

On Ground States of λ -Model on the Cayley Tree of order two

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Abstract. In the paper, we consider the λ -model with spin values $\{1, 2, 3\}$ on the Cayley tree of order two. We describe ground states of the model.

1. Introduction

The choice of Hamiltonian for concrete systems of interacting particles represents an important problem of equilibrium statistical mechanics [1]. The matter is that, in considering concrete real systems with many (in abstraction, infinitely many) degrees of freedom, it is impossible to account for all properties of such a system without exceptions. The main problem consists in accounting only for the most important features of the system, consciously removing the other particularities. On the other hand, the main purpose of equilibrium statistical mechanics consists in describing all limit Gibbs distributions corresponding to a given Hamiltonian [5]. This problem is completely solved only in some comparatively simple cases. In particular, if there are only binary interactions in the system then problem of describing the limit Gibbs distributions simplifies. One of the important models in statistical mechanics is Potts models. These models describe a special and easily defined class of statistical mechanics models. Nevertheless, they are richly structured enough to illustrate almost every conceivable nuance of the subject. In particular, they are at the center of the most recent explosion of interest generated by the confluence of conformal field theory, percolation theory, knot theory, quantum groups and integrable systems [6, 10]. The Potts model [14] was introduced as a generalization of the Ising model to more than two components. At present the Potts model encompasses a number of problems in statistical physics (see, e.g. [18]). Investigations of phase transitions of spin models on hierarchical lattices showed that they make the exact calculation of various physical quantities [2, 11, 12]. Such studies on the hierarchical lattices begun with development of the Migdal-Kadanoff renormalization group method where the lattices emerged as approximants of the ordinary crystal ones. In [13] the phase diagrams of the q -state Potts models on the Bethe lattices were studied and the pure phases of the the ferromagnetic Potts model were found. In [3, 4, 9] using those results, uncountable number of the pure phase of the 3-state Potts model were constructed. These investigations were based on a measure-theoretic approach developed in [15, 11]. The structure of the Gibbs measures of the Potts models has been investigated in [3].



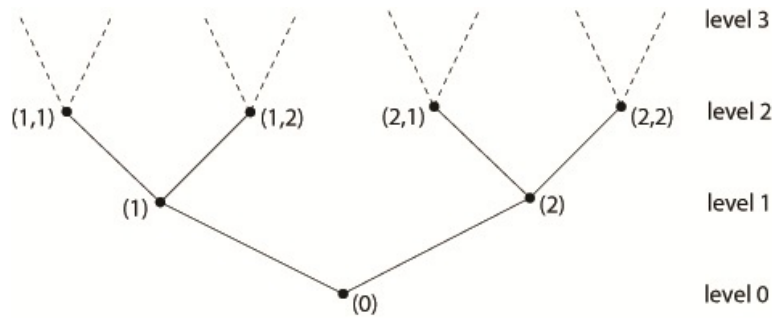


Figure 1. The first levels of Γ_+^2

It is natural to consider a model which is more complicated than Potts one, in [7] we proposed to study so-called λ -model on the Cayley tree (see also [16, 17]). In the mentioned paper, for special kind of λ -model, its disordered phase is studied (see [8]) and some its algebraic properties are investigated. In the present, we consider symmetric λ -model with spin values $\{1, 2, 3\}$ on the Cayley tree of order two. This model is much more general than Potts model, and exhibits interesting structure of ground states. We describe ground states of the model which will open new insight to the structure of the Gibbs measures of the model. Moreover, our results will allow to find Gibbs measures by means of counter methods for the λ -model on the Cayley tree (see [17] for more details).

2. Preliminaries

Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k + 1$ edges, except for the root x^0 , which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y . The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y .

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^n W_m, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

The set of direct successors of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n.$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k + 1$.

Now we are going to introduce a coordinate structure in Γ_+^k . Every vertex x (except for x^0) of Γ_+^k has coordinates (i_1, \dots, i_n) , here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0) (see Figure 1). Namely, the symbol (0) constitutes level 0 and the sites i_1, \dots, i_n form level n of the lattice. In this notation for $x \in \Gamma_+^k$, $x = \{i_1, \dots, i_n\}$ we have $S(x) = \{(x, i) : 1 \leq i \leq k\}$, here (x, i) means that (i_1, \dots, i_n, i) .

Let us define on Γ_+^k a binary operation $\circ : \Gamma_+^k \times \Gamma_+^k \rightarrow \Gamma_+^k$ as follows, for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m)$$

and

$$y \circ x = (j_1, \dots, j_m) \circ (i_1, \dots, i_n) = (j_1, \dots, j_m, i_1, \dots, i_n).$$

By means of the defined operation Γ_+^k becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations $\tau_g : \Gamma_+^k \rightarrow \Gamma_+^k$, $g \in \Gamma_k$ by

$$\tau_g(x) = g \circ x.$$

Let $G \subset \Gamma_+^k$ be a sub-semigroup of Γ_+^k and $h : V \rightarrow \mathbb{R}$ be a function. We say that h is a G -periodic if $h(\tau_g(x)) = h(x)$ for all $x \in V, g \in G$ and $l \in L$. Any Γ_+^k -periodic function is called *translation-invariant*. Put

$$G_m = \left\{ x \in \Gamma_+^k : d(x, x^0) \equiv 0 \pmod{m} \right\}, \quad m \geq 2.$$

One can check that G_m is a sub-semigroup with a unit.

In this paper, we consider the models where the spin takes values in the set $\Phi = \{1, 2, \dots, q\}$ and is assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^{\Gamma^k}$. The Hamiltonian the λ -model has the following form

$$H(\sigma) = \sum_{\langle x, y \rangle \in L} \lambda(\sigma(x), \sigma(y)) \quad (1)$$

where the sum is taken over all pairs of nearest-neighbor vertices $\langle x, y \rangle$, $\sigma \in \Omega$. From a physical point of view the interactions between particles do not depend on their locations, therefore from now on we will assume that λ is a symmetric function, i.e. $\lambda(u, v) = \lambda(v, u)$ for all $u, v \in \mathbb{R}$.

We note that λ -model of this type can be considered as generalization of the Potts model. The Potts model corresponds to the choice $\lambda(x, y) = -J\delta_{xy}$, where $x, y, J \in \mathbb{R}$.

In what follows, we restrict ourself to the case $k = 2$ and $\Phi = \{1, 2, 3\}$, and for the sake of simplicity, we consider the following function:

$$\lambda(i, j) = \begin{cases} \bar{a} & , if \quad |i - j| = 2, \\ \bar{b} & , if \quad |i - j| = 1, \\ \bar{c} & , if \quad i = j, \end{cases} \quad (2)$$

where $\bar{a}, \bar{b}, \bar{c} \in \mathbb{R}$ for some given numbers.

Remark 2.1. We point out the considered model is more general then well-known Potts model [18], since if $\bar{a} = \bar{b} = 0, \bar{c} \neq 0$, then this model reduces to the mentioned model.

3. Ground States

In this section, we describe ground state of the λ -model on a Cayley tree. For a pair of configurations σ and φ coinciding almost everywhere, i.e., everywhere except finitely many points, we consider the relative Hamiltonian $H(\sigma, \varphi)$ determining the energy differences of the configurations σ and φ :

$$H(\sigma, \varphi) = \sum_{\substack{\langle x, y \rangle \\ x, y \in V}} (\lambda(\sigma(x), \sigma(y)) - \lambda(\varphi(x), \varphi(y))) \quad (3)$$

For each $x \in V$, the set $\{x, S(x)\}$ is called a *ball*, and it is denoted by b_x . The set of all balls we denote by M .

We define the energy of the configuration σ_b on b as follows

$$U(\sigma_b) = \frac{1}{2} \sum_{\substack{\langle x, y \rangle \\ x, y \in V}} (\lambda(\sigma(x), \sigma(y)))$$

From (3), we got the following lemma.

Lemma 3.1. *The relative Hamiltonian (3) has the form*

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)).$$

Lemma 3.2. *The inclusion*

$$U(\varphi_b) \in \left\{ \frac{\alpha + \beta}{2} : \forall \alpha, \beta \in \{\bar{a}, \bar{b}, \bar{c}\} \right\} \quad (4)$$

holds for every configuration φ_b on b ($b \in M$).

A configuration φ is called a *ground state* of the relative Hamiltonian H if

$$U(\varphi_b) = \min \left\{ \frac{\alpha + \beta}{2} : \forall \alpha, \beta \in \{\bar{a}, \bar{b}, \bar{c}\} \right\} \quad (5)$$

for any $b \in M$

For any configuration σ_b , we have

$$U(\sigma_b) \in \{U_1, U_2, U_3, U_4, U_5, U_6\},$$

where

$$\begin{aligned} U_1 &= \bar{a}, U_2 = (\bar{a} + \bar{b})/2, U_3 = (\bar{a} + \bar{c})/2, \\ U_4 &= \bar{b}, U_5 = (\bar{b} + \bar{c})/2, U_6 = \bar{c}. \end{aligned} \quad (6)$$

We denote

$$A_m = \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid U_m = \min_{1 \leq k \leq 6} \{U_k\}\} \quad (7)$$

Using (7), we obtain

$$\begin{aligned} A_1 &= \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid \bar{a} \leq \bar{b}, \bar{a} \leq \bar{c}\}, & A_2 &= \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid \bar{a} = \bar{b} \leq \bar{c}\}, \\ A_3 &= \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid \bar{a} = \bar{c} \leq \bar{b}\}, & A_4 &= \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid \bar{b} \leq \bar{a}, \bar{b} \leq \bar{c}\}, \\ A_5 &= \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid \bar{b} = \bar{c} \leq \bar{a}\}, & A_6 &= \{(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \mid \bar{c} \leq \bar{a}, \bar{c} \leq \bar{b}\}. \end{aligned}$$

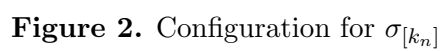
Now, we want to find ground states for each considered cases. To do so, we introduce some notation. For each sequence $\{k_0, k_1, \dots, k_n, \dots\}$, $k_n \in \{1, 2, 3\}$, $n \in \mathbb{N} \cup \{0\}$, we define a configuration σ on Ω by

$$\sigma(x) = k_\ell, \quad \text{if } x \in W_\ell, \ell \geq 0.$$

This configuration is denoted by $\sigma_{[k_n]}$.

If the sequence $\{k_0, k_1, \dots, k_n, \dots\}$ is n -periodic, (i.e. $k_{\ell+n} = k_\ell, \forall n \in \mathbb{N}$), then instead of $\{k_0, k_1, \dots, k_n, \dots\}$, we write $\{k_0, k_1, \dots, k_{n-1}\}$. Correspondingly, the associated configuration is denoted by $\sigma_{[k_0, k_1, \dots, k_{n-1}]}$

Theorem 3.3. *Let $(\bar{a}, \bar{b}, \bar{c}) \in A_1$, then there are only two G_2 -periodic ground states.*



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Now using Figure 3, for each $n \in \mathbb{N}$, one can construct configurations on Ω defined by:

$$\sigma_1^{(2)} = \sigma_{[1,3]}, \quad \sigma_2^{(2)} = \sigma_{[3,1]}.$$

Then, we can see that for any $b \in M$, one has

$$U(\sigma_{b_{1,2}}^{(2)}) = \min_{1 \leq k \leq 6} \{U_k\}$$

which means $\sigma_{1,2}^{(2)}$ is a ground state. Moreover, $\sigma_{1,2}^{(2)}$ is G_2 -periodic. Note that all ground states will coincide with these ones. \square

Theoram 3.4. *Let $(\bar{a}, \bar{b}, \bar{c}) \in A_2$, then the following statements hold:*

- (i) *for every $n \in \mathbb{N}$, there is G_n -periodic ground state;*
- (ii) *there is uncountable number of ground states.*

Proof. Let $(\bar{a}, \bar{b}, \bar{c}) \in A_2$, then one can see that for this triple, the minimal value is $(\bar{a} + \bar{b})/2$, which is achieved by the configurations on b given in Figure 5. (i) Now using Figure 5, for each

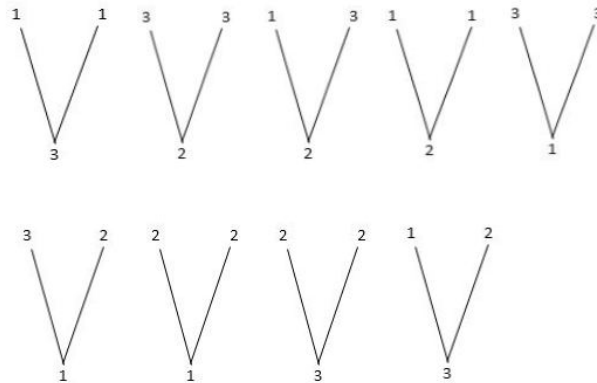


Figure 5. Configurations for A_2

$n \in \mathbb{N}$, one can construct configurations on Ω defined by

$$\sigma^{(2n)} = \sigma_{\underbrace{[1, (2, 3), \dots, (2, 3), 2]}_{2n}},$$

$$\sigma^{(2n+1)} = \sigma_{\underbrace{[1, (2, 3), \dots, (2, 3)]}_{2n+1}}.$$

Then, we can see that for any $b \in M$, one has

$$U(\sigma^{(\xi)}) = \min_{1 \leq k \leq 6} \{U_k\}, \quad \xi \in \{2n, 2n+1\}$$

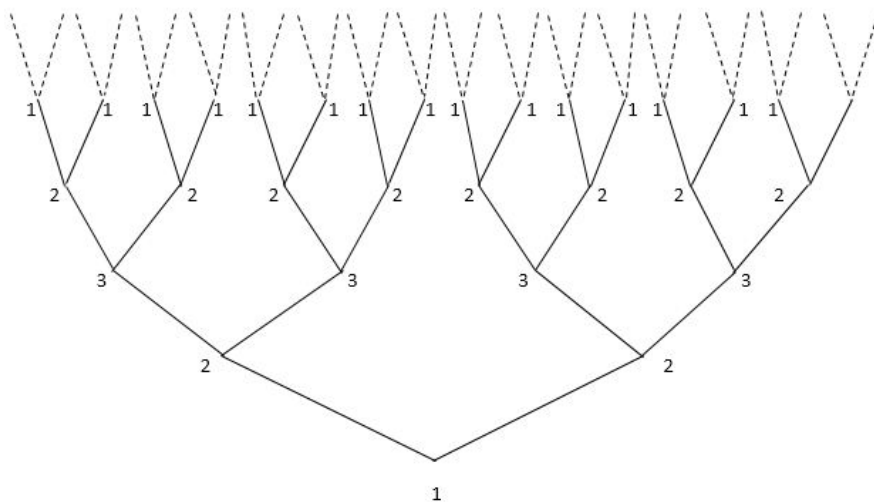


Figure 6. Configuration for $\sigma^{(2)}$

which means $\sigma^{(n)}$ is a G_n -periodic ground state.

(ii) To construct uncountable number of ground states, we consider the set

$$\Sigma_{1,2,3} = \{(\mathbf{t}_n) | \mathbf{t}_n \in \{1, 2, 3\}, \delta_{\{t_n, t_{n+1}\}} = 0, n \in \mathbb{N}\} \quad (8)$$

where δ is the Kronecker delta. One can see that the set $\Sigma_{1,2,3}$ is uncountable. Take any $\mathbf{t} = (t_n) \in \Sigma_{1,2,3}$. Let us construct a configuration by

$$\sigma^{(\mathbf{t})} = \begin{cases} 1, & x = (0), \\ \mathbf{t}_k, & x \in W_k, k \in V. \end{cases}$$

One can check that $\sigma^{(t)}$ is a ground state, and the correspondence $\mathbf{t} \in \Sigma_{1,2,3} \rightarrow \sigma^{(\mathbf{t})}$ shows that the set $\{\sigma^{(\mathbf{t})}, \mathbf{t} \in \Sigma_{1,2,3}\}$ is uncountable. This completes the proof. \square

Using the same argument as above we can prove the following results.

Theorem 3.5. *The following statements hold:*

- (I) if $(\bar{a}, \bar{b}, \bar{c}) \in A_3$, then there are three translation-invariant ground states and for every $n \in \mathbb{N}$, one can find G_n -periodic ground states;
- (II) if $(\bar{a}, \bar{b}, \bar{c}) \in A_4$, then for every $n \in \mathbb{N}$, there is a $G_{(3n+1)}$ -periodic ground state;
- (III) if $(\bar{a}, \bar{b}, \bar{c}) \in A_5$, then there are three translation-invariant ground states and there is uncountable number of ground states;
- (IV) if $(\bar{a}, \bar{b}, \bar{c}) \in A_6$, then there are only three transition-invariant ground states.

4. References

- [1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New-York-London: Academic Press)
- [2] Dorogovtsev S N, Goltsev A V and Mendes J F F 2004 *Eur Phys J B* **38** 177
- [3] Ganikhodjaev N N 1990 *Theor. Math. Phys.* **85** 1125
- [4] Ganikhodjaev N N, Mukhamedov F M and Mendes J F F 2006 *J Stat Mech* P08012
- [5] Georgii H O 1988 *Gibbs measures and phase transitions* (Berlin: Walter de Gruyter)

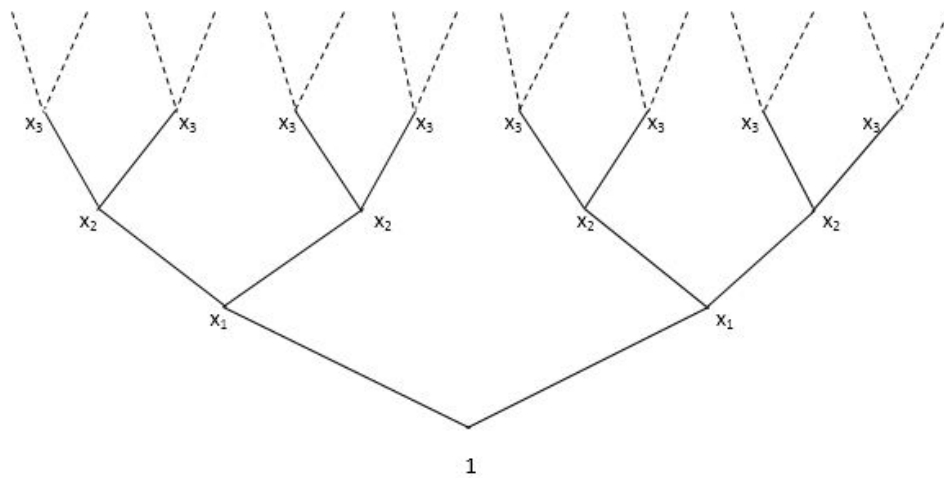


Figure 7. Configuration for $\sigma^{(t)}$

- [6] Marques M C 1988 *J Phys A: Math Gen* **21** 1061
- [7] Mukhamedov F M 2004 *Rep Math Phys* **53** 1
- [8] Mukhamedov F M and Rozikov U A 2000 *Theor. Math. Phys.* **124** 1202
- [9] Mukhamedov F, Rozikov U and Mendes J F F 2007 *J. Math. Phys.* **48** 013301
- [10] Nightingale M P and Schick M 1982 *J Phys A: Math Gen* **15** L39
- [11] Peruggi F 1984 *J. Math Phys* **25** 3303
- [12] Peruggi F 1984 *J. Math Phys* **25** 3316
- [13] Peruggi F, di Liberto F and Monroy G 1983 *J Phys A: Math Gen* **16** 811
- [14] Potts R B 1952 *Proc Cambridge Philos Soc* **48**(1952) 106
- [15] Preston C 1974 *Gibbs states on countable sets* (London: Cambridge University Press)
- [16] Rozikov U A 1998 *Siberian Math. Jour.* **39** 373
- [17] Rozikov U A 2013 *Gibbs Measures on Cayley Trees*, (Singapore: World Scientific)
- [18] Wu F Y 1982 *Rev Mod Phys* **54** 235