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Scattering of a flat top solitons of cubic - quintic nonlinear Shrödinger equation by a linear delta potential

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Abstract. The flat top soliton is a localized solution of cubic - quintic Nonlinear Shrödinger equation (C-Q NLSE) that can propagate preserving its shape. In this work we consider the interaction of soliton with weak external localized potentials. It is shown that depending on an initial velocity the soliton may be reflected or transmitted by potential. The approximate analytical results based on varational approach qualitatively confirmed by numerical simulations.

1. Introduction

The dynamics of soliton solutions of nonlinear partial differential equations attract attention of researches working in different fields of mathematics and physics. The important class of nonlinear equations consist of the nonlinear Schrödinger equation (NLSE) and its generalizations, which have a numerous applications in optics, condensed matter physics, Bose - Einstein condensates, biophysics and so on [1, 2, 3]. Different types of soliton solutions, such as bright solitons, dark solitons, gap solitons, soliton complexes have been found and their properties have been studied in details [1, 2, 3, 5]. It is well known that solitons are localized wave solutions that can propagate preserving their shape, and these properties makes them considerable in technological applications as a carriers of information. In this regard it is essential to study a stability and dynamics of solitons under the action of different type of unavoidable perturbations. Also the controlled management of soliton dynamics by application of external fields become a fascinating field of research [4]. One of the approaches to manage a solitons is a scattering on a localized potential barriers and wells [6, 7, 8, 9]. It was revealed that solitons of one dimensional NLSE behave like classical particles with internal structure in the process of scattering on the low energy barriers, namely they could be reflected or transmitted depending on the initial velocity of soliton, and also the shape oscillations can be excited. This phenomena can be theoretically justified by application of perturbative or variational methods [1, 4]. Soliton interacting with high energy barriers or with potential wells may show a nonclassical, wavelike actions. In particular there were evidences that the tunneling through the barrier or quantum reflection and capture on the potential well may happen. In this paper we concentrate on a one type of bright soliton of generalised NLSE, namely the flat top soliton solution of the one dimensinal

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cubic - quintic NLSE (C-Q NLSE) [12, 10]. The C-Q NLSE contains additional term taking into account the higher order, quintic nonlinearity. It has applications in nonlinear optics and Bose Einstein condensate theory. Our aim is to study the scattering of flat top soliton on the narrow potential barrier and well, which can be modeled by delta function. We apply the variational optimization method with super-Gaussian trial function, also the results of numerical experiments will be reported and compared with analytical predictions.

2. The main equations and variational approximation

The main equation we are considering is the one dimensional C-Q NLSE with external potential, and in dimensionless units it can be written in the following form [10]

$$i\psi_t = -\frac{1}{2}\psi_{xx} + U(x)\psi - \alpha|\psi|^2\psi + \beta|\psi|^4\psi. \tag{1}$$

Here $\psi(x,t)$ is the nonlinear wave field. In Bose-Einstein condensate theory it is a dimensionless mean field wave function of the condensate where two and three body interactions are taken into account, and in nonlinear optics it represents the slowly varying envelope of electric field for light propagating in optical media with cubic and quintic nonlinearity. $U(x) = U_0\delta(x)$ is the localized external delta potential and further we consider both the potential barrier ($U_0 > 0$) and well ($U_0 < 0$) , α and β are, respectively, the coefficients of cubic and quintic nonlinearity. The exact solution of C-Q NLSE (1)with $U(x) = 0$, in the case of self focusing cubic ($\alpha > 0$) and defocusing quintic ($\beta > 0$) nonlinearity takes the following form [10]

$$\psi(x,t) = \sqrt{\frac{3\alpha}{4\beta}}\,\frac{\tanh(\eta)\ \exp[i(qx-\mu t)]}{\sqrt{1+\mathrm{sech}(\eta)\cosh(x/a)}}, \quad \eta \equiv \sqrt{\frac{2\beta}{3}}, \quad a \equiv \frac{1}{\alpha}\frac{\eta}{\tanh(\eta)}, \tag{2}$$

where q and μ stand for the wave vector and chemical potential, and the normalization condition is taken into account,

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\, dx = 1. \tag{3}$$

The equation (1) does not have exact soliton solution, but if initially we locate soliton far from localized potential, it will be very close to a exact solution. The scattering process can be considered as an interaction of moving soliton with potential located at $x = 0$, while asymptotically when $t = 0$ we have free soliton moving to the right with the constant velocity v, located far left from potential. The outcome of scattering in general could be quite complicated, the soliton may lose the energy by radiating the small amplitude waves and its shape may be drastically changed, and eventually soliton may be destroyed if the interaction is strong enough. In this work we are going to consider the analytically tractable case of scattering when soliton preserves the identity after interaction with potential, but its parameters may become time dependent because of perturbations. One of the fruitful approaches to study the evolution of parameters of soliton is the variational approximation (VA). A progress in application of VA is strongly determined by the choice of trial function for optimization procedure. It should be simple to proceed with analytic calculations and the same time its shape should be close to the solitons form. For flat top soliton this objective can be achieved by the use of super- Gaussian trial function [10].

$$\psi(x,t) = A(t)\exp\left[-\frac{1}{2}\left(\frac{x-\xi(t)}{a(t)}\right)^{2m} + ib(t)(x-\xi(t))^2 + iv(t)(x-\xi(t)) + i\varphi(t)\right], \tag{4}$$

$A(t), a(t), b(t), \xi(t), v(t)$ and $\varphi(t)$ are the time dependent variational parameters, representing the amplitude, width, chirp, center of mass position, velocity and phase of the flat-top soliton,

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respectively. The parameter m is the super Gaussian index of soliton, and following the paper [10], we assume in the further calculation that it is constant. The normalization condition (3) with the trial function (4) give the following relation between A and a, $2A^2a\Gamma(1+M)=1,\ M=1/2m$, also by definition $v=\xi_t$. Here $\Gamma(x)$ is the gamma function.

The governing equation (1) can be derived from the following Lagrangian density

$$\mathcal{L}=\frac{i}{2}(\psi\psi_t^*-\psi^*\psi_t)+\frac{1}{2}|\psi_x|^2-\frac{\alpha}{2}|\psi|^4+\frac{\beta}{3}|\psi(x,t)|^6+U(x)|\psi|^2. \tag{5}$$

As a next step we calculate the averaged Lagrangian $L=\int\mathcal{L}dx$ using the ansatz (4) and the Lagrangian density (5).

$$\begin{aligned} L &= (a^2b_t+2a^2b^2)\frac{\Gamma(1+3M)}{3\Gamma(1+M)}+\frac{\Gamma(2-M)}{8M\Gamma(1+M)\,a^2}+\varphi_t-\frac{\alpha}{2^{M+2}\Gamma(1+M)\,a} \\ &+\frac{\beta}{4\cdot 3^{M+1}\Gamma^2(1+M)\,a^2}-\frac{{\xi_t}^2}{2}-\frac{U_0e^{-(\xi/a)^{(1/M)}}}{2a\Gamma(1+M)}, \end{aligned} \tag{6}$$

The evolution in time of variational parameters are governed by the Euler-Lagrange equations $d/dt(\partial L/\partial\nu_t)-\partial L/\partial\nu=0$ where $\nu\ \rightarrow\ a,\ b,\ \xi,\ \phi$. By application of Euler - Lagrange equations to L one gets the following coupled differential equations for the $a(t)$ and $\xi(t)$

$$\begin{aligned} a_{tt} &= \frac{3\Gamma(2-M)}{4a^3M\,\Gamma(1+3M)}+\frac{\beta}{2a^3\cdot 3^M\,\Gamma(1+M)\Gamma(1+3M)}- \\ &\frac{\alpha}{a^2 2^{M+2}\,\Gamma(1+3M)}+\frac{3U_0e^{-(\xi/a)^{(1/M)}}((1/M)(\xi/a)^{(1/M)}-1)}{2a\Gamma(1+3M)}, \end{aligned} \tag{7}$$

$$\xi_{tt}=-\frac{U_0e^{-(\xi/a)^{(1/2M)}}(\xi/a)^{((1/M)-1)}}{2a^2M\Gamma(1+M)}. \tag{8}$$

When the flat top soliton is located far from potential $|\xi|>>a$, it is clear from the equations the system (7)-(8) is decoupled and one has a soliton moving freely with initial velocity and fixed at initial moment equilibrium width a_0. The value of a_0 can be determined from equation

$$a_{tt}=\frac{3\Gamma(2-M)}{4a^3M\,\Gamma(1+3M)}+\frac{\beta}{2a^3\cdot 3^M\,\Gamma(1+M)\Gamma(1+3M)}-\frac{\alpha}{a^2 2^{M+2}\,\Gamma(1+3M)}, \tag{9}$$

This equation is analogous to the equation of motion of a unit mass particle in anharmonic potential $a_{tt}=-\partial U(a)/\partial a$, and a_0 corresponds to the point where potential $U(a)$ reaches its minimum. When soliton come close to the potential barrier, the system becomes coupled, and one should consider full system (7)-(8).The flat top soliton can be transmitted or reflected, also the oscillations of soliton shape can be excited. The examples of the numerical solution of variational equations (VE)(7)-(8) are shown in the Fig.(1). Here and in all our numerical simulations the initial parameter values of stationary flat top soliton are $a_0=2.78481$, $M=0.11146$, and the coefficients are $\alpha=100$, $\beta=400$ [10]. Initial position is $\xi(0)=-10$. Variational analysis predicts also that the oscillations of a width of soliton $a(t)$ can be excited as a result of interaction with localized potentials.

In the Fig.(2) the example of solution of VE(7)-(8) for the case of flat top soliton scattering on the potential well shown. In this case variational analysis predicts that the flat top soliton is always transmitted with some advancement of position with respect to free motion [7].

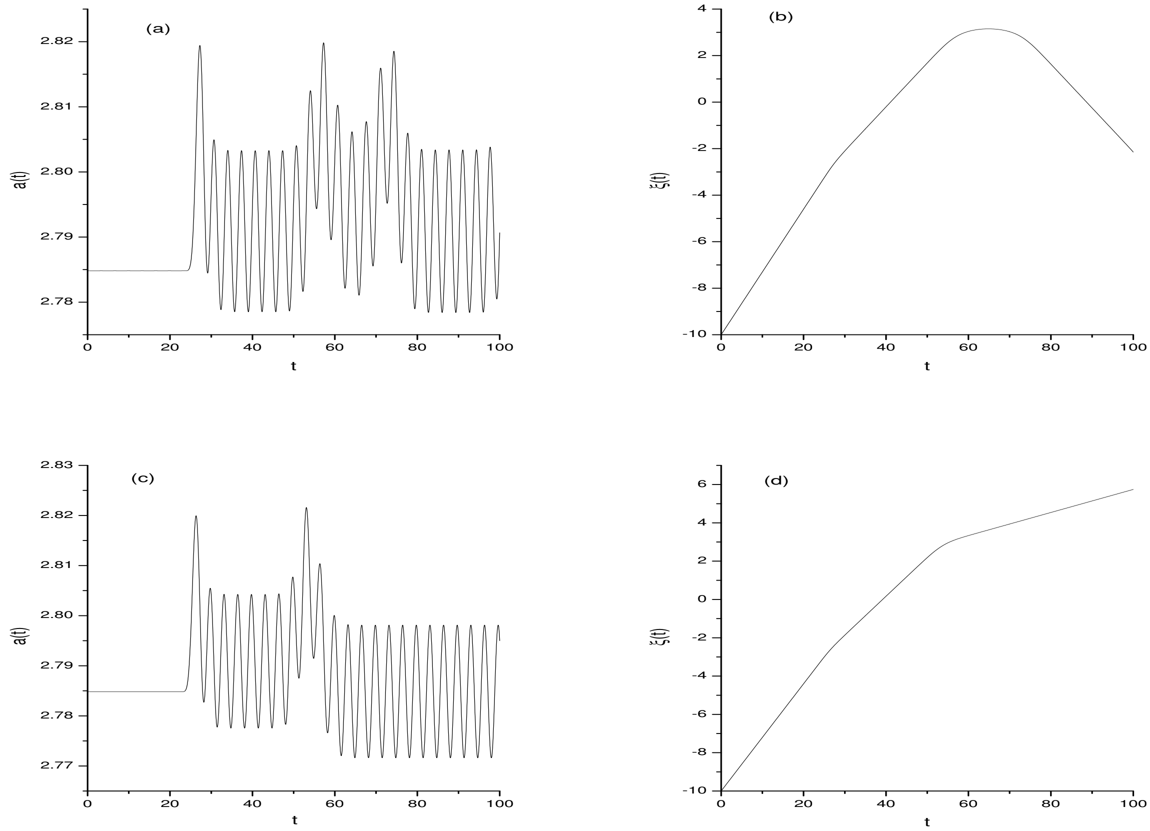
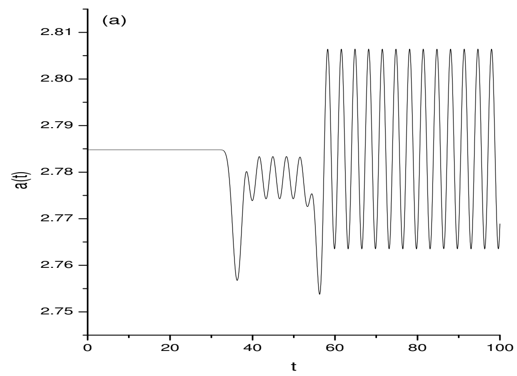


Figure 1. Scattering of a flat top soliton by weak potential barrier with $U_0 = 0.1$ at different velocities according to VE. Reflection at low velocity $v = 0.27$ (a) and (b) for width $a(t)$ and position $\xi(t)$ respectively as a function of time. Transmission at greater velocity $v = 0.28$ (c) and (d) again for $a(t)$ and $\xi(t)$ respectively.

3. Numerical simulations of flat top soliton scattering

In this section the results of numerical simulations of flat top soliton dynamics governed by C-Q NLSE (1) are presented and discussed. As a initial condition we have the flat top soliton with the same parameters as we have used for solution of VE, and the coefficients in C-Q NLSE are also the same [10]. Let us start with the case of weak potential barriers. As it follows from the analysis based on the varational equations, the soliton may be either transmitted or reflected depending upon the initial velocity of soliton. These conclusions are confirmed by the direct numerical solution of C-Q NLSE. In the Fig.(3) we present the examples of numerical solution of the C-Q NLSE . The comparison shows that the soliton behaves like particle with internal structure, and really it is transmitted or reflected, so the qualitative results of approximate analysis are supported by direct numerical simulations of governing equation. Nevertheless there are some quantitative discrepancies, for example the critical value of initial velocity v_{cr} at the point of transition between the transmission and reflection is $v_{cr} = 0.28$ from solution of variational equations and $v_{cr} = 0.31$ from solution of C-Q NLSE. It may be connected with the oversimplified trial function (4), wich does not take into account some properties of flat top soliton interaction with delta potential. The case of weak potential well is presented in the figure (4). It follows from the averaged equations and confirmed be direct solution of C-Q NLSE, that



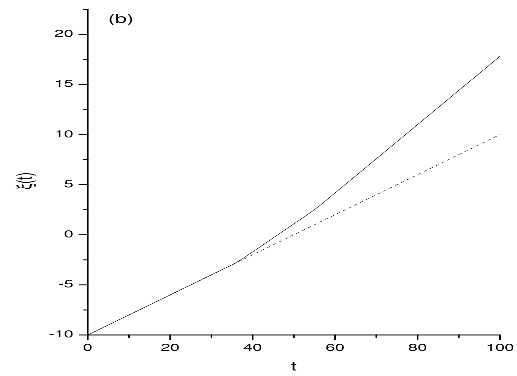
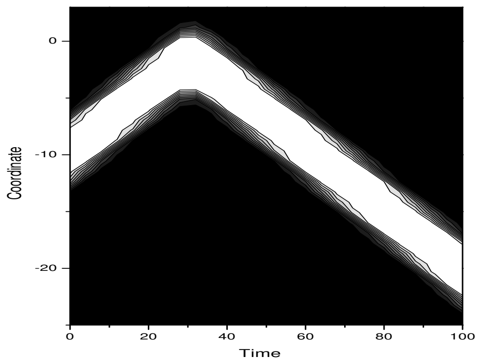


Figure 2. Scattering of a flat top soliton by weak potential well with $V_0 = -0.1$ at initial velocity$v = 0.2$ according to VE. (a) for evolution of width $a(t)$. (b) evolution of position $\xi(t)$ the solid line represents the case of interaction with potential and the dotted line the free motion with constant velocity $v(0)$.

flat top soliton with enough initial velocity is transmitted in the scattering process, with the acquired advancement in the position after scattering.



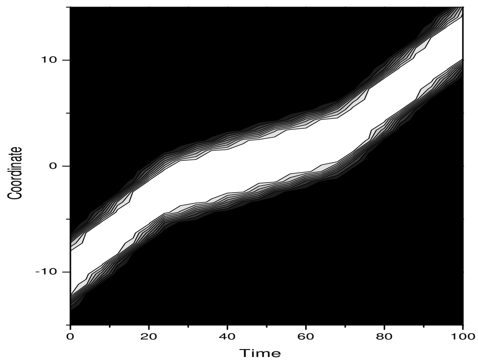


Figure 3. Scattering of flat top soliton on the potential barrier $U_0 = 0.1$ according to C-Q NLSE (1) . Left panel $v(0) = 0.28$, right panel $v(0) = 0.32$

In the case of strong potential barriers and wells the VE (4) based on the quasiparticle description are not able to reflect adequately the evolution of flat top soliton. In the Fig (5 we show the results of solution of C-Q NLSE. The left panel shows the example of scattering of soliton on strong potential barrier. The soliton in this case is splited up to two parts and it is partially reflected back and partially transmitted through the barrier. The right panel shows the quantum reflection of the flat top soliton from the potential well.

4. Conclusions

The scattering of a flat top soliton by delta potential barriers and wells, described by the C-Q NLSE (1), has been studied by variational approximation and numerical simulations. In the case

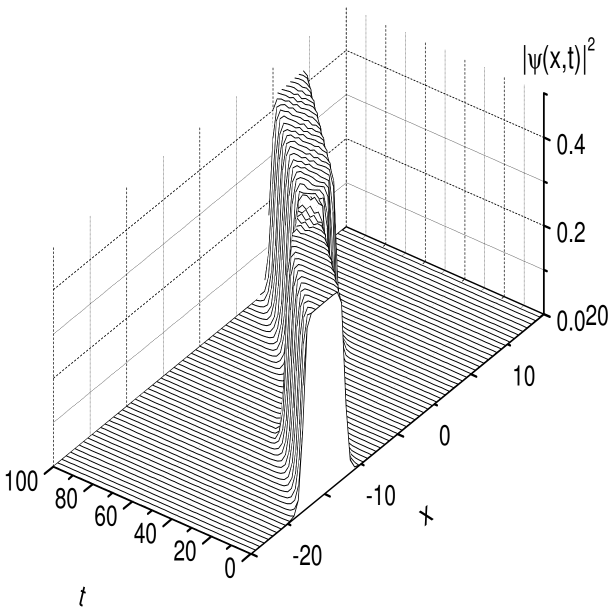


Figure 4. Scattering of flat top soliton on the potential well $U_0 = -0.1$ according to C-Q NLSE (1), $v(0) = 0.2$.

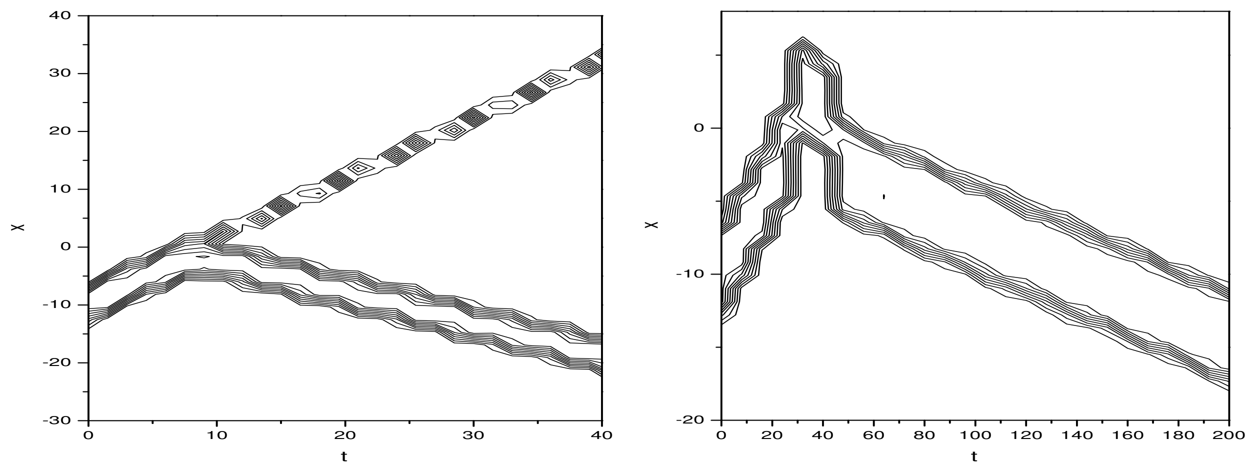


Figure 5. Left panel: Interaction of flat top soliton with potential barrier $U_0 = 1.1$, $v(0) = 1$. Right panel: Interaction of flat top soliton with potential well $U_0 = 0.35$, $V(0) = 0.3$

of week potential variational analysis with super Gaussian trial function may provide qualitative description of scattering process. Namely the soliton can be reflected from potential barrier or transmitted through it. The potential well cause only some advance in the position of soliton in comparison with free motion. In the case of strong potentials one have to rely on the numerical solution of C-Q NLSE (1). We give the examples of soliton splitting on the potential barriers and quantum reflection on the potential wells.

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