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Critical circle maps and thermodynamic formalism

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Abstract. Let X_k be the space of analytic, critical circle homeomorphisms with an irrational rotation number $\rho_k = [k, k, ..., k, ...]$. It is shown [S. Ostlund, D. Rand, J. Sethna and E. Siggia, *Physica D*,**8** (1983) 303] that, the renormalization operator $\mathcal{R}$ defined on X_k has a unique fixed point T_0. In this paper, we study the properties of orbits of a critical point and build a potential for T_0.

1. Introduction and results

Historically, ergodic theory was a branch of thermodynamics and statistical mechanics. Although we will not describe the precise relationships between thermodynamics and ergodic theory (see Ruelles book [10] in the references for this), the connections motivate several interesting constructions which turn out to have much wider applications. This is the heart of thermodynamic formalism.

The introduction of thermodynamic formalism within the mathematical field of dynamical systems occurred in the 1970s, and was primarily due to Y. Sinai, D. Ruelle and R. Bowen. In this paper we build the thermodynamic formalism for the circle maps with a critical point. The thermodynamic formalism for the unimodal Feigenbaum map was built by E. Vul, Y. Sinai and K. Khanin [1]. In fact, our work is an analogy of their work for the critical circle maps.

Let us define a set of real-analytic *commuting pairs* that corresponds to a set of real-analytic critical circle homeomorphisms the order of three. To do this, consider the pairs (ξ, η) satisfying the following conditions:

(a) both ξ and η are real-analytic, strictly increasing on $[\eta(0), 0)$ and $[0, \xi(0))$ respectively and $\xi(0) = \eta(0) + 1$;

(b) ξ and η commute at zero, that is $\eta(\xi(0)) = \xi(\eta(0))$;

(c) both ξ and η has the cubic critical point at zero, that is $\xi'(0) = \eta'(0) = \xi''(0) = \eta''(0) = 0$, $\xi'''(0)$, $\eta''(0)$ are non zero;

(d) $(\eta(\xi))'''(0) = (\xi(\eta))'''(0)$.

The pair (ξ, η) is called the critical commuting pair (shortly commuting pair). A commuting pair is called *non-generated* if $0 < \xi(0) < 1$. For a non-generated commuting pair (ξ, η) we define its the *height* $h(\xi, \eta) \in \mathbb{N}_0 \cup \{\infty\}$ as m if

$$\xi(\eta(0)) < 0,\ \xi^2(\eta(0)) < 0,\ ...,\ \xi^{m-1}(\eta(0)) < 0,\ \xi^m(\eta(0)) > 0$$

and $h(\xi,\eta)=\infty$ if no such m exists (note that in this case the map ξ has a fixed point) where ξ^i is the i^{th} iteration of ξ. A non-generated commuting pair (ξ,η) is called *renormalizable* if its height in finite and nonzero. Denote by X the set of all renormalizable commuting pairs. On X we define a renormalization operator $\mathcal{R}$ as follows

$$\mathcal{R}(\xi,\eta)=\left(\alpha_m\circ\xi^{m-1}\circ\eta\circ\alpha_m^{-1},\,\alpha_m\circ\xi^{m-1}\circ\eta\circ\xi\circ\alpha_m^{-1}\right)$$

where $\alpha_m(z)=\left(\xi^{m-1}(\eta(0))-\xi^m(\eta(0))\right)^{-1}(z)$. A commuting $(\xi,\eta)\in X$ is called *infinitely-renormalizable* if $\mathcal{R}^{n-1}(\xi,\eta)$ is renormalizable for all $n\in\mathbb{N}_0$. For a infinitely-renormalizable commuting (ξ,η) we define its *rotation number* $\rho(\xi,\eta)\in[0,1)$, by substituting its consecutive heights for *partial quotients* in the continued fractions expansion $\rho(\xi,\eta)=[m_1,m_2,...,m_n,...]$ where m_n is the height of $\mathcal{R}^{n-1}(\xi,\eta)$. Actually, the rotation number $\rho(\xi,\eta)$ of the pair (ξ,η) can be defined by dynamical approach as follows. Let $T_{\xi,\eta}$ be a circle homeomorphism generated by (ξ,η) defined on the unit circle $[\eta(0),\xi(0))$ as follows

$$T_{\xi,\eta}(x)=\begin{cases}\xi(x), & \text{if} \quad x\in[\eta(0),0),\\ \eta(x), & \text{if} \quad x\in[0,\xi(0)).\end{cases} \tag{1.1}$$

One can easily check that the homeomorphism $T_{\xi,\eta}$ is an analytic circle homeomorphism and has the cubic critical point at zero. It is well known since Poincaré (1885), that the orbit structure of $T_{\xi,\eta}$ is determined by some irrational constant $\rho(T_{\xi,\eta})$ so called the *rotation number* of $T_{\xi,\eta}$, in the following sense: for any point x of the circle, the mapping $T^j_{\xi,\eta}(x)\to j\rho(T_{\xi,\eta}) \mod 1,\ j\in\mathbb{Z}$ is orientation-preserving. One can easily verify that $\rho(T_{\xi,\eta})=\rho(\xi,\eta)$.

Further, consider the infinitely-renormalizable commuting pairs (ξ,η) whose the rotations numbers $\rho(\xi,\eta)$ equal to $\rho_k=[k,k,...,k,...]$ for some $k\geq 1$. The set of all such commuting pairs is denoted by X_k. R. Ostlund *et. al* [9] proved that the renormalization operator $\mathcal{R}$ acting on X_k has the unique fixed point (ξ_0,η_0). Let us recall the following definition.

Definition 1.1. Two orientation-preserving circle homeomorphisms T_1 and T_2 are said to be C^r conjugate, $r\geq 1$, if there exists an orientation-preserving circle homeomorphism $H\in C^r(\mathbb{S}^1)$ (if $r\geq 1$, then it is required that $H^{-1}\in C^r(\mathbb{S}^1)$ such that

$$H\circ T_1=T_2\circ H.$$

Denote by $E(T_0)$ the set of all circle homeomorphisms whose are C^1 conjugated with T_k and defined on the standard circle $\mathbb{S}^1$. It is well known (see [3]) that any two topological conjugated homeomorphisms have same rotation number. Therefore, the rotation numbers of homeomorphisms of $E(T_0)$ are the same and equal to ρ_k.

The purpose of this paper is to build thermodynamic formalism for the set $E(T_0)$. To formulate our main result, we introduce further notations. Let

$$\frac{p_n}{q_n}=\underbrace{[k,k,...,k]}_{n\text{ times}}$$

be the sequence of rational convergents of the continued fraction $\rho_k=[k,k,...,k,...]$. The coprime numbers p_n and q_n satisfy the recurrence relations $p_n=kp_{n-1}+p_{n-2}$ and $q_n=kq_{n-1}+q_{n-2}$ for $n\geq 1$, where, for convenience we set $p_0=0$, $q_0=1$ and $p_{-1}=1$, $q_{-1}=0$. Taking the critical point $x_0=0$, we define the n^{th} *fundamental segment* $I_0^n:=I_0^n(x_0)$ as the circle arc $[x_0,T_0^{q_n}(x_0)]$ if n is even and $[T_0^{q_n}(x_0),x_0]$ if n is odd. Certain number of images of fundamental segments I_0^{n-1} and I_0^n, under iterates of T_0, cover whole circle without overlapping beyond the endpoints and form the n^{th} *dynamical partition* of the circle

$$\mathbb{P}_n:=\mathbb{P}_n(x_0,T_0)=\left\{I_j^n:=T_0^j(I_0^n),\ 0\leq j<q_{n-1}\right\}\cup\left\{I_i^{n-1}:=T_0^i(I_0^{n-1}),\ 0\leq i<q_n\right\}.$$

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Obviously, the partition $\mathbb{P}_{n+1}$ is a refinement of the partition $\mathbb{P}_n$. Indeed, the "short" intervals I_j^n are members of $\mathbb{P}_{n+1}$ and each "long" interval $I_i^{n-1} \in \mathbb{P}_n,\ 0 \leq i < q_n$, is partitioned into $k+1$ intervals belonging to $\mathbb{P}_{n+1}$ such that

$$I_i^{n-1} = I_i^{n+1} \cup \bigcup_{s=0}^{k-1} I_{i+q_{n-1}+sq_n}^n. \tag{1.2}$$

Next, using the sequence of dynamical partition $(\mathbb{P}_n)_n$ we introduce a certain symbolic representation for the dynamics of T_0 as follows. Let $\mathcal{A} = \{a, 0, 1, ..., k\}$ be an *alphabet.* Consider the set of *infinite words* $\mathcal{L} = \{\vec{a} := (a_1, a_2, ..., a_n, ...) : \mid a_i \in \mathcal{A}\}$ corresponding to $\mathbb{S}^1 \setminus \mathcal{O}_{T_0}(x_0)$, where $\mathcal{O}_{T_0}(x_0) = \{x_0, T_0(x_0), ...\}$, and defined as follows. Take an arbitrary point $x \in \mathbb{S}^1 \setminus \mathcal{O}_{T_0}(x_0)$ we associate the unique word $\vec{a} := (a_1, a_2, ..., a_n, ...)$ defined inductively as: $a_n = a$ if $x \in I_j^n$, $0 \leq j < q_{n-1}$; $a_n = k - s$ if $x \in I_{i+q_{n-1}+sq_n}^n$, $0 \leq i < q_n$, $0 \leq s \leq k-1$; and $a_n = 0$ if $x \in I_i^{n+1}$, $0 \leq i < q_n$. Thus, we obtain a one-to-one correspondence

$$\mathbb{S}^1 \setminus \mathcal{O}_{T_0}(x_0) \xleftrightarrow{\psi} \mathcal{L}.$$

Notice that, the finite word $(a_1, a_2, ..., a_n)$ of the length n corresponds to an interval I^n of the dynamical partition $\mathbb{P}_n$. We set $I^n = I(a_1, a_2, ..., a_n)$. Denote by λ_0 a probability measure on the space of sequences $\mathcal{L}$ induced by Lebesgue measure on the circle, namely, $\lambda_0(a_1, a_2, ..., a_n) = |I(a_1, a_2, ..., a_n)|$. Consider another space of one-sided sequences,

$$\Omega_+ = \{\overrightarrow{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...) \mid \varepsilon_i \in \mathcal{A} \text{ and } \varepsilon_{i+1} = 0 \text{ iff } \varepsilon_i = a,\ i \geq 1\}.$$

Now, consider another $\widetilde{T} \in E(T_0)$. Since $\widetilde{T}$ and T_0 are topological conjugated the sets $\mathcal{O}_{T_0}(x_0)$ and $\mathcal{O}_{\widetilde{T}_0}(y_0)$ have the same order, where $y_0 = H(x_0)$ and H is the conjugation between $\widetilde{T}$ and T_0. Therefore, the symbolic representations of the points $x \in \mathbb{S}^1 \setminus \mathcal{O}_{T_0}(x_0)$ and $H(x) \in \mathbb{S}^1 \setminus \mathcal{O}_{\widetilde{T}_0}(y_0)$ are the same in $\mathcal{L}$ i.e., there exists a unique $\vec{a} \in \mathcal{L}$ such that $\vec{a} = \psi(x)$ and $\vec{a} = \psi(H(x))$. We set

$$\vec{\gamma}(\varepsilon) := \begin{cases} (0, a, 0, a, ...), & \text{if } \varepsilon = a \\ (a, 0, a, 0, ...), & \text{if } \varepsilon = 0 \end{cases}$$

$\Delta_i^n := H(I_i^n)$ and $V_1 := \Delta_1^1 \cup \Delta_1^2$. Our main result is the following result.

Theorem 1.2. *For any $T \in E(T_0)$ there exists a continuous (in the Tychonoff topology) mapping $U : \Omega_+ \to \mathbb{R}^1$ such that the following properties hold.*

(1) For any $\overrightarrow{\varepsilon} = (\varepsilon_1, ..., \varepsilon_p, \varepsilon_{p+1}, ..., \varepsilon_n, ...)$ and $\overrightarrow{b} = (\varepsilon_1, ..., \varepsilon_p, b_{p+1}, ..., b_n, ...)$ belonging to the space Ω_+, there exists a constant $C_1 = C_1(T) > 0$ such that

$$|U(\overrightarrow{\varepsilon}) - U(\overrightarrow{b})| \leq C_1 |\alpha_k|^{-p} \tag{1.3}$$

where $\alpha_k = (\xi^{k-1}(\eta(0)) - \xi^k(\eta(0)))^{-1}$.

(2) Let $I(a_1, ..., a_r, a_{r+1}, ..., a_n) \subset I(a_1, ..., a_r) \subset V_1,\ 1 \leq r < n$. Then

$$\frac{|I(a_1, ..., a_n)|}{|I(a_1, ..., a_r)|} = \Big(1 + \psi_1(a_1, ..., a_n)\Big) \cdot \exp\Big\{ \sum_{s=r+1}^{n} U(a_s, a_{s-1}, ..., a_r, ..., a_1, \overrightarrow{\gamma}(a_1)) \Big\}, \tag{1.4}$$

where $|\psi_1(a_1, ..., a_n)| \leq \text{const} \cdot |\alpha_k|^{-r}$.

Remark 1.3. • Theorem 1.2 generalizes the main theorem of [2].

- The second assertion of Theorem 1.2 implies that, the length of any interval $I^n \in \mathbb{P}_n$ can be expressed in the "Gibbs" form:

$$\text{const} \leq \frac{|I^n|}{\exp\left\{\sum\limits_{s=1}^{n} U(a_s, a_{s-1}, ..., a_r, ..., a_1, \overrightarrow{\gamma}(a_1))\right\}} \leq \text{Const}.$$

- Following the terminology of statistical mechanics, we call U the potential corresponding to the map T. Due to Theorem 1.2 the potential depends on the long-range variables exponentially weakly, i.e., on the statistical mechanical point of view, it is "good".
- One can easily verify that the potential U is unique and invariant under a smooth change of variable.

2. Metrical properties of the orbit of critical point

As we have mentioned above the homeomorphism T_0 is analytic, has the cubic critical point $x_0 = 0$ and its rotation number is irrational. Due to Yoccoz's [3] theorem T_0 and the linear rotation $T_{\rho_k}(x) = x + \rho_k (\text{mod } 1)$ are topological equivalent. Hence the topological properties of T_0 and T_{ρ_k} are the same. In this section we study the metric properties of the T_0-orbit of the critical point $x_0 = 0$. We note that the commuting pair (ξ_0, η_0) is a fixed point of renormalization operator $\mathcal{R}$ i.e.,

$$\begin{cases} \xi_0 = \alpha_k \circ \xi_0^{k-1} \circ \eta_0 \circ \alpha_k^{-1} \\ \eta_0 = \alpha_k \circ \xi_0^{k-1} \circ \eta_0 \circ \xi_0 \circ \alpha_k^{-1} \end{cases} \tag{2.1}$$

where $\alpha_k(z) = \alpha_k z$ and $\alpha_k = [\xi_0^{k-1}(\eta_0(0)) - \xi_0^k(\eta_0(0))]^{-1}$. We use this fact in the proof of the following theorem.

Theorem 2.1. *The following relations hold*

$$T_0^{q_n}(\alpha_k^{-n} x) = \alpha_k^{-n} \xi_0(x), \quad x \in [\eta_0(0), 0), \tag{2.2}$$

$$T_0^{q_n + q_{n-1}}(\alpha_k^{-n} x) = \alpha_k^{-n} \eta_0(x), \quad x \in [0, \xi_0(0)) \tag{2.3}$$

for all $n \geq 1$.

Proof. We prove the theorem by induction. Let $x \in [\eta_0(0), 0)$. If $n = 1$, then $q_1 = k$ and $\alpha_k^{-1}(x) \in [0, \xi(0))$. Therefore $T_0^{q_1}(\alpha_k^{-1} x) = T_0^k(\alpha_k^{-1} x) = T_0^{k-1}(T_0(\alpha_k^{-1} x)) = T_0^{k-1}(\eta_0(\alpha_k^{-1} x))$. Since $\eta_0(\alpha_k^{-1} x) \in [\eta_0(0), \eta_0(\xi(0)))$ we get

$$T_0^{k-1}(\eta_0(\alpha_k^{-1} x)) = \xi_0^{k-1}(\eta_0(\alpha_k^{-1} x)). \tag{2.4}$$

Relation (2.1) implies $\xi_0^{k-1}(\eta_0(\alpha_k^{-1} x)) = \alpha_k^{-1} \xi_0(x)$. Hence

$$T_0^{q_1}(\alpha_k^{-1} x) = \alpha_k^{-1} \xi_0(x). \tag{2.5}$$

Suppose, the equation (2.2) holds $k \leq n$. The relations (2.1) and $q_{n+1} = k q_n + q_{n-1}$ imply

$$\begin{aligned} T_0^{q_{n+1}}(\alpha_k^{-(n+1)} x) &= T_0^{k q_n + q_{n-1}}(\alpha_k^{-(n+1)} x) = T_0^{k q_n}(T_0^{q_{n-1}}(\alpha_k^{-(n-1)} \cdot \alpha_k^{-2} x)) \\ &= T_0^{k q_n}(\alpha_k^{-(n-1)} \xi_0(\alpha_k^{-2} x)) = T_0^{(k-1) q_n}(T_0^{q_n}(\alpha_k^{-n} \alpha_k \xi_0(\alpha_k^{-2} x))) \\ &= T_0^{(k-1) q_n}(\alpha_k^{-n} \xi_0(\alpha_k \xi_0(\alpha_k^{-2} x))) = T_0^{(k-1) q_n}(\alpha_k^{-n} \eta_0(\alpha_k^{-1} x)) \\ &= T_0^{(k-2) q_n}(T_0^{q_n}(\alpha_k^{-n} \eta_0(\alpha_k^{-1} x))) = T_0^{(k-2) q_n}(\alpha_k^{-n} \xi_0(\eta_k(\alpha_k^{-1} x))) \\ &\vdots \\ &= T_0^{q_n}(\alpha_k^{-n} \xi_0^{k-2} \circ \eta_0(\alpha_k^{-1} x)) = \alpha_k^{-n} \alpha_k^{-1} \xi_0(x) = \alpha_k^{-(n+1)} \xi_0(x). \end{aligned} \tag{2.6}$$

This proves the first assertion of theorem. The proof of the second assertion is similar. $\square$

As a consequence of Theorem 2.1 we have.

Corollary 2.2. *Let $x_0 = 0$ and $x_i = T_0^i x_0$, $i \geq 1$. We have*

$$x_{q_{n+1}} = \alpha_k^{-n} x_1$$

for all $n \geq 1$.

Let A_n be a subset of $\mathcal{O}_{T_0}(x_0)$ which is generation the n^{th}-dynamical partition i.e., $A_n = \{x_0, x_1, ..., x_{q_n+q_{n-1}}\}$. We set

$$A_n^{(1)} = A_n \cap [0, \xi_0(0)), \quad A_n^{(\ell+1)} = A_n \cap [x_\ell, x_{\ell+1}],$$

$$A_n^{(1,1)} = A_n \cap [0, x_{q_1+q_0}), \quad A_n^{(1,2)} = A_n \cap [x_{q_1+q_0}, \xi_0(0)),$$

where $1 \leq \ell \leq k$. It is clear, that $A_n = \bigcup_{\ell=1}^{k+1} A_n^{(\ell)}$ and $A_n^{(1)} = A_n^{(1,1)} \cup A_n^{(1,2)}$. Using the definition of $A_n^{(i)}$, $i = \overline{1, k+1}$ and the structure of dynamical partition one can show that

$$|A_n^{(i)}| = q_{n-1} + q_{n-2}, \, \text{ı} = \overline{1, k}, \quad |A_n^{(k+1)}| = q_{n-2} + q_{n-3}, \tag{2.7}$$

where $|\cdot|$ - the cardinality of the finite set. The next theorem describes the transition from A_n to A_{n+1}.

Theorem 2.3. *For all $n \geq 1$ the following relations hold.*

$$A_{n+1}^{(k+1)} = \alpha_k^{-1}\Big(A_n^{(1)} \cup \xi_0(0)\Big), \quad A_{n+1}^{(\ell+1)} = \xi_0^{\ell-1}\Big(\eta(A_{n+1}^{(1)})\Big), \, 1 \leq \ell \leq k-1,$$

$$A_n^{(1,1)} = \alpha_k^{-1}\Big(\bigcup_{i=1}^{k} A_n^{(i+1)}\Big), \quad A_n^{(1,2)} = \xi_0\Big(A_{n+1}^{(k+1)}\Big) = \xi_0\Big(\alpha_k^{-1}A_n^{(1)}\Big).$$

Proof. This theorem is also proved by induction. Assertion of the theorem can be easily verified for $n = 1$. We rewrite A_n and A_{n+1} as

$$A_n = A_{n-1} \cup (A_n \setminus A_{n-1}), \quad A_{n+1} = A_n \cup (A_{n+1} \setminus A_n).$$

By the induction hypothesis, the assertion of Theorem 2.3 is true for A_{n-1} and A_n. On the other hand, A_{n-1} is a subset of A_n and this a subset of A_{n+1}. Consequently, it is enough to prove the assertions of Theorem 2.3 for $A_n \setminus A_{n-1}$ and $A_{n+1} \setminus A_n$. Using the definition of A_n we get

$$A_{n+1} \setminus A_n = \Big\{x_{q_n+q_{n-1}}, \, x_{q_n+q_{n-1}+1}, ..., x_{q_{n+1}+q_n-1}\Big\},$$

$$A_n \setminus A_{n-1} = \Big\{x_{q_{n-1}+q_{n-2}}, \, x_{q_{n-1}+q_{n-2}+1}, ..., x_{q_{n-1}+q_n-1}\Big\},$$

$$|(A_{n+1}^{(j)} \setminus A_n^{(j)})| = kq_{n-1}, \; 1 \leq j \leq k, \quad |(A_{n+1}^{(k+1)} \setminus A_n^{(k+1)})| = kq_{n-2}.$$

Consider the first element $x_{q_n+q_{n-1}}$ of $A_{n+1} \setminus A_n$. By Theorem 2.1

$$x_{q_n+q_{n-1}} = \alpha_k^{-1} x_{q_{n-1}+q_{n-2}}. \tag{2.8}$$

So, the first element of $A_{n+1} \setminus A_n$ obtains from the first element of $A_n \setminus A_{n-1}$ by multiplying α_k^{-1}. We pass to the second element of $A_{n+1} \setminus A_n$. Here, depending on the sign of $x_{q_{n-1}+q_{n-2}}$ there are two possibilities: either $x_{q_{n-1}+q_{n-2}} \in [\eta_0(0), 0)$ or $x_{q_{n-1}+q_{n-2}} \in [0, \xi_0(0))$

(a) if $x_{q_{n-1}+q_{n-2}} \in [\eta_0(0), 0)$ then by Theorem 2.1 we have

$$x_{q_n+q_{n+1}+1} = T_0(x_{q_n+q_{n+1}}) = T_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}) = \xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}) \in [x_{q_1+q_0}, \xi_0(0)),$$

$$x_{q_n+q_{n+1}+2} = T_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}})) = \eta_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}})) \in [x_1, x_2),$$

$$x_{q_n+q_{n+1}+3} = T_0(\eta_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}))) = \xi_0(\eta_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}))) \in [x_2, x_3),$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$x_{q_n+q_{n+1}+k+1} = T_0(\xi_0^{k-2}(\eta_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}})))) = \xi_0^{k-1}(\eta_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}))) =$$
$$= \alpha_k^{-1}\eta_0(x_{q_{n-1}+q_{n-2}}) = \alpha_k^{-1} x_{q_{n-1}+q_{n-2}+1} \in [0, \xi_0(0)).$$

(b) if $x_{q_{n-1}+q_{n-2}} \in [\eta(0), 0)$ then again Theorem 2.1 we have

$$x_{q_n+q_{n-1}+1} = T_0(x_{q_n+q_{n+1}}) = T_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}) = \eta_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}) \in [x_1, x_2),$$

$$x_{q_n+q_{n-1}+2} = T_0(\eta_0(\alpha_k^{-1} x_{q_n+q_{n+1}})) = \xi_0(\eta_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}) \in [x_2, x_3)),$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$x_{q_n+q_{n-1}+k+1} = \xi_0^{k-1}(\eta_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}}) = \alpha_k^{-1}\xi_0(x_{q_{n-1}+q_{n-2}})) = \alpha_k^{-1} x_{q_{n-1}+q_{n-2}+1}.$$

Continue this process one can show that any element of $A_n \setminus A_{n-1}$ obtains by multiplying α_k^{-1} to an element of $A_{n+1} \setminus A_n$. Moreover, with the same manner as in (a) and (b) one can show that for each element of $x_{q_n+q_{n-1}+j}$, $0 \leq j \leq kq_n - 1$, there exists an element $x_{q_{n-1}} + q_{n-2} + i$ in

$$B_p := \left\{x_{q_{n-1}+q_{n-2}}, x_{q_{n-1}+q_{n-2}+1}, ..., x_{q_{n-1}+q_{n-2}+p}\right\}$$

where $p > 1$, such that: either $x_{q_n+q_{n-1}+j} = \alpha_k^{-1} x_{q_{n-1}+q_{n-2}+i}$, or $x_{q_n+q_{n-1}+j} = \xi_0^{j-t}(\eta_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}+2}))$, $t \leq j \leq t+k-2$, or $x_{q_n+q_{n-1}+j} = \xi_0^{j-t}(\eta_0(\xi_0(\alpha_k^{-1} x_{q_{n-1}+q_{n-2}+i})))$, $t \leq j \leq t+k-2$. Next we show that $B = A_n \setminus A_{n-1}$. To do this it is enough to show $p = kq_{n-1}$. A simply calculation shows that

$$|B \cap (0, \xi(0))| = |(A_{n+1} \setminus A_n) \cap [x_{q_1}, 0)| = q_{n-1} + q_{n-2};$$
$$|B \cap (x_{q_1}, 0]| = |(A_{n+1} \setminus A_n) \cap [x_{q_1+q_0}, \xi(0)]| = kq_{n-1};$$
$$|B \cap (x_{q_1+q_0}, \xi(0))| = |(A_{n+1} \setminus A_n) \cap [\eta(0), x_2]| = kq_{n-1};$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$|B \cap (x_{k-2}, x_{k-1})| = |(A_{n+1} \setminus A_n) \cap (x_{k-1}, x_k]| = kq_{n-1}.$$

Similarly

$$|B \cap [\eta(0), 0)| = |(A_{n+1} \setminus A_n) \cap [0, x_{q_1+q_0}]| = kq_{n-1} - kq_{n-2};$$
$$|B \cap (\eta(0), x_2]| = |(A_{n+1} \setminus A_n) \cap (x_2, x_3]| = kq_{n-1}.$$

Using these equations one can show that $|B \cap [\eta(0), \xi(0)]| = kq_{n-1} - kq_{n-2} + kq_{n-2} = kq_{n-1}$. This implies $|B| = |(A_{n+1} \setminus A_n)| = kq_{n-1}$. Theorem 2.3 is proved. □

Next we estimate the ratio of lengths of two intervals of dynamical partitions.

Lemma 2.4. *Let $I^n \in \mathbb{P}_n$ and $I^{n-m} \in \mathbb{P}_{n-m}$, $n > m$ such that $I^n \subset I^{n-m}$. There exists a constant $C = C(T_0) > 0$ such that*

$$\frac{I^n}{I^{n-m}} \leq C|\alpha_k|^{-m}, \quad |\alpha_k|^{-3n} \leq C|I_0^n|.$$

Proof. The proof of lemma follows closely that of [1] for the Feigenbaum map. ☐

To formulate our next result we define some subsets of $\mathbb{P}_n$ as follow

$$\mathbb{P}_n^{(1)} = \Big\{ I^n \in \mathbb{P}_n : \text{ such that } I^n \subset [\eta_0(0), 0] \Big\},$$

$$\mathbb{P}_n^{(2)} = \Big\{ I^n \in \mathbb{P}_n : \text{ such that } I^n \subset [0, \xi_0(0)] \Big\},$$

$$\mathbb{P}_{n,i} = \Big\{ I^n \in \mathbb{P}_n : \text{ such that } I^n \subset I_0^i \Big\},$$

where $i = (n-m-1), (n-m)$. The following theorem plays an important role to build the thermodynamical formalism for T_0.

Theorem 2.5. *The following relations hold, for any* $0 < m < n$.

$$\mathbb{P}_{n,n-m} = \alpha_k^{-(n-m)} \mathbb{P}_m^{(1)}, \qquad \mathbb{P}_{n,n-m-1} = \alpha_k^{-(n-m)} \mathbb{P}_m^{(2)}$$

if $(n-m)$ *is an odd number.*

$$\mathbb{P}_{n,n-m-1} = \alpha_k^{-(n-m)} \mathbb{P}_m^{(2)}, \qquad \mathbb{P}_{n,n-m} = \alpha_k^{-(n-m)} \mathbb{P}_m^{(1)}$$

if $(n-m)$ *is an even number.*

Proof. The proof of theorem follows from Theorem 2.3. ☐

Consider the interval $V_1^{n-m} = I_1^{n-m} \cup I_1^{n-m+1}$. Notice that this interval is a neighborhood of the point x_1. Further, taking two intervals I' and I such that $I' \subset I \subset V_1^{n-m}$ and $I' \in \mathbb{P}_{n+1}$, $I \in \mathbb{P}_n$ we provide an estimate for the ration $R_0 = |I'|/|I|$ and its "iterations" $R_i = |T_0^i(I')|/|T_0^i(I)|$. Denote

$$s_{n-m} = \begin{cases} q_{n-m+1}, & \text{if } I \subset I_1^{n-m}, \\ q_{n-m}, & \text{if } I \subset I_1^{n-m+1}. \end{cases}$$

The following lemma will be used below.

Lemma 2.6 ([8])**.** *There exists a constant* $C = C(T_0) > 0$ *and a natural number* $N_0 = N_0(T_0)$ *such that*

$$\left| \ln \frac{R_i}{R_0} \right| \leq C|\alpha_k|^{-m}, \quad 0 < i < s_{n-m} \tag{2.9}$$

for all $n > N_0$.

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2 i.e., we build the potential U for T_0. Note that the proof of Theorem 1.2 follows closely that of Theorem 1.1 in [2]. To prove the theorem first we define the prelimit potentials $U_m^{(n)}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ and $\widetilde{U}_m^{(n)}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ for $1 \leq m \geq n$ and then we show that these potentials converge to the same limit $U_m(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ exponentially fast as $n \to \infty$. Finally we show that the limit of $U_m(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ exists and unique as $m \to \infty$. Take $m \geq 1$ and fix it. Consider the $V_1^{(n-m)}$ neighborhood of x_1 for $n \geq m$. It is clear that $V_1^{(n-m)} \subset [\eta_0(0), x_3] \cup [x_2, \xi_0(0))$ for $n \geq m+4$. For definiteness we assume $(n-m)$- is even. The proof is similar for the case when $(n-m)$-is odd. It is obvious

$$I_1^{n-m} \subset [x_1, x_3] \quad \text{and} \quad I_1^{n-m+1} \subset [x_2, \xi(0)).$$

Consider two intervals $I' \in \mathbb{P}_n$ and $I \in \mathbb{P}_{n-1}$ such that $I' \subset I \subset I_1^{n-k}$. Let the words

$$I\Big(\underbrace{\overbrace{a,1,0,a,...,0,a}^{(n-m+1)\text{ times}},\varepsilon_m,...,\varepsilon_2,\varepsilon_1}_{(n-1)\text{ times}}\Big) \quad \text{and} \quad I\Big(\underbrace{\overbrace{a,1,0,a,...,0,a}^{(n-m+1)\text{ times}},\varepsilon_m,...,\varepsilon_2}_{(n-1)\text{ times}}\Big)$$

be the symbolic representations of I' and I respectively. Define the following functions

$$U_m^{(n)}(\varepsilon_1,\varepsilon_2,...,\varepsilon_m) = \ln\frac{I(a,1,0,a,...,0,a,\varepsilon_m,...,\varepsilon_2,\varepsilon_1)}{I(a,1,0,a,...,0,a,\varepsilon_m,...,\varepsilon_2)}.$$

and

$$\widetilde{U}_m^{(n)}(\varepsilon_1,\varepsilon_2,...,\varepsilon_m) = \ln\frac{I(1,0,a,0,...,a,\varepsilon_m,...,\varepsilon_2,\varepsilon_1)}{I(1,0,a,0,...,a,\varepsilon_m,...,\varepsilon_2)}.$$

It follows from the structure of dynamical partition and the definition of symbolical representation that

$$|I(1,0,a,0,...,a,\varepsilon_m,...,0,a)| = |I(1,0,a,0,...,a,\varepsilon_m,...,0)|.$$

It is easy to see that

$$U_m^{(n)}(a,0,\varepsilon_3,...,\varepsilon_m) = \widetilde{U}_m^{(n)}(a,0,\varepsilon_3,...,\varepsilon_m) = 0.$$

Using Theorem 2.5, we get:

$$I(a,1,0,a,...,0,a,\varepsilon_m,...,\varepsilon_2,\varepsilon_1) = \eta_0(I(0,a,0,a,...,0,a,\varepsilon_m,...,\varepsilon_2,\varepsilon_1)) =$$
$$= \eta_0(\alpha_k^{-(n-m-1)}(I(\varepsilon_m,...,\varepsilon_2,\varepsilon_1))), \tag{3.1}$$

where $I(\varepsilon_m,...,\varepsilon_2,\varepsilon_1) \in \mathbb{P}_m$. Similarly,

$$I(a,1,0,a,...,0,a,\varepsilon_m,...,\varepsilon_2) = \eta_0(\alpha_k^{-(n-m-1)}(I(\varepsilon_m,...,\varepsilon_2))) \tag{3.2}$$

where $I(\varepsilon_m,...,\varepsilon_2) \in \mathbb{P}_{m-1}$. Denote by (β_1,β_2) and (β_3,β_4) the intervals that correspond to $I(\varepsilon_m,...,\varepsilon_2,\varepsilon_1)$ and $I(\varepsilon_m,...,\varepsilon_2)$, respectively. It is obvious that $(\beta_1,\beta_2) \subset (\beta_3,\beta_4)$. Using the definition $U_m^{(n)}$ and (3.1)-(3.2)we get

$$U_m^{(n)}(\varepsilon_1,\varepsilon_2,...,\varepsilon_m) = \ln\frac{|\eta_0(\alpha_k^{-(n-m-1)}\beta_2) - \eta_0(\alpha_k^{-(n-m-1)}\beta_1)|}{|\eta_0(\alpha_k^{-(n-m-1)}\beta_4) - \eta_0(\alpha_k^{-(n-m-1)}\beta_3)|} =$$
$$= \ln\frac{\beta_2^3-\beta_1^3}{\beta_4^3-\beta_3^3} + \mathcal{O}(\alpha_k^{-3(n-m)}) := U_m(\varepsilon_1,\varepsilon_2,...,\varepsilon_m) + \mathcal{O}(|\alpha_k|^{-3(n-m)}). \tag{3.3}$$

With the same manner one can show

$$\widetilde{U}_m^{(n)}(\varepsilon_1,\varepsilon_2,...,\varepsilon_m) = U_m(\varepsilon_1,\varepsilon_2,...,\varepsilon_m) + O(|\alpha_k|^{-3(n-m)}). \tag{3.4}$$

Next we prove the existence of the limit of $U_m(\varepsilon_1,\varepsilon_2,...,\varepsilon_m)$ when $m \to \infty$. For this we take a word $\overrightarrow{\varepsilon} = (\varepsilon_1,...,\varepsilon_n,...) \in \Omega_+$ and fix it. For a given m consider the expression $U_m := U_m(\varepsilon_1,...,\varepsilon_m)$. Using (3.3) and (3.4), one can get

$$U_{\ell+m} = \ln\frac{|I(\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m}),\varepsilon_{\ell+m},...,\varepsilon_1))|}{|I(\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m}),\varepsilon_{\ell+m},...,\varepsilon_2))|} + O(|\alpha_k|^{-6n}), \tag{3.5}$$

IOP Conf. Series: Journal of Physics: Conf. Series **819** (2017) 012014 doi:10.1088/1742-6596/819/1/012014

where $\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m})$ is a $2n$- dimensional vector of the form

$$\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m}) = \begin{cases} (1,0,a,0,...,a,0), & \text{if} \quad \varepsilon_{\ell+m} = a, \\ (a,1,0,a,...,0,a), & \text{if} \quad \varepsilon_{\ell+m} = 0 \vee 1 \end{cases}$$

and the interval $\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m}, \varepsilon_{\ell+m-1}, ..., \varepsilon_1)$ t is an element of dynamical partition $\mathbb{P}_{2n+\ell+m}$. It follows from the definition of symbolical dynamics, that the interval $\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m}, \varepsilon_{\ell+m-1}, ..., \varepsilon_1)$ belongs to the trajectory of the interval $I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_{\ell+m}, ..., \varepsilon_1)$. More precisely

$$I(\overrightarrow{\gamma}_{2n}(\varepsilon_{\ell+m}), \varepsilon_{\ell+m}, ..., \varepsilon_{\ell+1}, \varepsilon_\ell, ..., \varepsilon_2, \varepsilon_1) = T_0^i(I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2, \varepsilon_1)) \tag{3.6}$$

where $0 \leq i < q_{2n+m}$ if $\varepsilon_\ell = a$ and $0 \leq i < q_{2n+m+1}$ if $\varepsilon_\ell = 0 \vee 1$. Using (2.9), we get

$$\left| \ln \left\{ \frac{|T_0^i(I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2, \varepsilon_1))|}{|T_0^i(I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2))|} \times \left(\frac{|I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2, \varepsilon_1)|}{|I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2)|} \right)^{-1} \right\} \right| \leq C|\alpha_k|^{-\ell}. \tag{3.7}$$

On the other hand, by (3.3) and (3.4) we have

$$\ln \frac{|I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2, \varepsilon_1)|}{|I(\overrightarrow{\gamma}_{2n+m}(\varepsilon_{\ell+m}), \varepsilon_\ell, ..., \varepsilon_2)|} = U_\ell + O(|\alpha_k|^{-(2n+m)}). \tag{3.8}$$

Combining (3.5), (3.7) and (3.8) we get

$$|U_{\ell+m} - U_\ell| \leq C|\alpha_k|^{-\ell}, \tag{3.9}$$

for sufficiently large ℓ and m. Hence $(U_n)_n$ is a Cauchy sequence. Let $U(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...)$ be the limit function of the sequence $(U_n)_n$. Taking the limit in (3.9) as $m \to \infty$, we get

$$|U(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...) - U(\varepsilon_1, \varepsilon_2, ..., \varepsilon_\ell)| \leq C|\alpha_k|^{-\ell}.$$

This proves the first assertion of Theorem 1.2. Next we prove the second assertion of Theorem 1.2. Consider the intervals $I(a_1, ..., a_n)$ and $I(a_1, ..., a_r)$ such that $I(a_1, ..., a_n) \subset I(a_1, ..., a_r) \subset [\eta_0(0), x_3] \cup [x_2, \xi_0(0)]$, $r < n$. Denote by (β_i, β_{i+1}) and (d_i, d_{i+1}) the intervals that correspond to $I(a_1, ..., a_{r+i})$ and $I(a_1, ..., a_{r+i-1})$ respectively, where $i \in [1, n-r]$. It is clear that $(\beta_i, \beta_{i+1}) \subset (d_i, d_{i+1})$. Using the same arguments as in (3.3), one can show that

$$U_{r+i}^{(n+r+i)}(a_{r+i}, ..., a_1) = \frac{\beta_{i+1}^3 - \beta_i^3}{d_{i+1}^3 - d_i^3} + O(|\alpha_k|^{-6n}) =$$

$$= \frac{I(a_1, ..., a_{r+i-1}, a_{r+i})}{I(a_1, ..., a_{r+i-1})} \cdot \frac{\beta_{i+1}^2 + \beta_{i+1}\beta_i + \beta_i^2}{d_{i+1}^2 + d_{i+1}d_i + d_i^2} + O(|\alpha_k|^{-6n}). \tag{3.10}$$

Since $[d_i, d_{i+1}] \subset [\eta_k(0), x_3] \cup [x_2, \xi_k(0)]$ and the rank of $[d_i, d_{i+1}]$ is $(r + i - 1)$ we get

$$\frac{\beta_{i+1}^2 + \beta_{i+1}\beta_i + \beta_i^2}{d_{i+1}^2 + d_{i+1}d_i + d_i^2} = 1 + O(|\alpha_k|^{-(r+i)}).$$

Substituting the right hand side of the above equation (3.10) we get

$$\frac{I(a_1, ..., a_{r+i-1}, a_{r+i})}{I(a_1, ..., a_{r+i-1})} = U_{r+i}^{(n+r+i)}(a_{r+i}, ..., a_1) + O(|\alpha_k|^{-(r+i)})$$

for sufficiently range n. Taking the products from both sides of this relation over $i = \overline{1, (n-r)}$, we obtain the second assertion of Theorem 1.2. Theorem 1.2 is completely proved.

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