The Ganikhodjaev Model of ABO Blood Groups

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Abstract. In 2010, N. Ganikhodjaev proposed the models of ABO and Rh blood groups of Malaysian people. Based on some numerical simulations, it was showed that the evolution of ABO blood groups of Malaysian people has a unique stable equilibrium. In this paper, we analytically prove that the Ganikhodjaev model of ABO blood groups has a unique fixed point.

1. Introduction

A blood group provides an ideal opportunity for the study of human variation without cultural prejudice. It can be easily classified for many different genetically inherited blood typing systems. We rarely take blood types into consideration in selecting mates. A few people know their own type today and no one did prior to 1900. As a result, differences in blood type frequencies around the world are most likely due to other factors than social discrimination. **ABO** blood group is the classification of human blood based on the inherited properties of red blood cells (erythrocytes) as determined by the presence or absence of the alleles **A** and **B**, which are carried on the surface of the red cells. Persons may thus have group **A** (which is carrying only **A** allele), group **B** (which is carrying only **B** allele), group **O** (which is neither carrying **A** allele nor **B** allele), and group **AB** (which is carrying both **A** and **B** alleles). The **ABO** blood group system has been discovered by the Austrian scientist Karl Landsteiner, who found three different blood types in 1900. He was awarded the Nobel Prize in Medicine in 1930 for his work. When we donate blood or have surgery, a small sample is usually taken in advance for at least **ABO** and **Rh** systems typing. We have learned a good deal about how common each of the **ABO** blood types is around the world. It is quite clear that the distribution patterns are complex. About 21% of all people in the world share the **A** blood group. The highest frequencies of **A** are found in small, unrelated populations, especially the Blackfoot Indians of Montana (30-35%), the Australian Aborigines (many groups are 40-53%). Overall in the world, the **B** blood group is the rarest **ABO** blood group. Only 16% of humanity have it. Note that it is highest in Central Asia and lowest among the indigenous peoples of the America and Australia. The **O** blood group is very common around the world. About 63% of humans share it. The group **O** is particularly high in frequency among the indigenous populations of Central and South America, where it approaches 100%. The rest of people in the world are sharing **AB** blood group.

A quadratic stochastic operator which was first studied by Bernstein [1] is a primary source for investigations of dynamical properties of population genetics [6, 7] in which it describes a distribution of a species for the next generation if the current distribution of these species was given. It also has a fascinating application in physics, economics (see [17]), and control system (see [14]). For example, in the paper [3], N. Ganikhodjaev described a mathematical model of the transmission of **ABO** blood groups as the quadratic stochastic operator. During the period 2007–2008, he collected the current blood group distributions of Malaysian people from more than 10,000 randomly chosen families in the states of Pahang and Kuala Lumpur. According to statistics, the current blood distribution in Malaysia was as follows: 20% from **A**; 29.3% from **B**; 8.4% from **AB**; 42.3% from **O**. Based on some numerical simulations, the future ABO blood group distribution of Malaysian people was predicted [3]. Later on this work was continued in the paper [4]. However, only numerical results were presented in both papers. In this paper, we present the analytical proof of the main result of [3].

2. Preliminaries

Let $\|\mathbf{x}\|_1 = \sum\limits_{k=1}^{m} |x_k|$ be a norm of a vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$. We say that $\mathbf{x} \geq 0$ (resp. $\mathbf{x} > 0$) if $x_k \geq 0$ (resp. $x_k > 0$) for all $k = \overline{1,m}$. Let $\mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 = 1,\ \mathbf{x} \geq 0\}$ be the $(m-1)-$dimensional standard simplex. An element of the simplex $\mathbb{S}^{m-1}$ is called *a stochastic vector*. Recall that a square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is called *stochastic* if every row is a stochastic vector. A square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is called *positive* if $p_{ij} > 0,\ \forall\ i,j = \overline{1,m}$. A cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is called *stochastic* if $\sum\limits_{k=1}^{m} p_{ijk} = 1,\ p_{ijk} \geq 0,\ \forall i,j,k = \overline{1,m}$. Every cubic stochastic matrix is associated with a quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ as follows

$$(\mathcal{Q}(\mathbf{x}))_k = \sum_{i,j=1}^{m} x_i x_j p_{ijk}, \quad \forall\ k = \overline{1,m}. \tag{2.1}$$

Let $\mathbf{Fix}(\mathcal{Q}) = \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathcal{Q}(\mathbf{x}) = \mathbf{x}\}$ be a fixed point set. Due to Brouwer's theorem, $\mathbf{Fix}(\mathcal{Q}) \neq \emptyset$. A fixed point of the quadratic stochastic operator is an equilibrium of the system.

In general, the main problem in the nonlinear operator theory is to study the asymptotic behavior of the nonlinear operator. This problem was not fully finished even in the class of quadratic stochastic operators (for more details see [5]). In [10], a special class of the nonlinear operators was studied as a generalization of a logistic mapping into the higher dimension. A fixed point set and an omega limiting set of the quadratic stochastic operators defined on the finite dimensional simplex were deeply studied in the series of papers [2, 4, 12, 15]. Ergodicity and chaotic dynamics of the quadratic stochastic operators on the finite dimensional simplex were studied in [9, 11, 13]. In [5, 8], it was given a long self-contained exposition of the recent achievements and open problems in the theory of the quadratic stochastic operators.

We recall some necessary notions and concepts which will be used in this paper.

The important tool to study a system of polynomial equations is a resultant.

Let us consider two polynomials

$$f(x) = c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4. \tag{2.2}$$
$$g(x) = d_0x^3 + d_1x^2 + d_2x + d_3. \tag{2.3}$$

The Sylvester matrix, denoted by $\mathrm{Syl}(f,g)$, is defined as follows:

$$\mathrm{Syl}(f,g,x) = \begin{bmatrix} c_0 & 0 & 0 & d_0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & d_1 & d_0 & 0 & 0 \\ c_2 & c_1 & c_0 & d_2 & d_1 & d_0 & 0 \\ c_3 & c_2 & c_1 & d_3 & d_2 & d_1 & d_0 \\ c_4 & c_3 & c_2 & 0 & d_3 & d_2 & d_1 \\ 0 & c_4 & c_3 & 0 & 0 & d_3 & d_2 \\ 0 & 0 & c_4 & 0 & 0 & 0 & d_3 \end{bmatrix}$$

The resultant of two polynomials f and g, denoted by $\mathrm{Res}(f, g)$, is the determinant of the Sylvester matrix $\mathrm{Res}(f, g) = \det(\mathrm{Syl}(f, g))$. After simple calculation, we have the following

$$\begin{aligned}\det(\mathrm{Syl}(f,g)) = {} & c_0c_4^2d_1^4 - 4c_0c_4^2d_0d_1^2d_2 + 4c_0c_4^2d_0^2d_1d_2 + 2c_0c_4^2d_0^2d_2^2 + c_0c_3c_4d_0d_1^2d_3 \\
& - 5c_0c_3c_4d_0^2d_2d_3 + 4c_0^2c_4d_0d_2d_3^2 - 5c_0c_1c_4d_0d_1d_3^2 - 4c_0^2c_4d_1d_2^2d_3 \\
& + 2c_0^2c_4d_1^2d_3^2 + 3c_0c_1c_4d_1^2d_2d_3 - 2c_0c_2c_4d_1^3d_3 + c_0^2c_4d_2^4 - c_0c_1c_4d_1d_2^3 \\
& + c_0c_2c_4d_1^2d_2^2 - c_0c_3c_4d_1^3d_2 + 3c_0c_3c_4d_0d_1d_2^2 - 2c_0c_2c_4d_0d_2^3 \\
& + 4c_0c_2c_4d_0d_1d_2d_3 - 3c_0c_2c_4d_0^2d_3^2 - c_0^2c_3d_2^3d_3 + 3c_0^2c_3d_1d_2d_3^2 - 3c_0^2c_3d_0d_3^3 \\
& + c_0c_1c_3d_1d_2^2d_3 - 2c_0c_1c_3d_1^2d_3^2 - c_0c_2c_3d_1^2d_2d_3 + c_0c_2c_3d_0d_1d_3^2 + c_0c_3^2d_1^3d_3 \\
& - 3c_0c_3^2d_0d_1d_2d_3 + 3c_0c_3^2d_0^2d_3^2 - c_0c_1c_3d_0d_2d_3^2 + 2c_0c_2c_3d_0d_2^2d_3 + c_0^3d_3^4 \\
& - 2c_0^2c_2d_1d_3^3 - c_0^2c_1d_2d_3^3 + c_0c_1^2d_1d_3^3 + c_0^2c_2d_2^2d_3^2 - c_0c_1c_2d_1d_2d_3^2 \\
& + 3c_0c_1c_2d_0d_3^3 + c_0c_2^2d_1^2d_3^2 - 2c_0c_2^2d_0d_2d_3^2 + c_0c_1c_4d_0d_2^2d_3 + c_1^2c_4d_0d_2^3 \\
& - 3c_1^2c_4d_0d_1d_2d_3 + 3c_1^2c_4d_0^2d_3^2 - c_1c_2c_4d_0d_1d_2^2 + 2c_1c_2c_4d_0d_1^2d_3 + c_1c_3c_4d_0d_1^2d_2 \\
& - c_1c_3c_4d_0^2d_1d_3 + 3c_1c_4^2d_0^2d_1d_2 - 3c_1c_4^2d_0^3d_3 + c_1c_2c_4d_0^2d_2d_3 - 2c_1c_3c_4d_0^2d_2^2 \\
& - c_1c_4^2d_0d_1^3 - c_1c_2^2d_0d_1d_3^2 + c_1c_2c_3d_0d_1d_2d_3 + 2c_1^2c_3d_0d_1d_3^2 - c_1c_3^2d_0d_1^2d_3 \\
& - 3c_1c_2c_3d_0^2d_3^2 + c_1^2c_2d_0d_2d_3^2 - c_1^2c_3d_0d_2^2d_3 + 2c_1c_3^2d_0^2d_2d_3 - c_1^3d_0d_3^3 + c_2^2c_4d_0^2d_2^2 \\
& - c_2^2c_4d_0^2d_1d_3 - c_2c_3c_4d_0^2d_1d_2 + 3c_2c_3c_4d_0^3d_3 + c_2c_4^2d_0^2d_1^2 - 2c_2c_4^2d_0^3d_2 + c_2^3d_0^2d_3^2 \\
& - c_2^2c_3d_0^2d_2d_3 + c_2c_3^2d_0^2d_1d_3 + c_3^2c_4d_0^3d_2 - c_3c_4^2d_0^3d_1 - c_3^3d_0^3d_3 + c_4^3d_0^4.\end{aligned} \tag{2.4}$$

Theorem 2.1 ([16]). *The resultant of two polynomials with coefficients in an integral domain is zero if and only if they have a common root in an algebraically closed field containing the coefficients.*

We now recall Sturm's theorem which expresses the number of distinct real roots of a polynomial located in an interval in terms of the number of changes of signs of the values of the Sturm's sequence at the end points of the interval.

The *Sturm sequence* of a polynomial $p(x)$ is the following sequence of polynomials of decreasing degree:

$$\begin{aligned} p_0(x) &:= p(x), \\ p_1(x) &:= p'(x), \\ p_2(x) &:= -\mathrm{rem}(p_0(x), p_1(x)) = p_1(x)q_0(x) - p_0(x), \\ p_3(x) &:= -\mathrm{rem}(p_1(x), p_2(x)) = p_2(x)q_1(x) - p_1(x), \\ &\ \vdots \\ 0 &= -\mathrm{rem} p_m(x), p_{m-1}(x). \end{aligned}$$

where $\mathrm{rem}(p_i, p_j)$ and q_i are the remainder and the quotient of the polynomial long division of p_i by p_j and m is the minimal number of polynomial divisions (never greater than $\deg(p)$) needed to obtain a zero remainder. That is, successively take the remainders with polynomial division and change their signs. Since $\deg(p_{i+1}) < \deg(p_i)$ for $0 \le i < m$, the algorithm terminates. The final polynomial, p_m, is the greatest common divisor of p and its derivative. If p is square free, it shares no roots with its derivative, hence p_m will be a non-zero constant polynomial. A sequence $p_0,\ p_1,\ p_2, \ldots,\ p_m$ is called *the canonical Sturm chain.* Let $\sigma(\xi)$ be the number of sign changes (ignoring zeroes) in the sequence $p_0(\xi), p_1(\xi), p_2(\xi), \ldots, p_m(\xi)$.

Theorem 2.2 (Sturm's Theorem, [16]). *The number of real zeros of $p(x)$ in (α, β) is $\sigma(\alpha) - \sigma(\beta)$.*

3. The Ganikhodjaev Model of ABO Blood Groups

During the period 2007-2008, Professor N. Ganikhodjaev collected the current blood group distributions of Malaysian people from more than 10 000 randomly chosen families from the states of Pahang and Kuala Lumpur. According to statistics, the current blood distribution in Malaysia was as follows: 20% from **A**; 29.3% from **B**; 8.4% from **AB**; 42.3% from **O**. The mathematical model of the transmission of **ABO** blood groups was described as the quadratic stochastic operator acting on 3D simplex (see [3, 4]).

$$\begin{cases} x'_A = 0.91x_A^2 + 0.6x_Ax_B + 0.92x_Ax_{AB} + 0.98x_Ax_O + 0.01x_B^2 \\ \qquad +0.20x_Bx_{AB} + 0.02x_Bx_O + 0.09x_{AB}^2 + 0.38x_{AB}x_O + 0.01x_O^2 \\ x'_B = 0.01x_A^2 + 0.76x_Ax_B + 0.32x_Ax_{AB} + 0.02x_Ax_O + 0.94x_B^2 \\ \qquad +1.26x_Bx_{AB} + 1.06x_Bx_O + 0.06x_{AB}^2 + 0.38x_{AB}x_O + 0.01x_O^2 \\ x'_{AB} = 0.01x_A^2 + 0.50x_Ax_B + 0.72x_Ax_{AB} + 0.02x_Ax_O + 0.01x_B^2 \\ \qquad +0.44x_Bx_{AB} + 0.02x_Bx_O + 0.84x_{AB}^2 + 0.40x_{AB}x_O + 0.01x_O^2 \\ x'_O = 0.07x_A^2 + 0.14x_Ax_B + 0.04x_Ax_{AB} + 0.98x_Ax_O + 0.04x_B^2 \\ \qquad +0.10x_Bx_{AB} + 0.90x_Bx_O + 0.01x_{AB}^2 + 0.84x_{AB}x_O + 0.97x_O^2 \end{cases}$$

Theorem 3.1. *The quadratic stochastic operator given above has a unique fixed point which belongs to* $\Omega = \{x \in S^3 : 0.08 < x_A < 0.15, 0.4 < x_B < 0.6, 0.03 < x_{AB} < 0.09, 0.2 < x_O < 0.4\}$.

Proof. For the convenience, we use $x = (x_A, x_B, x_{AB}, x_O) = (x_1, x_2, x_3, x_4)$. It is easy to see that any fixed point of the quadratic stochastic operator given above lies inside of the simplex, i.e., $x_1, x_2, x_3, x_4 > 0$. We now want to show that the following system of equations

$$\begin{aligned} x_1 = {} & 0.91x_1^2 + 0.01x_2^2 + 0.09x_3^2 + 0.01x_4^2 + 0.6x_1x_2 + 0.92x_1x_3 + 0.98x_1x_4 + 0.2x_2x_3 \\ & + 0.02x_2x_4 + 0.38x_3x_4. \\ x_2 = {} & 0.01x_1^2 + 0.94x_2^2 + 0.06x_3^2 + 0.01x_4^2 + 0.76x_1x_2 + 0.32x_1x_3 + 0.02x_1x_4 + 1.26x_2x_3 \\ & + 1.06x_2x_4 + 0.38x_3x_4. \\ x_3 = {} & 0.01x_1^2 + 0.01x_2^2 + 0.84x_3^2 + 0.01x_4^2 + 0.5x_1x_2 + 0.72x_1x_3 + 0.02x_1x_4 + 0.44x_2x_3 \\ & + 0.02x_2x_4 + 0.4x_3x_4. \\ x_4 = {} & 0.07x_1^2 + 0.04x_2^2 + 0.01x_3^2 + 0.97x_4^2 + 0.14x_1x_2 + 0.04x_1x_3 + 0.98x_1x_4 + 0.10x_2x_3 \\ & + 0.9x_2x_4 + 0.84x_3x_4. \end{aligned}$$

has a unique solution. Since $x_4 = 1 - x_1 - x_2 - x_3$, we obtain that

$$\begin{cases} x_1 = -0.06x_1^2 - 0.28x_3^2 - 0.38x_1x_2 - 0.42x_1x_3 - 0.18x_2x_3 + 0.96x_1 + 0.36x_3 + 0.01 \\ x_2 = -0.11x_2^2 - 0.31x_3^2 - 0.30x_1x_2 - 0.06x_1x_3 - 0.16x_2x_3 + 1.04x_2 + 0.36x_3 + 0.01 \\ x_3 = 0.45x_3^2 - 0.48x_1x_2 + 0.32x_1x_3 + 0.04x_2x_3 + 0.38x_3 + 0.01 \end{cases} . \tag{3.1}$$

Let $x_1 = x$ be a variable and $x_2 = a,\ x_3 = b$ be parameters.
It follows from the first equation that

$$x^2 + \left(\frac{19}{3}a + 7b + \frac{2}{3}\right)x + \frac{14}{3}b^2 - 6b + 3ab - \frac{1}{6} = 0 \tag{3.2}$$

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and it follows from the second and third equations that

$$x = \frac{0.11a^2 + 0.31b^2 + 0.16ab - 0.04a - 0.36b - 0.01}{-0.30a - 0.06b}, \tag{3.3}$$

$$x = \frac{-0.45b^2 + 0.62b - 0.04ab - 0.01}{0.48a + 0.32b}. \tag{3.4}$$

By equalizing equations (3.3) and (3.4), we get that

$$0.0528a^3 + 0.0722b^3 + 0.1a^2b + 0.0626ab^2 - 0.0192a^2 - 0.078b^2 + 0.0004ab - 0.0078a - 0.0038 = 0. \tag{3.5}$$

By substituting (3.4) into (3.2), we obtain that

$$\frac{778}{1875}a^2b^2 - 0.7368ab^3 - \frac{9892}{30000}b^4 + 1.2944ab^2 + 0.1204b^3 + 0.5696a^3b + 0.4896a^2b + \frac{353}{3750}ab + 0.4862b^2 - 0.0688a^2 - 0.0032a - \frac{109}{7500}b + 0.0001 = 0. \tag{3.6}$$

Consequently, the system of equations (3.1) has a solution (x_1, x_2, x_3) if and only if the system of equations (3.5) and (3.6) must have a solution (a, b) in which $x_2 = a,\ x_3 = b$ and x is defined by (3.4). Note that we are looking for all possible solutions (a, b) of the system of equations (3.5) and (3.6) in which $0 < a, b < 1$, $0 < a + b < 1$ such that

$$0 < \frac{-0.45b^2 + 0.62b - 0.04ab - 0.01}{0.48a + 0.32b} < 1, \tag{3.7}$$

$$0 < a + b + \frac{-0.45b^2 + 0.62b - 0.04ab - 0.01}{0.48a + 0.32b} < 1. \tag{3.8}$$

We define two polynomials

$$h_1(y) = \alpha_0 y^4 + \alpha_1 y^3 + \alpha_2 y^2 + \alpha_3 y + \alpha_4, \tag{3.9}$$

$$h_2(y) = \beta_0 y^3 + \beta_1 y^2 + \beta_2 y + \beta_3 \tag{3.10}$$

where

$$\alpha_0 = -\frac{9892}{30000},\ \alpha_1 = -0.7368a + 0.1204,\ \alpha_2 = \frac{1}{3}1.2448a^2 + 1.2944a + 0.4862,$$
$$\alpha_3 = 0.5696a^3 + 0.4896a^2 + \frac{1}{3}0.2824a - \frac{1}{3}0.0436,\ \alpha_4 = -0.0688a^2 - 0.0032a + 0.0001,$$
$$\beta_0 = 0.0722,\ \beta_1 = 0.0626a - 0.078,\ \beta_2 = 0.1a^2 + 0.0004a - 0.0038,$$
$$\beta_3 = 0.0528a^3 - 0.0192a^2 - 0.0078a$$

THE SIMPLE OBSERVATION: *a pair* (a, b) *is a solution of the system of equations* (3.5) *and* (3.6) *if and only if* $y = b$ *is a common root of the two polynomials defined by* (3.9) *and* (3.10).

Due to Theorem 2.1, two polynomials (3.9) and (3.10) have a common root if and only if their resultant is zero. Let us calculate the resultant of two polynomials (3.9) and (3.10):

$$\begin{aligned} p(a) := \mathrm{Res}(h_1, h_2) = &-0.000006886997576849 25a^{12} - 0.0000227411808131865a^{11} \\ &- 0.0000366739070389 2109a^{10} - 0.0000407570501403 9745a^9 \\ &- 0.0000286641071126 1251a^8 - 0.0000066362582593 9163a^7 \\ &+ 0.0000058444123789 1252a^6 + 0.0000052142130101 7348a^5 \\ &+ 0.0000016918694594 8385a^4 + 0.0000002498070287 0395a^3 \\ &+ 0.0000000151377298 5017a^2 + 0.0000000002489398 5432a \\ &+ 0.0000000000012153 5388. \end{aligned} \tag{3.11}$$

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We now want to show by means of Sturm's Theorem that p has a unique root in $[0, 1]$. Let us calculate the canonical Sturm chain of the polynomial (3.11)

$$
\begin{aligned}
p_0(a) \quad &= -0.0000068869975768492 5a^{12} - 0.0000227411808131865a^{11} \\
&\quad -0.0000366739070389210 9a^{10} - 0.0000407570501403974 5a^{9} \\
&\quad -0.0000286641071126125 1a^{8} - 0.0000066362582593916 3a^{7} \\
&\quad +0.0000058444123789125 2a^{6} + 0.0000052142130101734 8a^{5} \\
&\quad +0.0000016918694594838 5a^{4} + 0.0000002498070287039 5a^{3} \\
&\quad +0.0000000151377298501 7a^{2} + 0.0000000002489398543 2a \\
&\quad +0.00000000000121535388,
\end{aligned}
$$

$$
\begin{aligned}
p_1(a) \quad &= -0.000082643970922191a^{11} - 0.000250152988945051 5a^{10} \\
&\quad -0.000366739070389210 9a^{9} - 0.000366813451263577 05a^{8} \\
&\quad -0.000229312856900900 08a^{7} - 0.000046453807815741 41a^{6} \\
&\quad +0.000035066474273475 18a^{5} + 0.000026071065050867 35a^{4} \\
&\quad +0.000006767477837935 44a^{3} + 0.000000749421086111 86a^{2} \\
&\quad +0.0000000302754597003 4a + 0.0000000002489398543 3,
\end{aligned}
$$

$$
\begin{aligned}
p_2(a) \quad &= 0.0000003760914207604 9a^{10} + 0.0000017796154893465 7a^{9} \\
&\quad +0.0000011433497067346 3a^{8} - 0.0000024932363898563a^{7} \\
&\quad -0.0000039874325566579 4a^{6} - 0.0000022375193879532 4a^{5} \\
&\quad -0.0000005300806914382 9a^{4} - 0.0000000321710967232 7a^{3} \\
&\quad +0.0000000045701048709 8a^{2} + 0.0000000004660478548 2a \\
&\quad +0.00000000000449305429,
\end{aligned}
$$

$$
\begin{aligned}
p_3(a) \quad &= 0.000782250250513829 88a^{9} + 0.001343058777970894 79a^{8} \\
&\quad +0.000171405609584722 93a^{7} - 0.000955806815741985 41a^{6} \\
&\quad -0.000756899651829977 32a^{5} - 0.000217603198038689 65a^{4} \\
&\quad -0.000019825047887905 18a^{3} + 0.000000860416105221 8a^{2} \\
&\quad +0.0000001433480411951 1a + 0.0000000014344407361 4,
\end{aligned}
$$

$$
\begin{aligned}
p_4(a) \quad &= 0.0000000885867246882 29a^{8} + 0.0000002282160314564 09a^{7} \\
&\quad +0.0000002238055503526 12a^{6} + 0.0000001035748615055 88a^{5} \\
&\quad +0.0000000205126111779 95a^{4} + 0.0000000003847703357 18a^{3} \\
&\quad -0.0000000003253984063 21a^{2} - 0.0000000000257570454 72a \\
&\quad -0.00000000000241378479,
\end{aligned}
$$

$$
\begin{aligned}
p_5(a) \quad &= 0.000073249407132431 83a^{7} + 0.000172250020077575 11a^{6} \\
&\quad +0.000152143453017153 62a^{5} + 0.000065358416717023 26a^{4} \\
&\quad +0.000014032170620306 56a^{3} + 0.000001381148518325 99a^{2} \\
&\quad +0.0000000499558486244 4a + 0.0000000003970537055 8,
\end{aligned}
$$

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$$\begin{aligned}
p_6(a) &= 0.0000000698912522464a^6 + 0.0000001680100892789 8a^5 \\
&\quad +0.0000001421342356138 3a^4 + 0.0000000509765068198 8a^3 \\
&\quad +0.0000000076102704340 1a^2 + 0.0000000003980860314 1a \\
&\quad +0.0000000000034924495 9,
\end{aligned}$$

$$\begin{aligned}
p_7(a) &= -0.0000123931750673799 4a^5 - 0.0000197268714063645 1a^4 \\
&\quad -0.0000088516672091228 7a^3 - 0.0000013812638301889 2a^2 \\
&\quad -0.0000000681256611899 3a - 0.0000000005885711347 9,
\end{aligned}$$

$$\begin{aligned}
p_8(a) &= -0.0000000018666281114 6a^4 - 0.0000000026464191697 1a^3 \\
&\quad -0.0000000008999156814 5a^2 - 0.0000000000827526367 7a \\
&\quad -0.0000000000000796805 7,
\end{aligned}$$

$$\begin{aligned}
p_9(a) &= -0.0000001804214420304a^3 - 0.0000002077757697609 5a^2 \\
&\quad -0.0000000327635988931 9a - 0.0000000003319291565 9,
\end{aligned}$$

$$\begin{aligned}
p_{10}(a) &= -0.0000000000111586525 5a^2 - 0.0000000000108950655 8a \\
&\quad -0.0000000000001171512 6, \\
p_{11}(a) &= 0.0000000000000407817 6a + 0.0000000000000004099 9 \\
p_{12}(a) &= 0.0000000000000087490 9.
\end{aligned}$$

We now calculate the number of sign changes $\sigma(0)$ and $\sigma(1)$ in the sequences $\{p_0(0),\ p_1(0),\ p_2(0), \cdots,\ p_{12}(0)\}$ and $\{p_0(1),\ p_1(1),\ p_2(1), \cdots,\ p_{12}(1)\}$, respectively.

By simple calculation

$$\begin{aligned}
p_0(0) &= 0.0000000000012153538 9, \\
p_1(0) &= 0.0000000002489398543 3, \\
p_2(0) &= 0.0000000000044930542 9, \\
p_3(0) &= 0.0000000014344407361 4, \\
p_4(0) &= -0.0000000000024137847 9, \\
p_5(0) &= 0.0000000003970537055 8, \\
p_6(0) &= 0.0000000000034924495 9, \\
p_7(0) &= -0.0000000005885711347 9, \\
p_8(0) &= -0.0000000000000796805 7, \\
p_9(0) &= -0.0000000003319291565 9, \\
p_{10}(0) &= -0.0000000000001171512 6, \\
p_{11}(0) &= 0.0000000000000004099 9, \\
p_{12}(0) &= 0.0000000000000087490 9,
\end{aligned}$$

we then obtain that $\sigma(0) = 4$.

Again, by simple calculation

$$
\begin{aligned}
p_0(1) &= -0.00012934381117902623, \\
p_1(1) &= -0.00127343118358872744, \\
p_2(1) &= -0.00000597634286000726, \\
p_3(1) &= 0.00034758512315804309, \\
p_4(1) &= 0.00000664729152686279, \\
p_5(1) &= 0.00047846496898514639, \\
p_6(1) &= 0.0000004390239328741, \\
p_7(1) &= -0.00004242169174538096, \\
p_8(1) &= -0.00000000549651240509, \\
p_9(1) &= -0.00000042129273984113, \\
p_{10}(1) &= -0.00000000002217086939, \\
p_{11}(1) &= 0.00000000000004119175, \\
p_{12}(1) &= 0.00000000000000874909,
\end{aligned}
$$

we then obtain that $\sigma(1) = 3$.

Thus, due to Sturm's Theorem 2.2, the number of roots of the polynomial (3.11) in $[0, 1]$ is $\sigma(0) - \sigma(1) = 4 - 3 = 1$. Thus, the polynomial (3.11) has a unique root a_0 in $[0, 1]$ which belongs to the interval $(0.4, 0.6)$. Approximate value of a_0 can be calculated by using some software like MAPLE. Moreover, b is a common root of the polynomials (3.9) and (3.10) such that the inequalities (3.7) and (3.8) are satisfied. Some calculations shows that the inequalities (3.7) and (3.8) are satisfied for only one common root of the polynomials (3.9) and (3.10). In other words, the inequalities (3.7) and (3.8) are satisfied for a unique pair (a_0, b_0). By means of equation (3.4), we can find a unique value of x. It means that the system of equations (3.1) has a unique solution (x_0, a_0, b_0). Therefore, a point $(x_0, a_0, b_0, 1 - x_0 - a_0 - b_0)$ is a unique fixed point of the quadratic stochastic operator. This completes the proof $\square$

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