

# The Soliton Interaction in Weakly Nonlocal Nonlinear Media on the External Potentials.

N A B Aklan<sup>1</sup>, B A Umarov<sup>2</sup>

<sup>1</sup> Department of Computational and Theoretical Science, Kulliyah of Science, International Islamic University Malaysia, 25200 Kuantan, Malaysia

<sup>2</sup> Department of Physics, Kulliyah of Science, International Islamic University Malaysia, 25200 Kuantan, Malaysia

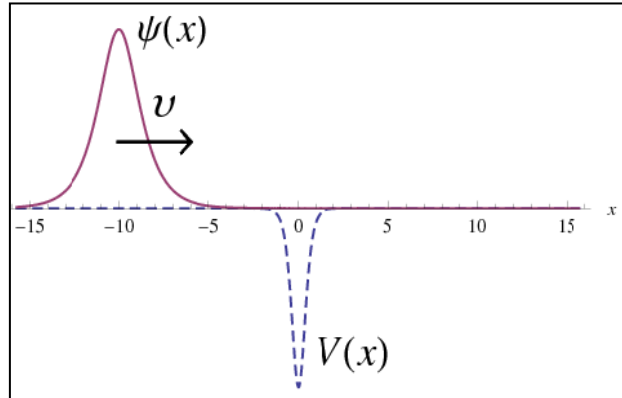
noramirah1987@gmail.com

**Abstract.** This paper is devoted to the analytical and numerical study of the solitons interaction and scattering on the external delta potential in the weakly nonlocal nonlinear media. Using generalized form of Nonlinear Schrödinger Equation (NLSE) which is Cubic-Quintic NLSE in weakly nonlocal nonlinear media, we applied the variational approximation method to derive the equations for soliton parameters evolution during scattering process. Then a direct numerical simulation of NLSE is used to check the validity of approximations, considering the soliton initially located far from potential. Depending on initial velocity of the soliton, the phenomenon of reflection and transmission of the soliton through the potential have been shown. The critical values of the velocity separating these two scenarios have been identified.

## 1. Introduction

The study of soliton interaction with the potential has been wide focus from many researchers regarding analyzing the result of the collision. The Nonlinear Schrodinger Equation (NLSE) describes the propagation of soliton and it is important to note that it is the competition between the effect of wave dispersion and nonlinearities. Soliton is a wave that can propagate in nonlinear media without any change of shape and velocity and displays particle like elastic collision with other solitons[1]. The NLSE is integrable when the particles move in one dimension of space. But in this paper, we are using the GNLSE, in form of Cubic-Quintic NLSE (CQNLSE) with the higher order terms. GNLSE is not integrable, therefore we consider the interaction of localized solitonlike solutions of GNLSE with external potentials. In general, the soliton solutions for GNLSE is not available and in case if they are available, they may have complicated form which is difficult for application in calculations. Thus, the numerical solution for soliton is used as initial condition and then the approximate semi-analytical method is applied. The soliton solutions in an implicit form have been studied in [2] for weakly nonlocal media with cubic nonlinearity. Similar study in nonlocal nonlinear media with cubic-quintic nonlinearity also been discussed in [3] but in consideration for exact analytical solutions in bright solitons and dark solitons.

This work is aimed to understand the conditions when the solitons interacts with the delta potential and study the phenomena of single soliton scattering on the external potential in CQNLSE. The incident of that phenomena is illustrated in figure below [4].



**Figure 1** The sketch of numerical experiments. Single soliton is set in motion towards the external potential  $V(x)$  with some velocity  $v$ . The soliton can be reflected, trapped or transmitted through the potential depending on the potential strength and initial velocity.

Derivation of the two-coupled equation for soliton's width and center-of-mass position will describe the scattering of single soliton after meeting the external potential  $V(x)$ . Reflection and transmission of matter-wave packets through potential in the presence of nonlinearity, when the wave packet's evolution is governed by the NLSE instead of the linear Schrodinger equation, bring a wide focus issue. Therefore, in this very paper we will try to understand these phenomena in the situation of soliton moving towards a potential by approximate analytical and numerical methods.

The paper is organized as follows. We introduce the model and governing equations, and problem statement in Section 2. Section 3 presents the variational approximation method, while Section 4 shows the numerical simulations of the dynamics of single soliton interacting with the localized potential. Finally, we conclude the paper by Section 5.

## 2. The Main Equations

The main equation of for this work is based on GNLSE which is Cubic-Quintic NLSE with weak nonlocality,

$$i\psi_t + \frac{1}{2}\psi_{xx} + \gamma|\psi|^2\psi + \mu|\psi|_{xx}^2\psi + \delta|\psi|^4\psi + V(x)\psi = 0 \quad (1)$$

where  $\psi(x,t)$  is the macroscopic wave function of the condensate,  $V(x)$  is the external potential and consider the signs of  $\gamma$  and  $\mu$  are not related to each other. We consider positive coefficient of nonlinearity for  $\gamma$ ,  $\delta$  and  $\mu$ , which corresponds to attractive interactions between atoms in condensate, of focusing nonlinearity in optics applications so that the system supports bright matter-wave solitons [5]. We study the scattering of a single soliton by delta potential

$$V(x) = U_0\delta(x+3) \quad (2)$$

where  $U_0 = \frac{\pi V_0}{\alpha}$  characterized by the strength  $U_0 > 0$ .

## 3. Variational Approximation Approach

The application of variational approximation approach method [6–8] is important to get the approximate system of ordinary differential equations for soliton parameters. This method is one of the important theoretical methods to analyze the nonintegrable soliton bearing equations. In this paper, by using variational approximation method, we consider the transmission of a single soliton in weakly nonlocal nonlinear media through some external potential. The proper choice of the trial function will lead to the success of the method; hence, we choose as a basis of our trial function a freely propagating

soliton when it is sufficiently far from the scattering potential. We assume the trial function as a Gaussian function with time dependent parameters

$$\psi(x, t) = Ae^{-\frac{(x-\xi)^2}{2a^2} + ib(x-\xi)^2 + iv(x-\xi) + i\varphi} \quad (3)$$

where  $A$ ,  $a$ ,  $\xi$ ,  $b$ ,  $v$ , and  $\varphi$  are the amplitude, width, center of mass position, chirp parameter, velocity and initial phase of the soliton respectively. The Cubic-Quintic NLSE of equation (1) is verified using the Langrangian Density

$$\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 - \frac{\gamma}{2}|\psi|^4 + \frac{\mu}{2}(|\psi|^2)^2 - \frac{\delta}{3}|\psi|^6 - V(x)|\psi|^2 \quad (4)$$

by Euler-Langrange equation.

At first, we reexpress equation (4) using trial function (3), then calculate the effective Langrangian with spatial integration of the Langrangian density  $L = \int_{-\infty}^{\infty} \mathcal{L} dx$ . This function will give full form of the total averaged Langrangian below

$$L = N \left[ \frac{1}{2}a^2b_t - \frac{1}{2}\xi_t^2 + \varphi_t + \frac{1}{4a^2} + a^2b^2 - \frac{U_0}{a\sqrt{\pi}}e^{-\frac{\xi^2}{a^2}} - \frac{\gamma N}{2\sqrt{2\pi}a} - \frac{\mu N}{2\sqrt{2\pi}a^3} - \frac{\delta N^2}{3\sqrt{3\pi}a^2} \right] \quad (5)$$

The norm of the wave function for this equation,  $N = \int_{-\infty}^{\infty} |\psi|^2 dx = \sqrt{\pi}A^2a$  is the conserved quantity and relative to the number of atoms in the condensate region where  $dN/dt = 0$ . Then we calculate for collective coordinate equations for variational parameters from Euler-Langrange equations  $d/dt(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = 0$ , where  $q_i$  is the time dependent collective coordinates  $a$ ,  $\xi$ ,  $b$ , and  $\varphi$ .

$$\begin{aligned} \frac{d}{dt} \cdot \frac{\partial L}{\partial a_t} - \frac{\partial L}{\partial a} &= 0, \\ \frac{d}{dt} \cdot \frac{\partial L}{\partial b_t} - \frac{\partial L}{\partial b} &= 0, \\ \frac{d}{dt} \cdot \frac{\partial L}{\partial \xi_t} - \frac{\partial L}{\partial \xi} &= 0, \end{aligned} \quad (6)$$

Through equation (6) we got,

$$b_t = \frac{1}{2a^4} - 2b^2 + \frac{U_0}{\sqrt{\pi}}e^{-\frac{\xi^2}{a^2}} \left( \frac{2\xi^2}{a^5} - \frac{1}{a^3} \right) - \frac{\gamma N}{2\sqrt{2\pi}a^3} - \frac{3\mu N}{2\sqrt{2\pi}a^5} - \frac{2\delta N^2}{3\sqrt{3\pi}a^4} \quad (7)$$

$$a_t = 2ab, \quad (8)$$

$$-\xi_t - \frac{2\xi U_0}{a^3\sqrt{\pi}}e^{-\frac{\xi^2}{a^2}} = 0 \quad (9)$$

and the main result of this work, which describes the soliton scattering by external potential  $V(x)$  are given below

$$a_{tt} = \frac{1}{a^3} + \frac{2U_0}{\sqrt{\pi}}e^{-\frac{\xi^2}{a^2}} \left( \frac{2\xi^2}{a^4} - \frac{1}{a^2} \right) - \frac{\gamma N}{\sqrt{2\pi}a^2} - \frac{3\mu N}{\sqrt{2\pi}a^4} - \frac{4\delta N^2}{3\sqrt{3\pi}a^3} \quad (10)$$

$$\xi_{tt} = -\frac{2\xi U_0}{a^3\sqrt{\pi}}e^{-\frac{\xi^2}{a^2}} \quad (11)$$

which are the two coupled equations for width and center-of-mass position. When  $U_0 = 0$ , where the external potential is absent and equation (10) and equation (11) are decoupled, we can find from equation (10) the approximate width of the stationary ( $a_{tt} = 0$ ) soliton solution of Cubic-Quintic NLSE

$$a_s = \sqrt{\frac{3\sqrt{3}(\sqrt{2} - 3A^2\mu)}{A^2(3\sqrt{3}\gamma + 4\sqrt{2}A^2\delta)}} \quad (12)$$

Perturbations around this stationary point may generate very small oscillations for the width. The velocity  $\xi_t$  in this case is the constant free parameter. In the presence of delta potential, the evolution of its width couples with the time evolution of position of the center of the soliton. The soliton's parameters are constant when the soliton is located far from inhomogeneity. Some qualitative results about soliton's evolution can be obtained if we neglect the effect of potential to the width of soliton. Then equation (11) describes the scattering of effective classical particle from the localized barrier.

$$\xi_{tt} = -\frac{2\xi U_0}{\sqrt{\pi}a^3} e^{-\frac{\xi^2}{a^2}} = \frac{dV_p(\xi)}{d\xi} \quad (13)$$

$$\xi_t = \sqrt{2V_p(\xi)} \quad (14)$$

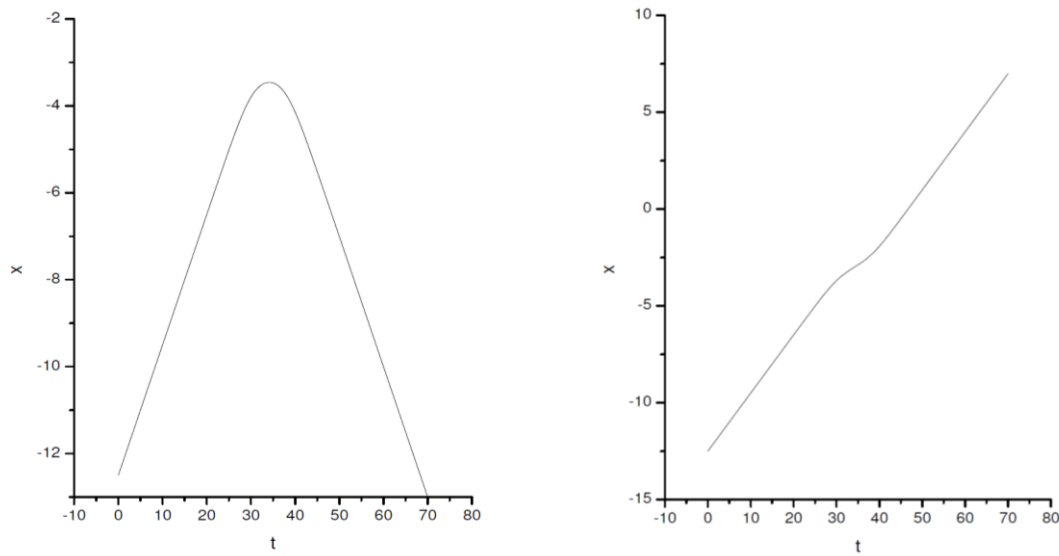
The equation (13) must be integrated for second time and reduced to equation (14) where  $V_p(\xi) = (U_0 / \sqrt{\pi}a) e^{-\xi^2/a^2} (1 + a^2 / \xi^2 (2 - a^2))$  is the effective potential representing the influence of the original localized delta potential to the solitons velocity. From equation (14), it illustrates the results of the particle after the collision. It can be either transmitted or reflected from the potential dependent on whether the velocity above or below of critical value  $V_c = 2(U_0 / \sqrt{\pi}a)$ .

#### 4. Numerical Simulations

In this section, the accuracy of the approximations was checked by direct numerical simulations of CQNLSE with soliton initially located far from potential since the variational approximation method gave the approximate results and is based on some assumptions. We are analyzing the approximation of equation (10) and equation (11) and compare with numerical solutions of the original equation (1). To find the soliton solution far from the potential we will use the Newton-Raphson and imaginary time numerical methods [7]. The accuracy of the numerical methods will be checked by repeating the calculations and by changing the step size and increasing the time of calculation until the results converge.

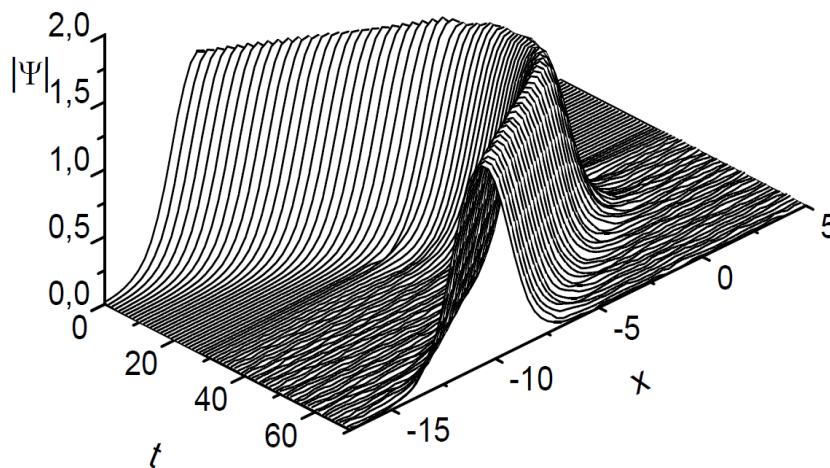
Soliton  $\psi(x)$ , which is the exact solution of equation (1) when  $V(x) = 0$ , is set in motion with some velocity  $v$  towards the potential barrier  $V(x)$  initially located far from the potential. According to the results attained from the numerical solutions of the ordinary differential equation (10) and equation (11), it shows the approximate time evolution of the width and position of the center of the soliton. [4,9,10] also show comparable results of soliton interaction in NLSE.

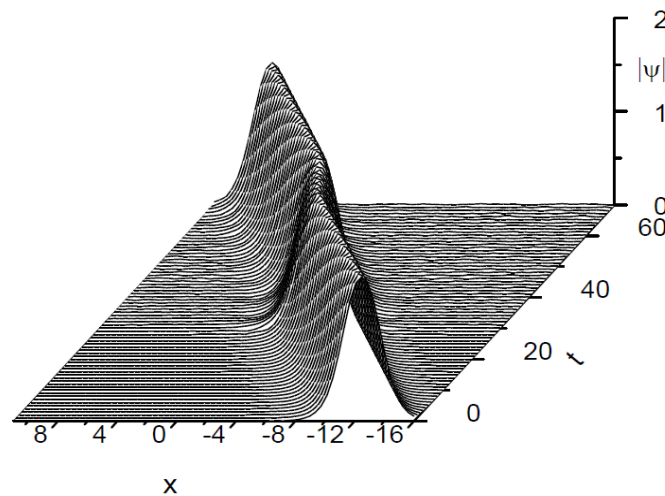
The parameters are set with constant coefficient for  $\gamma = 1$ ,  $\mu = -0.2$  and  $\delta = -0.1$  and also with initial width  $a(0) = a_s$  and  $A = 1.477$ , it is found that the soliton behaves like classical particles moves freely along with the constant width parameter of soliton. Soliton may be transmitted, trapped or reflected from the potential depending on the amplitude of the potential [11]. When it comes to the potential, both velocity and width will be affected by perturbation. The numerical solution results of variational equation (10) and equation (11) are presented in Figure 2 below



**Figure 2** Scattering of a soliton on the delta potential barrier  $V(x) \approx U_0 \delta(x+3)$  according to ODE systems for equation (10) and equation (11). Soliton is reflected (left panel) when  $U_0 = 0.11$  and is transmitted (right panel) when  $U_0 = 0.07$ .

According to the analysis based on ODE above, the soliton is either transmitted or reflected from the potential well depending upon the initial velocity of soliton. The examples of results of numerical solutions for equation (1) on the same parameters with  $A = 1.477$  is shown in the Figure 3 which had confirmed these conclusions.





**Figure 3** Scattering of the soliton on potential barrier according to CQNLSE (1). Above panel shows interaction of soliton with potential barrier when  $U_0 = 0.11$  and below panel with  $U_0 = 0.07$ .

## 5. Conclusions

The scattering of single soliton of CQNLSE in the presence of external delta potential have been studied. We noticed the existence of transmission and reflection of the soliton by the potential barrier. The variational approximation method is used to define the result variational equations and is compared with the direct numerical solution for accurate solutions.

We proposed to extend the research on soliton interactions with localized potential walls and wells for different shapes, and also other forms of generalized NLSE and expecting more interesting results. The results will be guidelines for possible future experiments with matter-wave solitons and optical solitons and their practical applications.

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