme on sub stencils. IS measure the smoothness of the numerical solution on a stencil to form a weight. We define the standard deviation of IS:

$$SDIS = \sum_{k=0}^{2} \left(IS_k - \overline{IS}\right)^2 \tag{3.2}$$

Where

$$\overline{IS} = \frac{\sum_{k=0}^{2} IS_k}{3} \tag{3.3}$$

When the stencil $S=\{I_{j-2},I_{j-1},I_j,I_{j+1},I_{j+2}\}$ is a flat region, namely

$$\boldsymbol{U}_{j-2}=\boldsymbol{U}_{j-1}=\boldsymbol{U}_j=\boldsymbol{U}_{j+1}=\boldsymbol{U}_{j+2} \tag{3.4}$$

We can easily find $SDIS=0$, and when the standard deviation of IS become great, we consider these points as a discontinuity. Though these stencils with huge gradient may not discontinuity, we both know that WENO scheme will degenerate to low order in complex region, so a low accuracy LxF scheme used to replace in these regions is sustainable.

3.2. Modified term limiter θ

Based on discussion above all, we find that $SDIS$ can be an effective indicator of huge gradient. In order to limit the scheme to low order accuracy, we add a term $\theta\left(\boldsymbol{F}_{j+1/2}^{LF}-\boldsymbol{F}_{j+1/2}\right)$ to flux, in which θ is a function of huge gradient measurement, so $SDIS$ can be the variable for function θ, namely $\theta=\theta\left(SDIS\right)$. θ satisfies the following requirements:

- Varying from 0 to 1.
- θ must be positive correlation with $SDIS$.
- Once $SDIS$ ensure a smoothness region, $\boldsymbol{F}_{j+1/2}^{LF}$ cannot influence flux $\boldsymbol{F}_{j+1/2}^{m}$ in case of losing excessive accuracy.

There are two possible choices of $\theta\left(SDIS\right)$ as displayed in figure 1:

$$f_1=1-\frac{1}{1+x} \tag{3.5}$$

$$f_2=1-e^{-x} \tag{3.6}$$

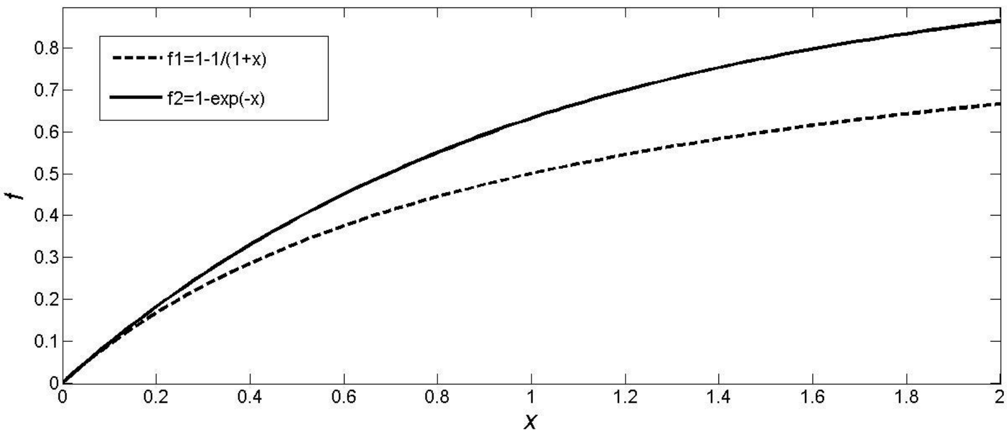

Figure 1. Functions: f_1-x and f_2-x

f_2 shows some better properties comparing to f_1: when huge gradient is recognized, a bigger weight is assigned to modify term $\theta\left(\boldsymbol{F}_{j+1/2}^{LF}-\boldsymbol{F}_{j+1/2}\right)$, which could keep the scheme more robust and a steep gradient of f_2 can easily distinguish smooth region. Above all, we use

$$\theta=1-e^{-SDIS} \tag{3.7}$$

In practical application, we use a modified function for a better robust:

$$\theta^m=\begin{cases}1-e^{-SDIS\times Ac}, when: SDIS\geq threshold\\ \Delta x^5, when: SDIS<threshold\end{cases} \tag{3.8}$$

where Ac is an amplified coefficient for adjust SDIS's effects to θ^m, and when SDIS is smaller than threshold, we consider the stencils are relative smooth, and modified term show little effects on flux. Because $\boldsymbol{F}_{j+1/2}^{WENO} - \boldsymbol{H}_{j+1/2} = O(\Delta x^5)$, we assign θ the same order as Δx^5.

4. Numerical test

We solve Euler system (2.1) as test:
Test 1: Lax problem with initial condition:

$$(\rho, u, p) = \begin{cases} (0.445, 0.698, 3.548), -1 < x < 0 \\ (0.500, 0.000, 0.571), 0 \le x < 1 \end{cases}$$

The numerical solution of density of LxF, FVS WENO and Hybrid schemes are shown in figure 2, figure 3 and figure 4 respectively.

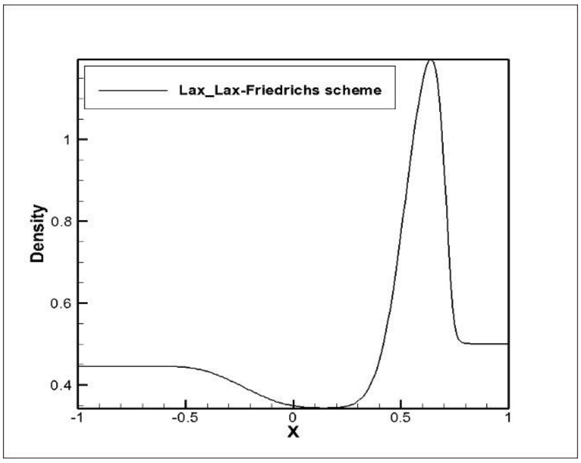

Figure 2. Lax problem: Lax-Freidrichs scheme, One-order Runge-Kutta; CFL=0.1; 200points; T=0.2;

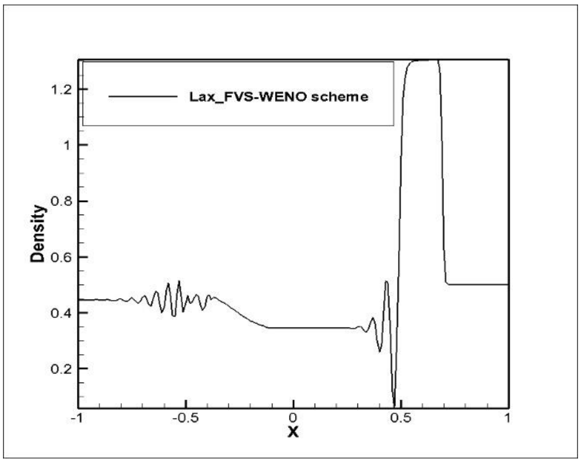

Figure 3. Lax problem: FVS-WENO scheme, One-order Runge-Kutta; CFL=0.1; 200points; T=0.2;

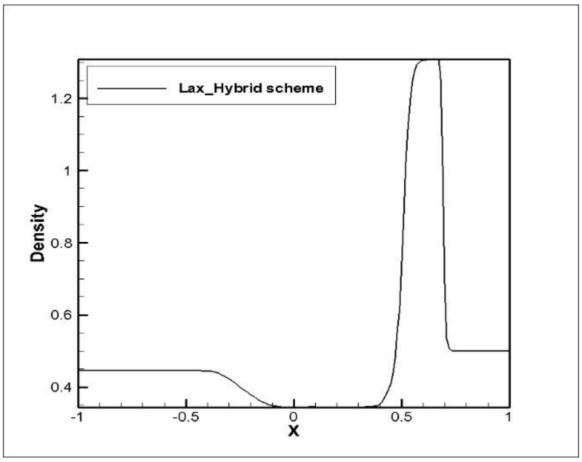

Figure 4. Lax problem: Hybrid-WENO scheme, One-order Runge-Kutta; CFL=0.1; 200points; T=0.2;

As shown in figure 2, figure 3 and figure 4, we can find that LxF scheme does not bring in oscillation but a large dissipation about discontinuity, WENO scheme show a high resolution of the discontinuity but a huge dispersion which would lead negative density that may cause blow up. Hybrid show a middle performance, an acceptable resolution, which is better than LxF scheme, and restrain the oscillation.
Test 2: Sod problem with initial condition:

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$$(\rho, \mathrm{u}, \mathrm{p}) = \begin{cases} (1.000, 0.750, 1.000), -1 < x < 0 \\ (0.125, 0.000, 0.100), 0 \le x < 1 \end{cases}.$$

The numerical solution of density of LxF, FVS WENO and Hybrid schemes are shown in figure 5, figure 6 and figure 7 respectively. Hybrid show the same property of resolution and oscillation restrained as Lax problem but a mild oscillation which means the hybrid scheme still need some correction.

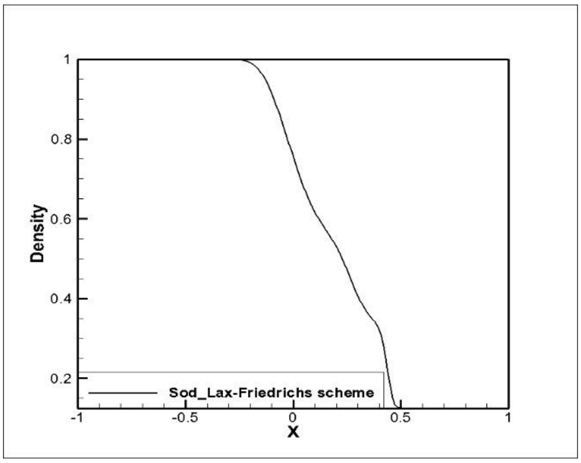

Figure 5. Sod problem: Lax-Freidrichs scheme, One-order Runge-Kutta; CFL=0.1; 200points; T=0.2;

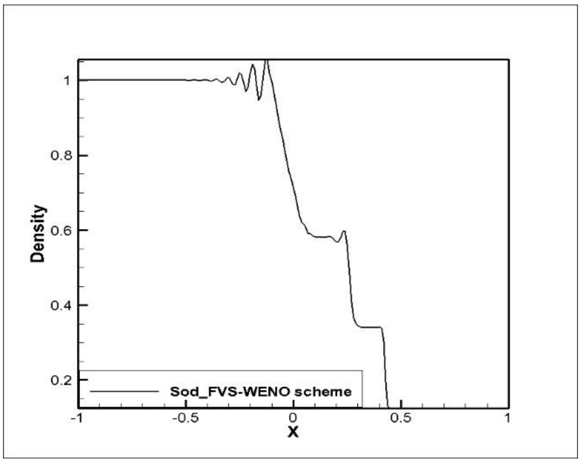

Figure 6. Sod problem: FVS-WENO scheme, One-order Runge-Kutta; CFL=0.1; 200points; T=0.2;

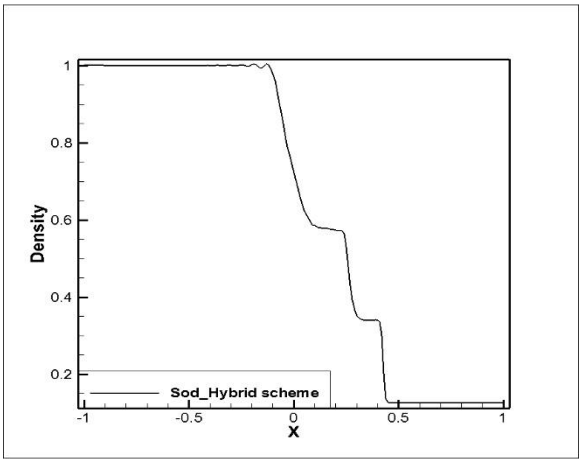

Figure 7. Sod problem: Hybrid-WENO scheme, One-order Runge-Kutta; CFL=0.1; 200points; T=0.2;

5. Conclusion

The property of FVS WENO scheme we have analysed in this paper show high dispersion because of its odd-order truncation error. We construct a hybrid scheme with the help of Lax-Friedrichs to overcome it. A limiter based on smoothness indicator is constructed for connecting WENO scheme with Lax-Friedrichs. Numerical simulations demonstrate that the hybrid show a high resolution and the ability to restrain oscillation. Further study will focus on a higher resolution of the hybrid of WENO and TVD and extend the hybrid scheme to multi-dimensions, more cases will be tested.

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