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Ising model on trees: $(k_0)-$ non translation-invariant Gibbs measures

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Abstract. For the Ising model on the Cayley tree of order k, we construct new sets of non translation-invariant Gibbs measures.

1. Introduction

One of the main problems for the Ising model is to describe all limiting Gibbs measures corresponding to this model. It is well known that for the Ising model, such measures form a nonempty, convex, and compact subset in the set of all probability measures. The problem of completely describing the element of this set is far from being completely solved. Some translation-invariant (see, e.g., [1], [8], [14]), periodic [3], [9], and continuum sets of non-periodic [8], [10] Gibbs measures for the Ising model on a Cayley tree have already been described.

In [6], [11]- [13] the notion of weakly periodic Gibbs measure is introduced. This notion generalizes the notion of periodic Gibbs measure.

To extend the set of periodic Gibbs measures corresponding to the Ising model, the notion of periodic measures was generalized to that of weakly periodic Gibbs measures in [6], [11]- [13], where under some conditions on parameters of the model (on the Cayley tree of order ≥ 6), the existence of such new measures was proved.

In [10], authors constructed continuum sets of non-periodic Gibbs measures for the Ising model on a Cayley tree.

In this paper we construct new sets of non periodic Gibbs measures for the Ising model on the Cayley tree of order k, which we called $(k_0)-$ translation-invariant, $(k_0)-$periodic and $(k_0)-$ non translation-invariant respectively.

This paper is organized as follows. In Section 2 we give necessary definitions and formulate the problem. Next in section 3, new sets Gibbs measures are constructed on the Cayley tree order k.

2. Definitions and the main problem.

Let $\tau^k = (V, L), k \geq 1$, be the Cayley tree of order k, i.e. an infinite graph such that every vertex of which is incident to exactly $k+1$ edges. Here V is the set of all vertices, L is the set of all edges of the tree τ^k. It is known that τ^k can be represented as a group G_k, which is the free product of $k+1$ cyclic groups of the second order [4], [8].

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For an arbitrary point $x^0 \in V$ we set $W_n = \{x \in V | d(x^0, x) = n\}$, $V_n = \bigcup_{m=0}^{n} W_m$, $L_n = \{< x, y > \in L | x, y \in V_n\}$, where $d(x, y)$ is the distance between the vertices x and y in the Cayley tree, i.e. the number of edges in the shortest path joining the vertices x and y.

Let $\Phi = \{-1, 1\}$ and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e. $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. Let $A \subset V$. We let Ω_A denote the space of configurations defined on the set A and taking values in Φ.

We consider the Hamiltonian of the Ising model:

$$H(\sigma) = -J \sum_{<x,y>\in L} \sigma(x)\sigma(y), \tag{2.1}$$

where $J \in R$, $\sigma(x) \in \Phi$ and $< x, y >$ are nearest neighbors.

Let $h_x \in R, x \in V$. For every n, we then define a measure μ_n on Ω_{V_n} by

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x\in W_n} h_x \sigma(x)\}, \tag{2.2}$$

where $\beta = \frac{1}{T}$ (T is temperature, $T > 0$), $\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$, Z_n^{-1} is the normalizing factor, and

$$H(\sigma_n) = -J \sum_{<x,y>\in L_n} \sigma(x)\sigma(y).$$

The compatibility condition for the measures $\mu_n(\sigma_n), n \geq 1$, is

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \tag{2.3}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $\mu_n, n \geq 1$ be a sequence of measures on the sets Ω_{V_n} that satisfy compatibility condition (2.3). By the Kolmogorov theorem, we then have a unique limit measure μ on $\Omega_V = \Omega$ (called the limit Gibbs measure) such that

$$\mu(\sigma_n) = \mu_n(\sigma_n)$$

for every $n = 1, 2, ...$ It is known that measures (2.2) satisfies the condition (2.3) if and only if the set $h = \{h_x, x \in G_k\}$ satisfies the condition

$$h_x = \sum_{y\in S(x)} f(h_y, \theta), \tag{2.4}$$

where

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\},$$

is the set of direct successors (children) of the point $x \in V$, $f(x, \theta) = \arctan(\theta \tanh x)$, $\theta = \tanh(J\beta)$, $\beta = \frac{1}{T}$ (see. [1], [8], [14]).

Namely, for any boundary condition satisfying the functional equation (2.4) there exists a unique Gibbs measure.

A boundary condition that satisfies (2.4) is called *compatible.*

Definition 2.1. *A set $h = \{h_x, x \in G_k\}$ of quantities is called $\widehat{G}_k$-periodic if $h_{xy} = h_x$, for all $x \in G_k$ and $y \in \widehat{G}_k$, where $\widehat{G}_k$ is a subgroup of G_k.*

Definition 2.2. *A Gibbs measure μ is said to be $\widehat{G}_k$- periodic if it corresponds to the $\widehat{G}_k$-periodic h. We call a G_k-periodic measure a translation-invariant measure.*

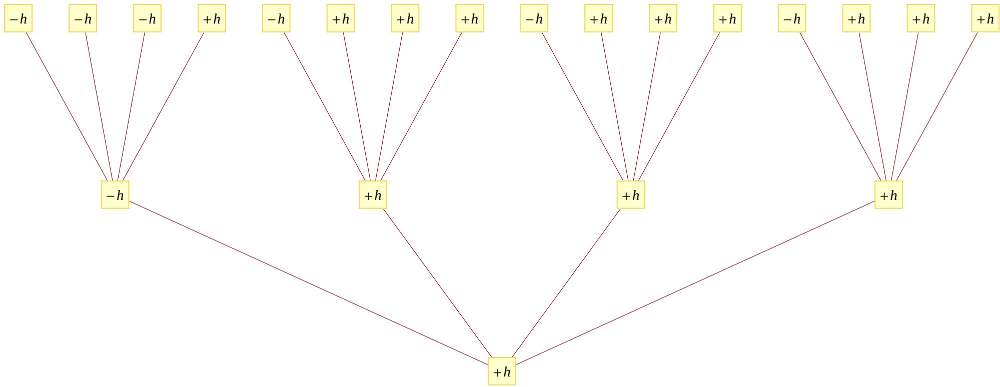

Figure 1. *Case* $k = 4, k_0 = 2,$ *and* $h = h_*^{(k_0)}$.

3. $(k_0)-$ non translation-invariant Gibbs measures

3.1. $(k_0)-$ translation-invariant Gibbs measures

We will consider ferromagnetic model of Ising (i.e. $J > 0$).

For translation-invariant set of quantities h equation (2.4) has the following form:

$$h = kf(h, \theta), 0 < \theta < 1. \tag{3.1}$$

The equation (3.1) has a unique solution $h = 0$, if $0 < \theta \leq \theta_c = \frac{1}{k}$, and three solutions denoted by $h = 0, \pm h_*^{(k)}, h_*^{(k)} > 0$, if $\theta_c < \theta < 1$ (see Theorem 2.2 [8]).

On the Cayley tree of order k, with $\pm h_*^{(k_0)}$ we construct a new solution of functional equation (2.4).

Let $k \geq 2, k_0 \geq 2$ such, that $(k - k_0)$ is even a positive number. For $x \in V$, by $S_{k_0}(x)$ we denote an arbitrary set of k_0 vertices of the set $S(x)$, and remaining $k - k_0$ vertices is denoted by $S_{k-k_0}(x)$.

We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in -h_*^{(k_0)}, +h_*^{(k_0)}$) as follows (see Figure 1):

(a_1) If on the vertex of the x, we have $h_x = +h_*^{(k_0)}$, then at each vertex from $S_{k_0}(x)$ we put the value $+h_*^{(k_0)}$, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of $+h_*^{(k_0)}$ and $-h_*^{(k_0)}$, so that its satisfy the following equality

$$\sum_{y \in S_{k-k_0}(x)} f(h_y, \theta) = 0. \tag{3.2}$$

(a_2) If on the vertex of the x, we have $h_x = -h_*^{(k_0)}$, then at each vertex from $S_{k_0}(x)$ we put the value $-h_*^{(k_0)}$, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of $+h_*^{(k_0)}$ and $-h_*^{(k_0)}$, so that its satisfy the equation (3.2).

Remark 3.1. *We can put to vertex $S_{k-k_0}(x)$ the values of $+h_*^{(k_0)}$ and $-h_*^{(k_0)}$ such that satisfy the equation (3.2). Because $|S_{k-k_0}(x)|$ is even and $f(-h, \theta) = -f(h, \theta)$ (see [1], [8]).*

The set of quantities h defined according to the rules (a_1), (a_2) is called $(k_0)-$ translation-invariant.

Proposition 3.2. *Any* $(k_0)-$ *translation-invariant set of quantities on the Cayley tree* τ^k *satisfies the functional equation (2.4).*

Proof. Let $h_x = h_*^{(k_0)}$, then

$$h_*^{(k_0)} = \sum_{y\in S(x)} f(h_y,\theta) = \sum_{y\in S_{k_0}(x)} f(h_y,\theta) + \sum_{y\in S_{k-k_0}(x)} f(h_y,\theta) = k_0 f(h_*^{(k_0)},\theta).$$

Let $h_x = -h_*^{(k_0)}$, then

$$-h_*^{(k_0)} = \sum_{y\in S(x)} f(h_y,\theta) = \sum_{y\in S_{k_0}(x)} f(h_y,\theta) + \sum_{y\in S_{k-k_0}(x)} f(h_y,\theta) = k_0 f(-h_*^{(k_0)},\theta).$$

Consequently, $(k_0)-$ translation-invariant set of quantities $h = \{h_x, x \in V\}$ satisfies the functional equation (2.4). □

A measure that corresponds to a $(k_0)-$translation-invariant set of quantities is called a $(k_0)-$translation-invariant Gibbs measure

Theorem 3.3. *Let* $k \geq 2, k_0 \geq 2$ *such, that* $(k - k_0)$ *is even a positive number, and* $T < T_{c,k_0}$. *Then, for the ferromagnetic Ising model on a Cayley tree of order k, there are exactly two* $(k_0)-$*translation-invariant Gibbs measures, were* $T_{c,k_0} = \frac{J}{arcth(\frac{1}{k_0})}$.

3.2. $(k_0)-$ *periodic Gibbs measures*

In this subsection we will consider antiferromagnetic model of Ising (i.e. $J < 0$).

Let $G_k^{(2)} = \{x \in G_k :$ the length of word x is even$\}$.

On Cayley tree of order k_0 any $G_k^{(2)}$ - periodic a set of quantities h has the form

$$h_x = \begin{cases} u & \text{if } x \in G_k^{(2)} \\ v & \text{if } x \in G_k \setminus G_k^{(2)}, \end{cases}$$

where the pair (u, v) satisfies the following system of equations

$$\begin{cases} u = k_0 f(v,\theta) \\ v = k_0 f(u,\theta), \end{cases} \tag{3.3}$$

In the antiferromagnetic case ($\theta \leq 0$) the system (3.3) has a unique solution $h_*^0 = (0, 0)$ if $\theta \geq -1/k$, and three distinct solutions $h_*^{\mp} = (-h_*, h_*)$, $h_*^0 = (0, 0)$, $h_*^{\pm} = (h_*, -h_*)$ if $-1 < \theta < -\theta_c$ (see Theorem 2.5 [8]).

Now, with $h_*^{\mp}$, we consider new solutions of functional equation (2.4).

We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \{-h_*, h_*\}$) as follows:

(a_3) If on the vertex of the x, we have $h_x = h_*$, then at each vertex from $S_{k_0}(x)$ we put the value $-h_*$, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of h_* and $-h_*$, so that satisfy (3.2).

(a_4) If on the vertex of the x, we have $h_x = -h_*$, then at each vertex from $S_{k_0}(x)$ we put the value h_*, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of h_* and $-h_*$, so that satisfy (3.2).

The set of quantities h defined according to the rules (a_3), (a_4), is called $(k_0)-$ periodic.

Similarly, using the $h_*^{\pm}$ we can construct another $(k_0)-$ periodic set of quantities h.

Proposition 3.4. *Any* $(k_0)-$ *periodic set of quantities on the Cayley tree* τ^k *satisfies the functional equation (2.4).*

Proof. Let $h_x = h_*$, then

$$h_* = \sum_{y \in S(x)} f(h_y, \theta) = \sum_{y \in S_{k_0}(x)} f(h_y, \theta) + \sum_{y \in S_{k-k_0}(x)} f(h_y, \theta) = k_0 f(-h_*, \theta).$$

Let $h_x = -h_*$, then

$$-h_* = \sum_{y \in S(x)} f(h_y, \theta) = \sum_{y \in S_{k_0}(x)} f(h_y, \theta) + \sum_{y \in S_{k-k_0}(x)} f(h_y, \theta) = k_0 f(h_*, \theta).$$

Consequently, $(k_0)-$peroidic set of quantities $h = \{h_x, x \in V\}$ satisfies the functional equation (2.4).

For $(k_0)-$peroidic set of quantities $h = \{h_x, x \in V\}$, constructed with $h_*^{\pm}$, similarly proved, that it is also satisfies the functional equation (2.4). □

Measures corresponds to a $(k_0)-$periodic set of quantities are called a $(k_0)-$periodic Gibbs measure.

As a result, we obtain the following theorem

Theorem 3.5. *Let $k \geq 2, k_0 \geq 2$ such, that $(k - k_0)$ is even a positive number, and $T < T_{c,k_0}$. Then, for the antiferromagnetic Ising model on a Cayley tree of order k, there are exactly two $(k_0)-$periodic Gibbs measures, where $T_{c,k_0} = \frac{J}{arcth(\frac{1}{k_0})}$.*

3.3. $(k_0)-$ non translation-invariant Gibbs measures

We will consider a ferromagnetic Ising model. It is known that for the ferromagnetic model of Ising there are exist continuum sets of Gibbs measures (see [1]- [4], [6]- [14]).

Let μ is any Gibbs measure. It is known that for any Gibbs measure μ, there exists unique set of quantities $h_\mu = \{h_x(\mu), x \in G_{k_0}\}$ on the Cayley tree of order k_0, which satisfying the following functional equation:

$$h_x(\mu) = \sum_{y \in S_{k_0}(x)} f(h_y(\mu), \theta), \tag{3.4}$$

where $f(h_y(\mu), \theta) = \arctan(\theta \tanh h_y(\mu))$.

We defined the set of quantities $h = \{h_x, x \in G_k\}$ on Cayley tree of order k, by rules a_1, a_2 and

a_5 If on the vertex of the x, we have $h_x = h_x(\mu)$, then at each vertex from $S_{k_0}(x)$ we put the value $h_{S(x)}(\mu)$, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of $h_*^{(k_0)}$ and $-h_*^{(k_0)}$, so that its satisfy to equation (3.2).

The set of quantities h, defined according to the rules (a_1), (a_2) and (a_5) is called $(k_0)-$ non translation-invariant.

Proposition 3.6. *Any $(k_0)-$ non translation-invariant set of quantities on the Cayley tree τ^k satisfies the functional equation (2.4).*

Proof. When $h_x = -h_*^{(k_0)}$ or $h_x = h_*^{(k_0)}$ the proof of Proposition 3.6 similar to proof of Proposition 3.4.

Let $h_x = h_x(\mu)$, then

$$h_x(\mu) = \sum_{y \in S(x)} f(h_y(\mu), \theta) = \sum_{y \in S_{k_0}(x)} f(h_y(\mu), \theta) + \sum_{y \in S_{k-k_0}(x)} f(h_y, \theta) = \sum_{y \in S_{k_0}(x)} f(h_y(\mu), \theta).$$

Consequently, $(k_0)-$ non translation-invariant set of quantities $h = \{h_x(\mu), x \in V\}$ satisfies the functional equation (2.4).

$\square$

Measures corresponds to a $(k_0)-$ non translation-invariant set of quantities are called a $(k_0)-$non translation-invariant Gibbs measure.

As a result, we obtain the following

Theorem 3.7. *Let $k \geq 2, k_0 \geq 2$ such, that $(k - k_0)$ is even a positive number, and $T < T_{c,k_0}$. Then, for the ferromagnetic Ising model on a Cayley tree of order k, there are continuum $(k_0)-$non translation-invariant Gibbs measures, where $T_{c,k_0} = \frac{J}{arcth(\frac{1}{k_0})}$.*

Remark 3.8. *In the Theorem 3.3 and the Theorem 3.7 (Theorem 3.5) conditions $T < T_{c,k_0}$ necessary for the existence translation-invariant (periodic) Gibbs measures on the Cayley tree of order k_0.*

Remark 3.9. *For antiferromagnetic Ising model can be constructed continuum set of new Gibbs measures by the rules a_3, a_4 and similar to a_5, which can be called $(k_0)-$non periodic Gibbs measures.*

Remark 3.10. *Note that these measures, which constructed in the Theorem 3.3, the Theorem 3.5 and the Theorem 3.5 are different from the previously known ones.*

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