

# On Genetic and Evolution Algebras

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**Abstract.** The genetic and evolution algebras generally are non-associative algebra. The concept of evolution and genetic algebras were introduced to answer the question what non-Mendelian genetics offers to mathematics. This paper we review some results of evolution and genetic algebras.

## 1. Introduction

Due to the interacting between Mendelian genetics and mathematics, the new object was introduced to mathematics in the 20s and 30s of the last century. The basic laws for inheritance, which are summarized as Mendel's Law of Segregation and Mendel's Law of Independent Assortment was established by Mendel. These laws were mathematically formulated by Serebrowsky [20], who was also the first to give an algebraic interpretation of the " $\times$ " sign, which indicated sexual reproduction. Later Glivenkov [8] used the notion of Mendelian algebras in his work. Also Kostitzin [10] independently introduced a "symbolic multiplication" to express Mendel's laws. In his several papers Etherington [4]-[6] introduced the formal language of abstract algebra to the study of the genetics. These algebras, in general, are non-associative.

However, in the beginning of the XX century in genetics there were discovered several examples of inheritances, where traits do not segregate in accordance with Mendel's laws. In the present day, non-Mendelian genetics is a basic language of molecular genetics. Non-Mendelian inheritance plays an important role in several disease processes. Naturally, the question arises: what non-Mendelian genetics offers to mathematics? The evolution algebras, introduced in [21] serves as the answer to this question.

The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebra; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with various branches of mathematics, such as graph theory, group theory, stochastic processes, mathematical physics etc. Since evolution algebras are not defined by identities, they can not belong to any well-known classes of non-associative algebras, as Lie, alternative and Jordan algebras.

The foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed, with pointers to some further research topics was given in book [21]. In mathematical genetics, a genetic algebra is a (possibly non-associative) algebra used to model inheritance in genetic. In application of genetics this algebra often has a basis corresponding to genetically different gametes, and the structure constant of the algebra encode the probabilities of producing offspring of various types. There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.) , whose investigation has



provided a number of significant contributions to theoretical population genetics [1, 12, 19, 22]. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called genetic. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. In the theory of non-associative algebras, particularly, in genetic algebras, the Lie algebra of derivations of a given algebra is one of the important tools for studying its structure. Note that problems of population genetics can be traced back to Bernstein's work [1] where evolution operators were studied. Such kind of operators are mostly described by quadratic stochastic operators (QSO) [12]. In [7, 14], it was given along self-contained exposition of the recent achievements and open problems in the theory of the QSO.

## 2. Some results in genetic and evolution algebras

In This section we give them main result in the genetic and evolution algebras.

**Definition 2.1.** [21] Let  $E$  be a vector space over a field  $K$  with defined multiplication  $\cdot$  and a basis  $\{e_1, e_2, \dots\}$  such that

$$e_i \cdot e_j = 0, \quad i \neq j,$$

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \quad i \geq 1,$$

then  $E$  is called evolution algebra and basis  $\{e_1, e_2, \dots\}$  is said to be natural basis.

From the above definition it follows that evolution algebras are commutative (therefore, flexible).

Let  $E$  be a finite dimensional evolution algebra with natural basis  $\{e_1, \dots, e_n\}$ , then

$$e_i \cdot e_i = \sum_{j=1}^n a_{ij} e_j, \quad 1 \leq i \leq n,$$

where remaining products are equal to zero.

The matrix  $A = (a_{ij})_{i,j=1}^n$  is called matrix of the algebra  $E$  in natural basis  $\{e_1, \dots, e_n\}$ .

**Definition 2.2.** [2] An element  $a$  of evolution algebra  $E$  is called nil if there exists  $n(a) \in \mathbb{N}$  such that  $(\dots \underbrace{(a \cdot a) \cdot a}_{n(a) \text{ times}}) \cdot a = 0$ . Evolution algebra  $E$  is called nil if any element of the

algebra is nil.

We introduce the following sequences:

$$E^{(1)} = E, \quad E^{(k+1)} = E^{(k)} E^{(k)}, \quad k \geq 1$$

$$E^{<1>} = E, \quad E^{<k+1>} = E^{<k>} E, \quad k \geq 1$$

$$E^1 = E, \quad E^k = \sum_{i=1}^{k-1} E^i E^{k-i}, \quad k \geq 1$$

Note that is not difficult to prove the following inclusions for  $k \geq 1$ :

$$E^{<k>} \subseteq E^k, \quad E^{(k+1)} \subseteq E^{2^k}.$$

Also, note that since  $E$  is commutative algebra we obtain  $E^k = \sum_{1 \leq i \leq k-i} E^i E^{k-i}$ .

**Definition 2.3.** [2] An evolution algebra  $E$  is called

- (i) solvable if there exists  $n \in \mathbb{N}$  such that  $E^{(n)} = 0$  and the minimal such number is called index of solvability;
- (ii) right nilpotent if there exists  $n \in \mathbb{N}$  such that  $E^{<n>} = 0$  and the minimal such number is called index of right nilpotency;
- (iii) nilpotent if there exists  $n \in \mathbb{N}$  such that  $E^n = 0$  and the minimal such number is called index of nilpotency.

**Theorem 2.4.** [2] The following statements are equivalent:

- a) The matrix of an evolution algebra  $E$  can be transformed by natural basis permutation to

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}; \quad (2.3)$$

- b) Evolution algebra  $E$  is right nilpotent algebra;
- c) Evolution algebra  $E$  is nil algebra.

The following theorem represents the criteria of nilpotency of finite dimensional evolution algebra.

**Theorem 2.5.** [2] Let  $E$  be an  $n$ -dimensional evolution algebra. Then  $E$  is nilpotent if and only if the matrix of evolution algebra  $A$  can be transformed by the natural basis permutation to form (2.3). Moreover, the index of nilpotency of evolution algebra  $E$  is not greater than  $2^{n-1} + 1$ .

**Corollary 2.6.** [2] For finite dimensional complex evolution algebra notions as nil, nilpotent and right nilpotent algebras are equivalent. However, the indexes of nility, right nilpotency and nilpotency do not coincide in general.

**Theorem 2.7.** [3] Any 2-dimensional non abelian complex evolution algebra  $E$  is isomorphic to one of the following pairwise non isomorphic algebras:

(i)  $\dim E^2 = 1$

- $E_1 : e_1 e_1 = e_1,$
- $E_2 : e_1 e_1 = e_1, \quad e_2 e_2 = e_1,$
- $E_3 : e_1 e_1 = e_1 + e_2, \quad e_2 e_2 = -e_1 - e_2,$
- $E_4 : e_1 e_1 = e_2.$

(ii)  $\dim E^2 = 2$

- $E_5 : e_1 e_1 = e_1 + a_2 e_2, \quad e_2 e_2 = a_3 e_1 + e_2, \quad 1 - a_2 a_3 \neq 0, \text{ where } E_5(a_2, a_3) \cong E'_5(a_3, a_2),$
- $E_6 : e_1 e_1 = e_2, \quad e_2 e_2 = e_1 + a_4 e_2, \text{ where for } a_4 \neq 0, \quad E_6(a_4) \cong E_6(a'_4) \Leftrightarrow \frac{a'_4}{a_4} = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3} \text{ for some } k = 0, 1, 2.$

Following [11] we consider a complex evolution algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with a basis  $\{e_1, e_2, \dots, e_n\}$  and the table of multiplications given by

$$\begin{cases} e_i \cdot e_i = a_i e_{\pi(i)}, & 1 \leq i \leq n, \\ e_i \cdot e_j = 0, & i \neq j, \end{cases}$$

where  $\pi$  is an element of the group of permutations  $S_n$ .

An evolution algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  is said to be *evolution algebra of permutations*.

by a *cycle permutation* we mean a permutation in which a part of symbols  $\{l_1, l_2, \dots, l_t\} \subseteq \{1, 2, \dots, n\}$  are cyclic permuted and the rest ones are stationary, i.e.,  $l_1 \rightarrow l_2 \rightarrow \dots \rightarrow l_t \rightarrow l_1$ , and we denote  $\pi = (l_1, l_2, \dots, l_t)$ .

**Example 2.8.** Consider the following evolution algebra:

$$E_n : \begin{cases} e_i \cdot e_i = e_{i+1}, & 1 \leq i \leq n-1, \\ e_n \cdot e_n = e_1, \\ e_i \cdot e_j = 0, & i \neq j. \end{cases}$$

Evidently, the algebra  $E_n$  is evolution algebra of permutations of the form  $E_{n,\pi}(1, 1, \dots, 1)$ , with  $\pi = (1, 2, 3, \dots, n)$ .

**Example 2.9.** Evolution algebra defined as follows:

$$EN_n : \begin{cases} e_i \cdot e_i = e_{i+1}, & 1 \leq i \leq n-1, \\ e_n \cdot e_n = 0, \\ e_i \cdot e_j = 0, & i \neq j, \end{cases}$$

is the algebra of permutations of the form  $E_{n,\pi}(1, 1, \dots, 1, 0)$  with  $\pi = (1, 2, 3, \dots, n)$ .

**Theorem 2.10.** [11] Let  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  be an evolution algebra of permutations with the following conditions:

- (i)  $a_i \neq 0$  for all  $i$  ( $1 \leq i \leq n$ ),
- (ii)  $\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_r$ , where  $\pi_1 = (l_1, l_2, \dots, l_{k_1})$ ,  $\pi_2 = (m_1, m_2, \dots, m_{k_2})$ ,  $\dots$ ,  $\pi_r = (p_1, p_2, \dots, p_{k_r})$  are independent cycles and  $k_1 + k_2 + \dots + k_r = n$ .

Then

$$E_{n,\pi}(a_1, a_2, \dots, a_n) \cong E_{k_1,\pi_1}(b_1, b_2, \dots, b_{k_1}) \oplus E_{k_2,\pi_2}(c_1, c_2, \dots, c_{k_2}) \oplus \dots \oplus E_{k_r,\pi_r}(d_1, d_2, \dots, d_{k_r}).$$

**Theorem 2.11.** [11] Any evolution algebra of permutation  $E_{n,\tau}(a_1, a_2, \dots, a_n)$  with  $\tau = (l_1, l_2, \dots, l_n)$  and condition  $a_i \neq 0$  for all  $i$  ( $1 \leq i \leq n$ ) is isomorphic to the algebra  $E_{n,\pi}(a_1, a_{\pi(1)}, \dots, a_{\pi^{n-1}(1)})$  with  $\pi = (1, 2, \dots, n)$ .

**Theorem 2.12.** [11] Any evolution algebra of permutation  $E_{n,\tau}(a_1, a_2, \dots, a_n)$  with  $\tau = (l_1, l_2, \dots, l_n)$  and condition  $a_i \neq 0$  for all  $i$  ( $1 \leq i \leq n$ ) is isomorphic to the algebra  $E_n$ .

**Theorem 2.13.** [11] Any evolution algebra of permutation  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with  $\pi = (l_1, l_2, \dots, l_n)$  and condition  $a_i = 0$  for some  $i \in \{1, 2, \dots, n\}$  is isomorphic to the algebra  $EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}$ .

**Theorem 2.14.** [11] An arbitrary evolution algebra of permutations  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  is isomorphic to a direct sum of algebras  $E_{p_1}, E_{p_2}, \dots, E_{p_s}, EN_{k_1}, EN_{k_2}, \dots, EN_{k_r}$ , i.e.,

$$E_{n,\pi}(a_1, a_2, \dots, a_n) \cong E_{p_1} \oplus E_{p_2} \oplus \dots \oplus E_{p_s} \oplus EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}.$$

**Definition 2.15.** Let  $\mathcal{A}$  be an algebra over  $\mathbb{K}$ . Assume that  $\mathcal{A}$  admits a basis  $\{e_1, \dots, e_n\}$ . Such that the multiplication constant  $P_{ij,k}$  with respect to this basis, are given by

$$e_i \cdot e_j = \sum_{k=1}^n P_{ij,k} e_k.$$

We say that  $\mathcal{A}$  is a genetic algebra if the multiplication constants  $P_{ij,k}$  satisfy

$$(i) \quad P_{ij,k} \geq 0$$

$$(ii) \quad \sum_{k=1}^n P_{ij,k} = 1.$$

In that case, the basis  $\{e_1, \dots, e_n\}$  is called a natural basis.

Note that some relations between evolution and cubic matrices have been investigated in [17, 18].

**Theorem 2.16.** [15] *Let  $\mathbb{A}$  be a genetic algebra generated by the basis  $e_1, e_2$  with structure constant  $(p_{ij,k})$ . Then  $\mathbb{A}$  is an evolution algebra with respect to new basis  $f_1, f_2$  if and only if one of the following condition satisfied:*

$$i) \ p_{21,1} = p_{12,1} \neq \frac{p_{11,1} + p_{22,1}}{2}$$

$$ii) \ p_{11,1} = p_{12,1} = p_{22,1}$$

**Theorem 2.17.** [16] *Let  $\mathbf{g}$  be a three dimensional genetic algebra generated with heredity coefficients  $\{p_{ij,k}\}$ . Then  $\mathbf{g}$  is an evolution algebra with respect to a new basis  $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$  if and only if the following conditions are satisfied:*

(i) *there are numbers  $k, l, m$  with  $k + l + m \neq 0$  such that*

$$\begin{aligned} k(p_{21,n} - p_{31,n}) + l(p_{22,n} - p_{32,n}) + m(p_{23,n} - p_{33,n}) &= 0, \\ k(p_{11,n} - p_{31,n}) + l(p_{12,n} - p_{32,n}) + m(p_{13,n} - p_{33,n}) &= 0, \quad n \in \{1, 2, 3\}; \end{aligned}$$

(ii) *the following two vectors*

$$\begin{aligned} (p_{21,1} - p_{23,1} - p_{31,1} + p_{33,1}, p_{21,2} - p_{23,2} - p_{31,2} + p_{33,2}, p_{21,3} - p_{23,3} - p_{31,3} + p_{33,3}), \\ (p_{22,1} - p_{23,1} - p_{32,1} + p_{33,1}, p_{22,2} - p_{23,2} - p_{32,2} + p_{33,2}, p_{22,3} - p_{23,3} - p_{32,3} + p_{33,3}) \end{aligned}$$

*are parallel;*

(iii) *the following two vectors*

$$\begin{aligned} (p_{11,1} - 2p_{13,1} + p_{33,1}, p_{11,2} - 2p_{13,2} + p_{33,2}, p_{11,3} - 2p_{13,3} + p_{33,3}), \\ (p_{12,1} - p_{13,1} - p_{32,1} + p_{33,1}, p_{12,2} - p_{13,2} - p_{32,2} + p_{33,2}, p_{12,3} - p_{13,3} - p_{32,3} + p_{33,3}) \end{aligned}$$

*are parallel;*

(iv) *there are two numbers  $s, t$  such that*

$$\begin{aligned} p_{11,n} + (t + s)p_{12,n} - (2 + t + s)p_{13,n} \\ + tsp_{22,n} - (s + t + 2st)p_{23,n} - (1 + s + t + st)p_{33,n} = 0, \quad n \in \{1, 2, 3\}. \end{aligned}$$

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