

The Fundamental Basis Theorem of Geometry from an algebraic point of view

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Abstract. An algebraic analog of the Fundamental Basis Theorem of geometry is offered with a pure algebraic proof involving the famous Waring's problem for polynomials. Unlike the geometry case the offered system of invariant differential operators is commuting, which is a new result even in the classical geometry of surfaces. Moreover the algebraic analog works in more general settings than does the Fundamental Basis Theorem of geometry.

1. Introduction

Let $m \leq n$ be any fixed natural numbers, H be any subgroup of the general linear group $GL(n, \mathbf{R})$, $\mathbf{B} \subset \mathbf{R}^m$ be an open unit ball and

$$\mathbf{x}(\mathbf{t}) = (x_1(t_1, t_2, \dots, t_m), x_2(t_1, t_2, \dots, t_m), \dots, x_n(t_1, t_2, \dots, t_m))$$

be infinitely smooth m -parametric variable surface in \mathbf{R}^n , where \mathbf{R} is the field of real numbers. The surface $\mathbf{x}(\mathbf{t})$ is assumed to be written in the row form.

Definition 1.1. An infinitely smooth function $f^\partial(\mathbf{x}(\mathbf{t}))$ of $\mathbf{x}(\mathbf{t}) = (x_1(\mathbf{t}), x_2(\mathbf{t}), \dots, x_n(\mathbf{t}))$ and its finite number of derivatives with respect to $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots, \frac{\partial}{\partial t_m}$ is said to be (\widehat{G}, H) -invariant of the surface if the equality

$$f^\delta(\mathbf{x}(\mathbf{s}(\mathbf{t}))h) = f^\partial(\mathbf{x}(\mathbf{t}))$$

holds true for any $h \in H$, $\mathbf{t} \in \mathbf{B}$ and $\mathbf{s} \in \widehat{G}$, where $\widehat{G} = \text{Dif}(\mathbf{B})$ stands for the group of diffeomorphisms of \mathbf{B} , ∂ stands for $(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots, \frac{\partial}{\partial t_m})$, $\mathbf{s}(\mathbf{t}) = (s_1(\mathbf{t}), s_2(\mathbf{t}), \dots, s_m(\mathbf{t}))$ and $\delta^i = \frac{\partial}{\partial s_i(\mathbf{t})}$, $i = 1, 2, \dots, m$.

One of the important problems of differential geometry is the description of all such invariant functions $\mathbf{R}(\mathbf{x})^{(\widehat{G}, H)}$. It is called the set of differential invariant functions of m -dimensional surfaces with respect to the motion group H . The Fundamental Basis Theorem of geometry, first formulated by Lie, states that there exist (\widehat{G}, H) -invariant differential operators $\delta^1, \delta^2, \dots, \delta^m$ and a finite system of the (\widehat{G}, H) -invariant functions such that locally any (\widehat{G}, H) -invariant function is a function of this finite system of elements and their finite number of derivatives with respect to the $\delta^1, \delta^2, \dots, \delta^m$. Note that the Fundamental Basis Theorem does not state the existence of such commuting system of invariant differential operators. But it states that the commutators $\delta^i \delta^j - \delta^j \delta^i$, $i, j = 1, 2, \dots, m$, can be represented as linear combinations of $\delta^1, \delta^2, \dots, \delta^m$. For the



more detailed information about the Fundamental Basis Theorem of geometry one can see [1, 2]. We note also that the main method of geometry, in dealing with differential invariant functions and invariant differentials, are the "The Moving Frame Method" and its generalizations [3].

In the current paper an algebraic analog of the Fundamental Basis Theorem of geometry is offered with a pure algebraic proof involving the famous Waring's problem for polynomials. Moreover it is shown that the system of invariant differential operators $\delta^1, \delta^2, \dots, \delta^m$ can be chosen commuting which is a new result even in classical differential geometry of surfaces. In addition our approach works in more general settings than does "The Moving Frame Method".

To formulate an algebraic analog of the Fundamental Basis Theorem note that the above change of parameters $\mathbf{t} = (t_1, \dots, t_m)$ can be presented in the following form

$$\frac{\partial}{\partial t_i} = \sum_{j=1}^m \frac{\partial s_j(\mathbf{t})}{\partial t_i} \frac{\partial}{\partial s_j},$$

i.e. $\delta = g^{-1}\partial$, where g is matrix with the elements $g_j^i = \frac{\partial s_j(\mathbf{t})}{\partial t_i}$, $i, j = 1, \dots, m$, and ∂ (δ) is the column vector with the "coordinates" $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots, \frac{\partial}{\partial t_m}$ (respectively, $\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \dots, \frac{\partial}{\partial s_m}$).

Moreover every infinitely smooth parameterized surface $\mathbf{x} : \mathbf{B} \rightarrow \mathbf{R}^n$ can be regarded as an element of differential module $(F^n; \partial^1, \partial^2, \dots, \partial^m)$, with the coordinate-wise action of $\partial^i = \frac{\partial}{\partial t_i}$ on elements of F^n , where $F = C^\infty(\mathbf{B})$ is the differential ring of infinitely smooth functions relative to differential operators $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots, \frac{\partial}{\partial t_m}$. If elements of this module are written in the row form the above considered transformations look like

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \mapsto \mathbf{x}h, \quad \partial \mapsto g^{-1}\partial,$$

where g is a matrix with elements $g_j^i = \frac{\partial s_j(\mathbf{t})}{\partial t_i}$, $i, j = 1, 2, \dots, m$, $h \in H$, $\mathbf{s} \in \widehat{G}$.

Therefore the following algebraic analogue of the above problem is natural.

Let $(F; \partial)$ be any characteristic zero differential field, where ∂ is a column-vector of commuting system of differential operators $\partial^1, \partial^2, \dots, \partial^m$ of F ,

$$C = \{a \in F : \partial^i a = 0 \text{ for } i = 1, 2, \dots, m\}$$

be its constant field, H be a subgroup of $GL(n, C)$ and

$$GL^\partial(m, F) = \{g \in GL(m, F) : \partial^i g_k^j = \partial^j g_k^i \text{ for } i, j, k = 1, 2, \dots, m\}.$$

One can verify easily that whenever $g \in GL^\partial(m, F)$ the system $\delta^1, \delta^2, \dots, \delta^m$ is also a commuting system of differential operators of F , where $\delta = g^{-1}\partial$. This is an analogue of gauge transformations (change of variables) for the abstract differential field $(F; \partial)$.

In general the set $GL^\partial(m, F)$ is not a group with respect to the ordinary product of matrices as far as it is not closed with respect to that product. But by the use of it a natural groupoid [4] can be constructed with the base $\{g^{-1}\partial : g \in GL^\partial(m, F)\}$.

Further let x_1, \dots, x_n be differential algebraic independent variables over F , \mathbf{x} stand for the row vector with coordinates x_1, x_2, \dots, x_n , $C\langle \mathbf{x}, \partial \rangle$ be the field of ∂ -differential rational functions in \mathbf{x} over C and $G = GL^\partial(m, F)$.

Definition 1.2. An element $f^\partial\langle \mathbf{x} \rangle \in C\langle \mathbf{x}, \partial \rangle$ is called to be (G, H) -invariant if the equality

$$f^{g^{-1}\partial}\langle \mathbf{x}h \rangle = f^\partial\langle \mathbf{x} \rangle$$

holds true for any $g \in G, h \in H$.

We denote by $C\langle \mathbf{x}; \partial \rangle^{(G,H)}$ the field of all such (G,H) -invariant ∂ -differential rational functions. From the algebraic point of view this field is a natural analog of $R(\mathbf{x})^{(\widehat{G},H)}$ and as such it is worthy to be described.

The organization of the paper is as follows. In the next section, with its own notations, a generalization of classical Faa' di Bruno formula is presented via the symmetric tensor product of matrices. This generalization is used to prove an algebraic analog of the Fundamental Basis Theorem of geometry in the next section.

In the hypersurface case a similar result have been obtained in [7] by another algebraic method, the finite group H case is considered in [8]. The used notions of differential algebra can be found in [5].

2. A generalization of Faa' di Bruno formula

The following classical Faa' di Bruno formula on higher order derivatives of the composite functions is well known: For smooth functions of one variable $f = f(x)$, $g = g(t)$ and natural number m the following equality

$$\frac{d^m}{dt^m} f(g(t)) = \sum_{k=1}^m f^{(k)}(g(t)) \sum_{|\alpha|=k, \|\alpha\|=m} \frac{m!}{\alpha!} \left(\frac{g'(t)}{1!}\right)^{\alpha_1} \left(\frac{g''(t)}{2!}\right)^{\alpha_2} \dots \left(\frac{g^{(m)}(t)}{m!}\right)^{\alpha_m}$$

holds true, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_m!$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_m$, $\|\alpha\| = \alpha_1 + 2\alpha_2 + \dots + m\alpha_m$. It can be proved by induction on m taking into account the additive, the Leibniz and chain rule properties of the derivative. Of course the fact that elements $g'(t), g''(t), \dots$ commute with each other is used silently.

Note that if one uses notations $d := \frac{d}{dt}$, $g := g'(t)$, $\delta := \frac{d}{dg(t)} = \frac{d}{dx}$ then $d = g\delta$ (the chain rule) and the above mentioned formula can be written in the form

$$\frac{d^m}{m!} = \sum_{k=1}^m \left(\sum_{|\alpha|=k, \|\alpha\|=m} \frac{k!}{\alpha!} \left(\frac{g}{1!}\right)^{\alpha_1} \left(\frac{dg}{2!}\right)^{\alpha_2} \dots \left(\frac{d^{m-1}g}{m!}\right)^{\alpha_m} \right) \frac{\delta^k}{k!}.$$

In this section we are going to show that "the same formula" is true in partial derivatives (multivariate) case but to do it we need the symmetric tensor product. The main properties of it with proofs can be found in [9, 10]. Here some needed definitions and properties are presented without proofs, except for some results which are very closely related to the proof of the generalized Faa' di Bruno formula.

Let n be any positive integer and I_n stand for all row n -tuples with nonnegative integer entries with the following linear order: $\beta = (\beta_1, \beta_2, \dots, \beta_n) < \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ if and only if $|\beta| < |\alpha|$ or $|\beta| = |\alpha|$ and $\beta_1 > \alpha_1$ or $|\beta| = |\alpha|$, $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$ etcetera. We write $\beta \ll \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, 2, \dots, n$, $\binom{\alpha}{\beta}$ stands for $\frac{\alpha!}{\beta!(\alpha-\beta)!}$.

We allow n to be zero also and in this case it is accepted that $I_n = \{0\}$.

Consider any associative algebra \mathbf{A} , with 1, over a field of zero characteristic and let \mathbf{C} stand for the center of \mathbf{A} . For any nonnegative integer numbers p', p let $M_{n',n}(p', p; \mathbf{A}) = M(p', p; \mathbf{A})$ stand for all " $p' \times p$ " size matrices $A = (A_{\alpha'}^{\alpha})_{|\alpha'|=p', |\alpha|=p}$ (α' presents row, α presents column and $\alpha' \in I_{n'}, \alpha \in I_n$) with entries from \mathbf{A} .

The ordinary size of such matrix is $\binom{p'+n'-1}{n'-1} \times \binom{p+n-1}{n-1}$. Over this kind of matrices in addition to the ordinary sum and product of matrices we consider the following symmetric product \odot as well [9, 10]:

Definition 2.1. If $A \in M(p', p; \mathbf{A})$ and $B \in M(q', q; \mathbf{A})$ then

$$A \odot B = C \in M(p' + q', p + q; \mathbf{A})$$

such that for any $|\alpha'| = p' + q'$, $|\alpha| = p + q$, where $\alpha' \in I_{n'}$, $\alpha \in I_n$,

$$C_{\alpha}^{\alpha'} = (A \odot B)_{\alpha}^{\alpha'} = \sum_{\beta, \beta'} \binom{\alpha}{\beta} A_{\beta}^{\beta'} B_{\alpha-\beta}^{\alpha'-\beta'}$$

where the sum is taken over all $\beta' \in I_{n'}$, $\beta \in I_n$, for which $|\beta'| = p'$, $\beta' \ll \alpha'$, $|\beta| = p$, $\beta \ll \alpha$.

We use the symmetric tensor product as well when all components of A are linear maps from \mathbf{A} to \mathbf{A} and all components of B are also linear maps from \mathbf{A} to \mathbf{A} or all components of it are in \mathbf{A} . In the first case $A_{\beta}^{\beta'} B_{\alpha-\beta}^{\alpha'-\beta'}$ stands for composition $A_{\beta}^{\beta'} \circ B_{\alpha-\beta}^{\alpha'-\beta'}$ and in the second case it stands for $A_{\beta}^{\beta'} (B_{\alpha-\beta}^{\alpha'-\beta'})$.

In future $A^{\odot m}$ means the m -th power of matrix A with respect to the product \odot .

Proposition 2.2. a) $A \odot B = B \odot A$ whenever all entries of A or B are in \mathbf{C} .

b) $(\lambda_1 A + \lambda_2 B) \odot C = \lambda_1 (A \odot C) + \lambda_2 (B \odot C)$ and $A \odot (\lambda_1 B + \lambda_2 C) = \lambda_1 (A \odot B) + \lambda_2 (A \odot C)$ whenever $\lambda_1, \lambda_2 \in \mathbf{C}$.

c) $(A \odot B) \odot C = A \odot (B \odot C)$.

d) $A(B \odot H) = (AB) \odot H$, for any row $H \in M(0, p; \mathbf{A})$.

e) For any natural m , $v \in M(1, 0; \mathbf{C})$ and $h \in M(0, 1; \mathbf{C})$ one has

$$(v^{\odot m})_0^{\alpha'} = \binom{m}{\alpha'} v^{\alpha'}, \quad (h^{\odot m})_{\alpha}^0 = m! h^{\alpha}$$

, where h^{α} stands for $h_1^{\alpha_1} h_2^{\alpha_2} \dots h_n^{\alpha_n}$.

Let $\partial = (\partial^1, \partial^2, \dots, \partial^{n'})$, $\delta = (\delta^1, \delta^2, \dots, \delta^n)$ be column vectors of commuting systems of differential operators of the algebra \mathbf{A} for which $\partial = g\delta$, where $g_j^i \in \mathbf{C}$, $\partial^k g_j^i = \partial^i g_j^k$ for all $i, k = 1, 2, \dots, n'$, $j = 1, 2, \dots, n$.

In ordinary (one variable) derivative d case the change of the variable means a change of d according to $d \mapsto a^{-1}d$, where $a \in F$. In this case to prove the Faa' di Bruno formula one needs only additive property of d , the Leibniz law and that elements $\{d^k a\}_{k=1,2,\dots}$ commute with each other. Therefore to prove similar formula in partial (multivariate) derivatives case one should have "an appropriate product" for which the above listed properties of d hold true for the partial derivatives. The following result says that for "an appropriate product" one can take the symmetric tensor product.

Lemma 2.3. The following are true.

a) $\partial \odot (A \odot B) = (\partial \odot A) \odot B + A \odot (\partial \odot B)$ (the Leibniz law).

b) $\partial \odot (A(p', q)B(q, p)) = (\partial \odot A)B + \frac{1}{q+1} (A \odot g)(\delta \odot B)$ whenever

$$\delta^k (B_{\beta}^{\alpha-e_k}) = \frac{\alpha_k}{q+1} (\delta \odot B)_{\beta}^{\alpha} = \frac{\alpha_k}{|\alpha|} (\delta \odot B)_{\beta}^{\alpha}$$

for all $k = 1, 2, \dots, n$, $\alpha \in I_n$, $|\alpha| = q + 1$ and $|\beta| = p$. Here AB stands for the ordinary product of matrices A and B , all components of $e_i \in I_n$ are zero except for i -th which is 1.

c) If $\delta^k (B_{\gamma}^{\alpha-e_k}) = \frac{\alpha_k}{|\alpha|} (\delta \odot B)_{\gamma}^{\alpha}$ for all $k = 1, 2, \dots, n$, $\alpha \in I_n$, $|\alpha| = q + 1$ and $|\gamma| = p$ then the matrix $\delta \odot B$ also has similar property:

$$\delta^k (\delta \odot B)_{\gamma}^{\beta-e_k} = \frac{\beta_k}{|\beta|} (\delta \odot (\delta \odot B))_{\gamma}^{\beta}$$

for all $k = 1, 2, \dots, n$, $\beta \in I_n$, $|\beta| = q + 2$ and $|\gamma| = p$.

d) $\partial \odot (A \frac{\delta^{\odot m}}{m!}) = (\partial \odot A) \frac{\delta^{\odot m}}{m!} + (A \odot g) \frac{\delta^{\odot m+1}}{(m+1)!}$ (the generalized chain rule).

Proof. We use component-wise checking.

$$\text{a. } (\partial \odot (A \odot B))_{\alpha}^{\alpha'} = \sum_{i=1}^{n'} \partial^i (A \odot B)_{\alpha}^{\alpha' - e_i} = \sum_{i=1}^{n'} \partial^i \sum_{\beta', \beta} \binom{\alpha}{\beta} A_{\beta}^{\beta'} B_{\alpha - \beta}^{\alpha' - e_i - \beta'} =$$

$$\sum_{i=1}^{n'} \sum_{\beta', \beta} \binom{\alpha}{\beta} (\partial^i A_{\beta}^{\beta'}) B_{\alpha - \beta}^{\alpha' - e_i - \beta'} + \sum_{i=1}^{n'} \sum_{\beta', \beta} \binom{\alpha}{\beta} A_{\beta}^{\beta'} \partial^i B_{\alpha - \beta}^{\alpha' - e_i - \beta'} =$$

$$\sum_{\beta', \beta} \binom{\alpha}{\beta} \sum_{i=1}^{n'} (\partial^i A_{\beta}^{\beta' - e_i}) B_{\alpha - \beta}^{\alpha' - \beta'} + \sum_{\beta', \beta} \binom{\alpha}{\beta} A_{\beta}^{\beta'} \sum_{i=1}^{n'} \partial^i B_{\alpha - \beta}^{\alpha' - e_i - \beta'} =$$

$$\sum_{\beta', \beta} \binom{\alpha}{\beta} (\partial \odot A)_{\beta}^{\beta'} B_{\alpha - \beta}^{\alpha' - \beta'} + \sum_{\beta', \beta} \binom{\alpha}{\beta} A_{\beta}^{\beta'} (\partial \odot B)_{\alpha - \beta}^{\alpha' - \beta'} =$$

$$((\partial \odot A) \odot B)_{\alpha}^{\alpha'} + (A \odot (\partial \odot B))_{\alpha}^{\alpha'} = ((\partial \odot A) \odot B + A \odot (\partial \odot B))_{\alpha}^{\alpha'}$$

$$\text{b. } (\partial \odot (AB))_{\alpha}^{\alpha'} = \sum_{i=1}^{n'} \partial^i (AB)_{\alpha}^{\alpha' - e_i} = \sum_{i=1}^{n'} \partial^i (\sum_{\gamma} A_{\gamma}^{\alpha' - e_i} B_{\alpha}^{\gamma}) =$$

$$\sum_{\gamma} \sum_{i=1}^{n'} ((\partial^i A_{\gamma}^{\alpha' - e_i}) B_{\alpha}^{\gamma} + A_{\gamma}^{\alpha' - e_i} (\partial^i B_{\alpha}^{\gamma})) = \sum_{\gamma} (\partial \odot A)_{\gamma}^{\alpha'} B_{\alpha}^{\gamma} + \sum_{\gamma} \sum_{i=1}^{n'} A_{\gamma}^{\alpha' - e_i} (\sum_{j=1}^n g_j^i \delta^j B_{\alpha}^{\gamma})$$

Note that $\sum_{\gamma} (\partial \odot A)_{\gamma}^{\alpha'} B_{\alpha}^{\gamma} = ((\partial \odot A) B)_{\alpha}^{\alpha'}$ and

$$\sum_{\gamma} \sum_{i=1}^{n'} A_{\gamma}^{\alpha' - e_i} (\sum_{j=1}^n g_j^i \delta^j B_{\alpha}^{\gamma}) = \sum_{\gamma} \sum_{i=1}^{n'} \sum_{j=1}^n A_{\gamma}^{\alpha' - e_i} g_j^i \frac{\gamma_j + 1}{|\gamma| + 1} (\delta \odot B)_{\alpha}^{\gamma + e_j} =$$

$$\sum_{\gamma} \sum_{i=1}^{n'} \sum_{j=1}^n A_{\gamma}^{\alpha' - e_i} g_j^i \frac{\gamma_j + 1}{q + 1} (\delta \odot B)_{\alpha}^{\gamma + e_j} = \sum_{\gamma} \sum_{i=1}^{n'} \sum_{j=1}^n A_{\gamma - e_j}^{\alpha' - e_i} g_j^i \frac{\gamma_j}{q + 1} (\delta \odot B)_{\alpha}^{\gamma} =$$

$$\sum_{\gamma} \sum_{i=1}^{n'} \sum_{j=1}^n \binom{\gamma}{e_j} A_{\gamma - e_j}^{\alpha' - e_i} g_j^i \frac{1}{q + 1} (\delta \odot B)_{\alpha}^{\gamma}$$

$$= \sum_{\gamma} (A \odot g)_{\gamma}^{\alpha'} \frac{1}{q + 1} (\delta \odot B)_{\alpha}^{\gamma} = \frac{1}{q + 1} ((A \odot g) (\delta \odot B))_{\alpha}^{\alpha'}$$

c. Proof of it is similar to the b) case.

d. It is a consequence of the second relation as far as $\delta^{\odot m}$ has property

$$\delta^k ((\delta^{\odot m})^{\alpha - e_k}) = \frac{\alpha_k}{m + 1} (\delta^{\odot (m+1)})^{\alpha}$$

for any $k = 1, 2, \dots, n$, $\alpha \in I_n$ and $|\alpha| = m + 1$. □

So we have the following properties of ∂ : Additive property, Leibniz law (Lemma 2.3 a)), the generalized chain rule (Lemma 2.3 d)) and

$$(\partial^{\odot k} g)_{k=0,1,2,\dots}$$

commute with each other with respect to \odot . In the last case we are using the fact that the center \mathbf{C} is invariant with respect to derivatives. Therefore without repeating routine induction on m , as in one variable case, one more time we can formulate the following generalization of well known classical Faa' di Bruno formula [11].

Theorem 2.4. Let \mathbf{A} be any associative algebra, with 1, over a field of zero characteristic and let \mathbf{C} stand for the center of \mathbf{A} . If $\partial = (\partial^1, \partial^2, \dots, \partial^{n'})$, $\delta = (\delta^1, \delta^2, \dots, \delta^n)$ are column vectors of commuting system of differential operators of the algebra \mathbf{A} for which $\partial = g\delta$, where $g = (g_j^i)_{i=1,2,\dots,n';j=1,2,\dots,n}$, $g_j^i \in \mathbf{C}$, $\partial^k g_j^i = \partial^i g_j^k$ for all $i, k = 1, 2, \dots, n'$, $j = 1, 2, \dots, n$ then for any natural m the following equality is true

$$\frac{\partial^{\odot m}}{m!} = \sum_{k=1}^m \sum_{|\alpha|=k, \|\alpha\|=m} \frac{(\frac{\partial^{\odot 0}}{1!} \odot g)^{\odot \alpha_0}}{\alpha_0!} \odot \frac{(\frac{\partial^{\odot 1}}{2!} \odot g)^{\odot \alpha_1}}{\alpha_1!} \odot \dots \odot \frac{(\frac{\partial^{\odot m-1}}{m!} \odot g)^{\odot \alpha_{m-1}}}{\alpha_{m-1}!} \frac{\delta^{\odot k}}{k!}$$

where $\frac{\partial^{\odot 0}}{0!} \odot g = 1 \odot g = g$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{m-1})$, $\|\alpha\| = 1\alpha_0 + 2\alpha_1 + \dots + m\alpha_{m-1}$.

One can check directly that in $n = n' = 1$ case whenever $|\alpha| = k$, $\|\alpha\| = m$ then

$$\frac{k!}{\alpha!} \left(\frac{g}{1!}\right)^{\alpha_0} \left(\frac{dg}{2!}\right)^{\alpha_1} \dots \left(\frac{d^{m-1}g}{m!}\right)^{\alpha_{m-1}} = \frac{(\frac{\partial^{\odot 0}}{1!} \odot g)^{\odot \alpha_0}}{\alpha_0!} \odot \frac{(\frac{\partial^{\odot 1}}{2!} \odot g)^{\odot \alpha_1}}{\alpha_1!} \odot \dots \odot \frac{(\frac{\partial^{\odot m-1}}{m!} \odot g)^{\odot \alpha_{m-1}}}{\alpha_{m-1}!}$$

at any $1 \leq k \leq m$, where $d = \partial = g\delta$. For it one should take into account that $\frac{\partial^{\odot i-1}}{i!} \odot g$ should be treated as " $i \times 1$ " size matrix, though it's ordinary size is 1×1 , and notice

$$\begin{aligned} \left(\frac{\partial^{\odot i-1}}{i!} \odot g\right)^{\odot \alpha_{i-1}} &= \left(\frac{d^{i-1}g}{i!}\right)^{\alpha_{i-1}} \alpha_{i-1}!, \\ \left(\frac{\partial^{\odot i-1}}{i!} \odot g\right)^{\odot \alpha_{i-1}} \odot \left(\frac{\partial^{\odot j-1}}{j!} \odot g\right)^{\odot \alpha_{j-1}} &= \left(\frac{d^{i-1}g}{i!}\right)^{\alpha_{i-1}} \left(\frac{d^{j-1}g}{j!}\right)^{\alpha_{j-1}} (\alpha_{i-1} + \alpha_{j-1})! \end{aligned}$$

3. The Fundamental Basis Theorem of Geometry

As we have agreed in the introduction $(F; \partial)$ stands for the characteristic zero differential field with the given commuting system of differential operators $\partial^1, \partial^2, \dots, \partial^m$ of F , C is its constant field, H be a subgroup of $GL(n, C)$ and

$$GL^\partial(m, F) = \{g \in GL(m, F) : \partial^i g_k^j = \partial^j g_k^i \text{ for } i, j, k = 1, 2, \dots, m\}.$$

Further it is assumed that the system of differential operators $\partial^1, \partial^2, \dots, \partial^m$ is linear independent over F .

Definition 3.1. An element $f^\partial \langle \mathbf{x} \rangle \in C \langle \mathbf{x}; \partial \rangle$ (A differential operator $d(\partial, \mathbf{x}) : C \langle \mathbf{x}; \partial \rangle \rightarrow C \langle \mathbf{x}; \partial \rangle$) is said to be (G, H) -invariant if the equality

$$f^{g^{-1}\partial} \langle \mathbf{x}h \rangle = f^\partial \langle \mathbf{x} \rangle \quad (\text{respectively, } d(g^{-1}\partial, \mathbf{x}h) = d(\partial, \mathbf{x}))$$

holds true for any $g \in G$, $h \in H$.

Definition 3.2. An element $f^\partial \langle \mathbf{x} \rangle \in C \langle \mathbf{x}; \partial \rangle$ is said to be (G, H) -relative invariant if there exist integer numbers k, l such that the equality

$$f^{g^{-1}\partial} \langle \mathbf{x}h \rangle = \det(g)^k \det(h)^l f^\partial \langle \mathbf{x} \rangle$$

holds true for any $g \in G, h \in H$.

Proposition 3.3. If $f_1^\partial \langle \mathbf{x} \rangle, f_2^\partial \langle \mathbf{x} \rangle$ are two nonzero relative invariants with corresponding $(k_1, l_1), (k_2, l_2)$ and $k_1 \neq 0, k_2 \neq 0$ then for the column vector

$$M_1^\partial \langle \mathbf{x} \rangle = \frac{\partial \odot f_2^\partial \langle \mathbf{x} \rangle}{k_2 f_2^\partial \langle \mathbf{x} \rangle} - \frac{\partial \odot f_1^\partial \langle \mathbf{x} \rangle}{k_1 f_1^\partial \langle \mathbf{x} \rangle} \quad \text{one has } M_1^\partial \langle \mathbf{x} \rangle = g M_1^\partial \langle \mathbf{x}h \rangle \text{ for any } g \in G, h \in H.$$

Proof. Application of ∂ to the both sides of the equality

$$f_i^{g^{-1}\partial}\langle \mathbf{x}h \rangle = \det(g)^{k_i} \det(h)^{l_i} f_i^\partial\langle \mathbf{x} \rangle$$

results in

$$g\delta \odot f_i^\delta\langle \mathbf{y} \rangle = \det(h)^{l_i} k_i \det(g)^{k_i-1} (\partial \odot \det(g)) f_i^\partial\langle \mathbf{x} \rangle + \det(h)^{l_i} \det(g)^{k_i} (\partial \odot f_i^\partial\langle \mathbf{x} \rangle),$$

where $\delta = g^{-1}\partial$, $\mathbf{y} = \mathbf{x}h$. Hence, assuming $k_i \neq 0$, one has

$$g \frac{\delta \odot f_i^\delta\langle \mathbf{y} \rangle}{k_i f_i^\delta\langle \mathbf{y} \rangle} = \frac{\partial \odot \det(g)}{\det(g)} + \frac{\partial \odot f_i^\partial\langle \mathbf{x} \rangle}{k_i f_i^\partial\langle \mathbf{x} \rangle}.$$

Therefore for the column vector $M_1^\partial\langle \mathbf{x} \rangle = \frac{\partial \odot f_2^\partial\langle \mathbf{x} \rangle}{k_2 f_2^\partial\langle \mathbf{x} \rangle} - \frac{\partial \odot f_1^\partial\langle \mathbf{x} \rangle}{k_1 f_1^\partial\langle \mathbf{x} \rangle}$ one has

$$M_1^{g^{-1}\partial}\langle \mathbf{x}h \rangle = g^{-1} M_1^\partial\langle \mathbf{x} \rangle$$

for any $g \in G, h \in H$. □

Theorem 3.4. (*The Fundamental Basis Theorem of Geometry*) *There exists a commuting system of (G, H) -invariant differential operators*

$$\delta = (\delta^1(\partial, \mathbf{x}), \delta^2(\partial, \mathbf{x}), \dots, \delta^m(\partial, \mathbf{x}))$$

such that $C\langle \mathbf{x}; \partial \rangle^{(G, H)}$ is a finitely generated δ -differential field over C .

Proof. The proof is constructive in terms of the system of invariant differential operators. For any natural k , $g \in G$ and $\delta = g^{-1}\partial$ due to **Theorem 2.4** one has the following representation.

$$\begin{pmatrix} \partial \\ \frac{\partial \odot 2}{2!} \\ \vdots \\ \frac{\partial \odot k}{k!} \end{pmatrix} = \begin{pmatrix} g & 0 & 0 & \cdots & 0 \\ \frac{\partial \odot g}{2!} & \frac{g \odot 2}{2!} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ \frac{\partial \odot k-1 \odot g}{k!} & * & * & \cdots & \frac{g \odot k}{k!} \end{pmatrix} \begin{pmatrix} \delta \\ \frac{\delta \odot 2}{2!} \\ \vdots \\ \frac{\delta \odot k}{k!} \end{pmatrix} \quad (3.1)$$

The number of equations (rows) in this equality is

$$\binom{m-1+1}{m-1} + \binom{m-1+2}{m-1} + \cdots + \binom{m-1+k}{m-1} = \binom{m+k}{m} - 1.$$

The number of columns of $\mathbf{x}^{\odot l}$ is $\binom{n-1+l}{n-1}$.

Let us show that whenever $1 \leq l_1^k < l_2^k < \cdots < l_{j_k}^k$ is a sequence of integers such that

$$\binom{m+k}{m} - 1 = \sum_{i=1}^{j_k} \binom{n-1+l_i^k}{n-1} \quad (3.2)$$

one can construct a nontrivial (G, H) -relative invariant. Indeed in this case for $h \in H$, $\mathbf{x}h = \mathbf{y}$ due to $\frac{(\mathbf{x}h)^{\odot p}}{p!} = \frac{\mathbf{x}^{\odot p} h^{\odot p}}{p!}$ for any natural p , one has

$$\left(\frac{\mathbf{x}^{\odot j_1}}{j_1!}, \frac{\mathbf{x}^{\odot j_2}}{j_2!}, \dots, \frac{\mathbf{x}^{\odot j_k}}{j_k!} \right) \text{Diag} \left(\frac{h^{\odot j_1}}{j_1!}, \frac{h^{\odot j_2}}{j_2!}, \dots, \frac{h^{\odot j_k}}{j_k!} \right) = \left(\frac{\mathbf{y}^{\odot j_1}}{j_1!}, \frac{\mathbf{y}^{\odot j_2}}{j_2!}, \dots, \frac{\mathbf{y}^{\odot j_k}}{j_k!} \right).$$

Due to Lemma 2.3 b) application of (3.1) to this equality results in

$$\left(\left(\begin{array}{c} \partial \\ \frac{\partial^{\odot 2}}{2!} \\ \vdots \\ \frac{\partial^{\odot k}}{k!} \end{array} \right) \odot \left(\frac{\mathbf{x}^{\odot j_1}}{j_1!}, \frac{\mathbf{x}^{\odot j_2}}{j_2!}, \dots, \frac{\mathbf{x}^{\odot j_k}}{j_k!} \right) \right) \text{Diag} \left(\frac{h^{\odot j_1}}{j_1!}, \frac{h^{\odot j_2}}{j_2!}, \dots, \frac{h^{\odot j_k}}{j_k!} \right) =$$

$$\left(\begin{array}{ccccc} g & 0 & 0 & \dots & 0 \\ \frac{\partial^{\odot 2} g}{2!} & \frac{g^{\odot 2}}{2!} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial^{\odot k-1} \odot g}{k!} & * & * & \dots & \frac{g^{\odot k}}{k!} \end{array} \right) \left(\left(\begin{array}{c} \delta \\ \frac{\delta^{\odot 2}}{2!} \\ \vdots \\ \frac{\delta^{\odot k}}{k!} \end{array} \right) \odot \left(\frac{\mathbf{y}^{\odot j_1}}{j_1!}, \frac{\mathbf{y}^{\odot j_2}}{j_2!}, \dots, \frac{\mathbf{y}^{\odot j_k}}{j_k!} \right) \right).$$

By taking the determinant of both sides of this equality one has

$$\det(h)^{\sum_{i=1}^k \binom{n+j_i-1}{n}} f_k^\partial \langle \mathbf{x} \rangle = \det(g)^{\binom{m+k}{m+1}} f_k^\delta \langle \mathbf{y} \rangle,$$

where $f_k^\partial \langle \mathbf{x} \rangle$ stands for the $\det \left(\begin{array}{c} \partial \odot X \\ \frac{\partial^{\odot 2}}{2!} \odot X \\ \vdots \\ \frac{\partial^{\odot k}}{k!} \odot X \end{array} \right)$, $X = \left(\frac{\mathbf{x}^{\odot j_1}}{j_1!}, \frac{\mathbf{x}^{\odot j_2}}{j_2!}, \dots, \frac{\mathbf{x}^{\odot j_k}}{j_k!} \right)$, which means that

$f_k^\partial \langle \mathbf{x} \rangle \in C \langle \mathbf{x}; \partial \rangle$ is a relative invariant:

$$f_k^{g^{-1}\partial} \langle \mathbf{x}h \rangle = \det(g)^{-\binom{m+k}{m+1}} \det(h)^{\sum_{i=1}^k \binom{n+j_i-1}{n}} f_k^\partial \langle \mathbf{x} \rangle, \tag{3.3}$$

for any $g \in G$ and $h \in H$. Note that to get it we have used the equality

$$\det \left(\frac{h^{\odot p}}{p!} \right) = \det(h)^{\binom{n+p-1}{n}}$$

for any square matrix $h \in M_{n,n}(1, 1; F)$ and natural p [10].

Let us now justify the existence of (infinitely) many values of k for which equality (3.2) holds true for some $1 \leq l_1^k < l_2^k < \dots < l_{j_k}^k$.

At any fixed $n > 1$ the number $\binom{n-1+k}{n-1}$ is the value of $n-1$ -order polynomial (without a prime divisor)

$$p[t] = \binom{n-1+t}{n-1} = \frac{(t+n-1)(t+n-2)\dots(t+1)}{(n-1)!}$$

at $t = k$. The positive solution of Waring's problem for polynomials [6] guarantees the exists of a natural number s such that every, i.e. not only of the form $\binom{m+k}{m} - 1$, natural number, except finite of them, can be represented as a sum of values of this polynomial at s positive integers. In our case we do not even need j_k in (3.2) to be same (s) for different values of k . Therefore one can find a sequence $1 \leq k_1 < k_2 < \dots < k_{m+1}$ of values of k for each of which (3.2) type equality holds true. For each $k = k_i$ we can construct the corresponding relative invariant

$f_{k_i}^\partial \langle \mathbf{x} \rangle$ and have the sequence of (G, H) -relative invariants $f_{k_1}^\partial \langle \mathbf{x} \rangle, f_{k_2}^\partial \langle \mathbf{x} \rangle, \dots, f_{k_{m+1}}^\partial \langle \mathbf{x} \rangle$. By the use of them one can construct a matrix $M^\partial \langle \mathbf{x} \rangle$ consisting of columns

$$M_i^\partial \langle \mathbf{x} \rangle = \frac{-\partial \odot f_{k_{i+1}}^\partial \langle \mathbf{x} \rangle}{\binom{m+k_{i+1}}{m+1} f_{k_{i+1}}^\partial \langle \mathbf{x} \rangle} + \frac{\partial \odot f_{k_1}^\partial \langle \mathbf{x} \rangle}{\binom{m+k_1}{m+1} f_{k_1}^\partial \langle \mathbf{x} \rangle},$$

$i = 1, 2, \dots, m$. Due to the construction the obtained matrix $M^\partial \langle \mathbf{x} \rangle \in GL^\partial(m, C \langle \mathbf{x}; \partial \rangle)$ is a nonsingular matrix for which the equality $M^{g^{-1}\partial} \langle \mathbf{x}h \rangle = g^{-1}M^\partial \langle \mathbf{x} \rangle$ holds true. It implies that the commuting system of differential operators

$$\delta = (\delta^1(\partial, \mathbf{x}), \delta^2(\partial, \mathbf{x}), \dots, \delta^m(\partial, \mathbf{x})) = M^\partial \langle \mathbf{x} \rangle^{-1} \partial$$

is (G, H) -invariant. Therefore $C \langle \mathbf{x}; \partial \rangle^{(G, H)}$ is a finitely generated δ -differential field over C as a subfield of the finitely generated δ -differential field $C \langle \mathbf{x}, M^\partial \langle \mathbf{x} \rangle; \delta \rangle$ [5]. \square

This theorem does not provide a method to find a finite system of generators for the δ -differential field $C \langle \mathbf{x}; \partial \rangle^{(G, H)}$ over C . We hope that the obtained system of invariant differential operators can be used to reduce this problem to finding a system of generators of the field of (not differential) invariants of an action of H as we have done it for patches case in [13]. Note also that the **Theorem 3.4** holds true even if one changes G to any sub groupoid of $GL^\partial(m, F)$.

Remark 3.5. *It is said that the Waring's problem for polynomials is valid in the following more stronger form as well: If an integer valued at nonnegative integers polynomial $p[t]$ with positive leading coefficient has nor prime divisor then there exists a natural number s such that every natural number, except finite of them, can be represented as a sum of values of this polynomial at s distinct positive integers. In this case the existence of representation (3.2) for all k , except finite of them, is an immediate consequence of it.*

Example 3.6. *Now as an application of the above presented results let us consider two dimensional surfaces in \mathbf{R}^3 that is $m = 2, n = 3$ case, where H may be any subgroup of $GL(3, \mathbf{R})$. In this case one has the following two sequences.*

$$\left(\binom{m+k}{m} - 1 \right)_{k=1,2,3,\dots} = \{2, 5, 9, 14, 20, 27, 35, 44, 54, \dots\},$$

$$\binom{n-1+i}{n-1}_{i=1,2,3,\dots} = \{3, 6, 10, 15, 21, 28, 36, 45, 55, \dots\}.$$

Representing the elements of the first sequence as the sums of different elements of the second one, when it is possible, one has:

1. $9 = 3 + 6$, which implies that $k = 3, l_1^3 = 1, l_2^3 = 2$ and due to (3.3) for $f_3^\partial \langle \mathbf{x} \rangle = \det \left(\begin{array}{c} \partial \odot (\mathbf{x}; \frac{\mathbf{x}^{\odot 2}}{2!}) \\ \frac{\partial^{\odot 2}}{2!} \odot (\mathbf{x}; \frac{\mathbf{x}^{\odot 2}}{2!}) \\ \frac{\partial^{\odot 3}}{3!} \odot (\mathbf{x}; \frac{\mathbf{x}^{\odot 2}}{2!}) \end{array} \right)$ the equality $f_3^{g^{-1}\partial} \langle \mathbf{x}h \rangle = \det(g)^{-10} \det(h)^5 f_3^\partial \langle \mathbf{x} \rangle$ is valid.

2. $27 = 6 + 21$, which implies that $k = 6, l_1^6 = 2, l_2^6 = 5$ and due to (3.3) for

$$f_6^\partial \langle \mathbf{x} \rangle = \det \left(\begin{array}{c} \partial \odot \left(\frac{\mathbf{x}^{\odot 2}}{2!}, \frac{\mathbf{x}^{\odot 5}}{5!} \right) \\ \frac{\partial^{\odot 2}}{2!} \odot \left(\frac{\mathbf{x}^{\odot 2}}{2!}, \frac{\mathbf{x}^{\odot 5}}{5!} \right) \\ \frac{\partial^{\odot 3}}{3!} \odot \left(\frac{\mathbf{x}^{\odot 2}}{2!}, \frac{\mathbf{x}^{\odot 5}}{5!} \right) \\ \frac{\partial^{\odot 4}}{4!} \odot \left(\frac{\mathbf{x}^{\odot 2}}{2!}, \frac{\mathbf{x}^{\odot 5}}{5!} \right) \\ \frac{\partial^{\odot 5}}{5!} \odot \left(\frac{\mathbf{x}^{\odot 2}}{2!}, \frac{\mathbf{x}^{\odot 5}}{5!} \right) \\ \frac{\partial^{\odot 6}}{6!} \odot \left(\frac{\mathbf{x}^{\odot 2}}{2!}, \frac{\mathbf{x}^{\odot 5}}{5!} \right) \end{array} \right)$$

the equality $f_6^{g^{-1}\partial}(\mathbf{x}h) = \det(g)^{-56} \det(h)^{39} f_6^\partial(\mathbf{x})$ is true.

3. $44 = 6 + 10 + 28$, which implies that $k = 8, l_1^8 = 1, l_2^8 = 3, l_3^8 = 6$ and due to (3.3) for $f_8^\partial(\mathbf{x})$ the equality $f_8^{g^{-1}\partial}(\mathbf{x}h) = \det(g)^{-120} \det(h)^{67} f_8^\partial(\mathbf{x})$ holds true.

So in this case one has column vectors

$$M_1^\partial(\mathbf{x}) = \frac{\partial \odot f_6^\partial(\mathbf{x})}{-56 f_6^\partial(\mathbf{x})} - \frac{\partial \odot f_3^\partial(\mathbf{x})}{-10 f_3^\partial(\mathbf{x})}, \quad M_2^\partial(\mathbf{x}) = \frac{\partial \odot f_8^\partial(\mathbf{x})}{-120 f_8^\partial(\mathbf{x})} - \frac{\partial \odot f_3^\partial(\mathbf{x})}{-10 f_3^\partial(\mathbf{x})},$$

the second order square matrix $M^\partial(\mathbf{x})$ consisting of these two columns and the commuting system of invariant differential operators $\delta = (\delta_1, \delta_2)$ defined by $\delta = M^\partial(\mathbf{x})^{-1}\partial$.

So even in this simple case expressions for commuting system of invariant differential operators δ_1, δ_2 in terms of initial ∂_1, ∂_2 and \mathbf{x} are quite complicated (nontrivial). May be it is the reason that even in $m = 2, n = 3$ case the classical Fundamental Basis theorem didn't guarantee the existence of a commuting system of invariant differential operators with the needed properties.

4. Conclusion

In the paper a qualitative improvement of the Fundamental Basis theorem is provided by showing that the needed invariant system of differential operators can be chosen commuting. A method to construct the needed system of invariant differential operators is presented as well. The approach of the paper to the problem is pure algebraic and is applicable in more general settings than do geometric methods.

5. Acknowledgement

I would like to thank Greg Martin for informing me about paper [6]. This research is supported by the MOHE Malaysia under grant FRGS14-153-0394.

References

- [1] Olver P J 2009 *Diff. Geom. Appl.* **27** 230–239.
- [2] Olver P J 2011 *Comtemp. Math.* **54** 995–121.
- [3] Olver P J and Pohjanpelto J 2008 *Canadian J. Math.*, **60** 1336–1386
- [4] Weinstein A *Notices of the AMS*, **43** 744–752
- [5] Kolchin E R 1973 *Differential Algebra and Algebraic Groups* (New York: Academic Press)
- [6] Kamke E 1921 *Math. Ann.* **83** 85–112
- [7] Bekbaev U 1999 In book: *Proc. Pengintegrasian Teknologi dalam Sains Matematik* (Penang, USM Press) 58–64.
- [8] Bekbaev U 2005 *Bull. of Malays. Math. Sci. Soc.* **28**(2) 55–60
- [9] Bekbaev U 2013 *Inter. J. Pure Appl. Math* **84**(5) 463–476
- [10] Bekbaev U 2010 *arXiv:1010.2579*.
- [11] Bekbaev U 2012 In book: *Inter. Conf. Operator algebras and related topics* (Tashkent, National University of Uzbekistan Press)
- [12] Bekbaev U 2015 *arXiv: 1505.03623*.
- [13] Bekbaev U 2015 *AIP Conf. Proc.* **1660** 090048