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On Translation Invariant Quantum Markov Chains associated with Ising-XY models on a Cayley tree

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Abstract. In this paper, we consider the Ising-XY model with competing interactions on the Cayley tree of order two. This model can be seen as a non-commutative (i.e. J-XY-interactions on next-neighbor vertices) perturbation of the classical Ising model on the Cayley tree. For the considered model we establish the existence of three translation-invariant quantum Markov chains. We notice that if the XY-interactions vanish, i.e. $J = 0$, then one gets the Ising model. If the classical Ising model vanishes in the considered model, then we obtain XY-model for which it turns out there exists only one translation invariant QMC.

1. Introduction

In [1] a quantum analogues of Markov chains were constructed and they were called by quantum Markov chains (QMC) which are defined on infinite tensor product algebras. The reader is referred to [5, 17] and the references cited therein, for recent developments of the theory and the applications.

In [3]-[6],[8, 19] it was attempted to construct quantum analogues of classical Markov fields. In [7] it has been proposed a definition of quantum Markov states and chains, which extend all known ones. We point out that one of the basic open problems in quantum probability is the construction of a theory of quantum Markov fields, that is quantum process with multi-dimensional index set. This program concerns the generalization of the theory of Markov fields (see [15],[18])) to non- commutative setting, naturally arising in quantum statistical mechanics and quantum field theory.

First attempts to investigate QMC over such trees was done in [14], such studies were related to investigation of thermodynamic limit of valence-bond-solid models on a Cayley tree [16]. The mentioned considerations naturally suggest the study of the following problem: the extension to fields of the notion of generalized QMC. In [13, 12], we have introduced a hierarchy of notions of Markovianity for states on discrete infinite tensor products of C^*–algebras and for each of these notions we constructed some explicit examples. In [21, 20, 24, 25] noncommutative

extensions of classical Markov fields, associated with Ising and Potts models on a Cayley tree, were investigated.

In this paper, we consider the Ising-XY model with competing interactions on the Cayley tree of order two. This model can be seen as a non-commutative (i.e. J-XY-interactions on next-neighbor vertices) perturbation of the classical Ising model on the Cayley tree. For the considered model we are going to establish the existence of translation invariant quantum Markov chains (see [9]). We notice that if the XY-interactions vanish, i.e. $J = 0$, then one gets the Ising model for which the corresponding QMC has been studied in [10], and shown the existence of the phase transition in the sense of [9]. We point out that if one considers the Ising type next neighbor interactions, then the corresponding QMC has been investigated in [22, 23]. In the present paper, we establish the existence (under some conditions) of three translation invariant QMC for the considered model. If the classical Ising model vanishes in the considered model, then we obtain XY-model for which it turns out there exists only one translation invariant QMC.

2. Preliminaries

Let $\Gamma_+^k = (L, E)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (i.e. each vertex of Γ_+^k has exactly $k+1$ edges, except for the root x^0, which has k edges). Here L is the set of vertices and E is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l =< x, y >$ if there exists an edge connecting them. A collection of the pairs $< x, x_1 >, \ldots, < x_{d-1}, y >$ is called a *path* from the point x to the point y. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

Recall a coordinate structure in Γ_+^k: every vertex x (except for x^0) of Γ_+^k has coordinates $(i_1, \ldots, i_n)$, here $i_m \in \{1, \ldots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \ldots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice (see Fig. 1).

Let us set

$$W_n = \{x \in L \,:\, d(x, x_0) = n\}, \qquad \Lambda_n = \bigcup_{k=0}^{n} W_k, \qquad \Lambda_{[n,m]} = \bigcup_{k=n}^{m} W_k, \ (n < m)$$

$$E_n = \big\{ < x, y > \in E \,:\, x, y \in \Lambda_n \big\}, \qquad \Lambda_n^c = \bigcup_{k=n}^{\infty} W_k$$

For $x \in \Gamma_+^k$, $x = (i_1, \ldots, i_n)$ denote

$$S(x) = \{(x, i) :\ 1 \leq i \leq k\}.$$

Here (x, i) means that $(i_1, \ldots, i_n, i)$. This set is called a set of *direct successors* of x.

Two vertices $x, y \in V$ is called *one level next-nearest-neighbor vertices* if there is a vertex $z \in V$ such that $x, y \in S(z)$, and they are denoted by $> x, y <$. In this case the vertices x, z, y was called *ternary* and denoted by $< x, z, y >$.

Let us define on Γ_+^k a binary operation $\circ : \Gamma_+^k \times \Gamma_+^k \to \Gamma_+^k$ as follows: for any two elements $x = (i_1, \ldots, i_n)$ and $y = (j_1, \ldots, j_m)$ put

$$x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m) \tag{1}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \ldots, i_n) \circ (0) = (i_1, \ldots, i_n). \tag{2}$$

By means of the defined operation Γ_+^k becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations $\tau_g : \Gamma_+^k \to \Gamma_+^k$, $g \in \Gamma_+^k$ by

$$\tau_g(x) = g \circ x. \tag{3}$$

It is clear that $\tau_{(0)} = id$.

The algebra of observables $\mathcal{B}_x$ for any single site $x \in L$ will be taken as the algebra M_d of the complex $d \times d$ matrices. The algebra of observables localized in the finite volume $\Lambda \subset L$ is then given by $\mathcal{B}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}_x$. As usual if $\Lambda^1 \subset \Lambda^2 \subset L$, then $\mathcal{B}_{\Lambda^1}$ is identified as a subalgebra of $\mathcal{B}_{\Lambda^2}$ by tensoring with unit matrices on the sites $x \in \Lambda^2 \setminus \Lambda^1$. Note that, in the sequel, by $\mathcal{B}_{\Lambda,+}$ we denote the positive part of $\mathcal{B}_\Lambda$. The full algebra $\mathcal{B}_L$ of the tree is obtained in the usual manner by an inductive limit

$$\mathcal{B}_L = \overline{\bigcup_{\Lambda_n} \mathcal{B}_{\Lambda_n}}.$$

In what follows, by $\mathcal{S}(\mathcal{B}_\Lambda)$ we will denote the set of all states defined on the algebra $\mathcal{B}_\Lambda$.

Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras. Recall [2] that a *quasi-conditional expectation* with respect to the given triplet is a completely positive (CP) unital linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{E}(ca) = c\mathcal{E}(a)$, for all $a \in \mathcal{A}$, $c \in \mathcal{C}$.

Definition 2.1 ([13]). *A state φ on $\mathcal{B}_L$ is called a forward quantum Markov chain (QMC), associated to $\{\Lambda_n\}$, if for each Λ_n, there exist a quasi-conditional expectation $\mathcal{E}_{\Lambda_n^c}$ with respect to the triplet*

$$\mathcal{B}_{\Lambda_{n+1}^c} \subseteq \mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_{\Lambda_{n-1}^c} \tag{4}$$

and a state $\hat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$ such that for any $n \in \mathbb{N}$ one has

$$\hat{\varphi}_{\Lambda_n^c}|\mathcal{B}_{\Lambda_{n+1}\setminus\Lambda_n} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c}|\mathcal{B}_{\Lambda_{n+1}\setminus\Lambda_n} \tag{5}$$

and

$$\varphi = \lim_{n\to\infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c} \tag{6}$$

in the weak- topology.*

Note that (5) is an analogue of the DRL equation from classical statistical mechanics [15, 18, 26], and QMC is thus the counterpart of the infinite-volume Gibbs measure.

3. Construction of Quantum Markov Chains on Cayley tree

In this section we are going to provide a construction of a forward quantum Markov chain which contain competing interactions (see [9, 11]). In this section we recall some notations.

Let us rewrite the elements of W_n in the following order, i.e.

$$\overrightarrow{W_n} := \left(x_{W_n}^{(1)}, x_{W_n}^{(2)}, \cdots, x_{W_n}^{(|W_n|)}\right).$$

In what follows, by $^\circ\prod$ we denote an ordered product, i.e.

$$^\circ\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n,$$

where elements $\{a_k\} \subset \mathcal{B}_L$ are multiplied in the indicated order. This means that we are not allowed to change this order.

Note that each vertex $x \in L$ has interacting vertices $\{x, (x,1), \dots, (x,k)\}$. Assume that each edges $< x, (x,i) >$ $(i = 1, \dots, k)$ operators $K_{<x,(x,i)>} \in \mathcal{B}_x \otimes \mathcal{B}_{(x,i)}$ is assigned, respectively. Moreover, for each competing vertices $> (x,i), (x,i+1) <$ and $< x, (x,i), (x,i+1) >$ $(i = 1, \dots, k)$ the following operators are assigned:

$$L_{>(x,i),(x,i+1)<} \in \mathcal{B}_{(x,i)} \otimes \mathcal{B}_{(x,i+1)}, \quad M_{(x,(x,i),(x,i+1))} \in \mathcal{B}_x \otimes \mathcal{B}_{(x,i)} \otimes \mathcal{B}_{(x,i+1)}.$$

We would like to define a state on $\mathcal{B}_{\Lambda_n}$ with boundary conditions $\omega_0 \in \mathcal{B}^+_{(0)}$ and $\{h^x \in \mathcal{B}^+_x : x \in L\}$.

For each $n \in \mathbb{N}$ denote

$$A_{x,(x,1),\dots,(x,k)} = \big(\circ \prod_{i=1}^{k} K_{x,(x,i)}\big)\big(\circ \prod_{i=1}^{k} L_{>(x,i),(x,i+1)<}\big)\big(\circ \prod_{i=1}^{k} M_{(x,(x,i),(x,i+1))}\big), \tag{7}$$

$$K_{[m,m+1]} := \prod_{x \in \overrightarrow{W}_m} A_{x,(x,1),\dots,(x,k)}, \quad 1 \le m \le n, \tag{8}$$

$$\mathbf{h}_n^{1/2} := \prod_{x \in \overrightarrow{W}_n} (h^x)^{1/2}, \quad \mathbf{h}_n = \mathbf{h}_n^{1/2}(\mathbf{h}_n^{1/2})^* \tag{9}$$

$$\mathbf{K}_n := \omega_0^{1/2} \prod_{m=1}^{n-1} K_{[m,m+1]} \mathbf{h}_n^{1/2} \tag{10}$$

$$\mathcal{W}_{n]} := \mathbf{K}_n \mathbf{K}_n^* \tag{11}$$

One can see that $\mathcal{W}_{n]}$ is positive.

In what follows, by $\mathrm{Tr}_\Lambda : \mathcal{B}_L \to \mathcal{B}_\Lambda$ we mean normalized partial trace (i.e. $\mathrm{Tr}_\Lambda(\mathbb{1}_L) = \mathbb{1}_\Lambda$, here $\mathbb{1}_\Lambda = \bigotimes_{y \in \Lambda} \mathbb{1}$), for any $\Lambda \subseteq_{\mathrm{fin}} L$. For the sake of shortness we put $\mathrm{Tr}_{n]} := \mathrm{Tr}_{\Lambda_n}$.

Let us define a positive functional $\varphi^{(n)}_{w_0,\mathbf{h}}$ on $\mathcal{B}_{\Lambda_n}$ by

$$\varphi^{(n)}_{w_0,\mathbf{h}}(a) = \mathrm{Tr}(\mathcal{W}_{n+1]}(a \otimes \mathbb{1}_{W_{n+1}})), \tag{12}$$

for every $a \in \mathcal{B}_{\Lambda_n}$. Note that here, Tr is a normalized trace on $\mathcal{B}_L$ (i.e. $\mathrm{Tr}(\mathbb{1}_L) = 1$).

To get an infinite-volume state φ on $\mathcal{B}_L$ such that $\varphi\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n)}_{w_0,\mathbf{h}}$, we need to impose some constrains to the boundary conditions $\{w_0, \mathbf{h}\}$ so that the functionals $\{\varphi^{(n)}_{w_0,\mathbf{h}}\}$ satisfy the compatibility condition, i.e.

$$\varphi^{(n+1)}_{w_0,\mathbf{h}}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n)}_{w_0,\mathbf{h}}. \tag{13}$$

Theorem 3.1. *[11] Assume that for every $x \in L$ and triple $\{x, (x,i), (x,i+1)\}$ $(i = 1, \dots, k-1)$ the operators $K_{<x,(x,i)>}$, $L_{>(x,i),(x,i+1)<}$, $M_{(x,(x,i),(x,i+1))}$ are given as above. Let the boundary conditions $w_0 \in \mathcal{B}_{(0),+}$ and $\mathbf{h} = \{h_x \in \mathcal{B}_{x,+}\}_{x \in L}$ satisfy the following conditions:*

$$\mathrm{Tr}(\omega_0 h^{(0)}) = 1, \tag{14}$$

$$\mathrm{Tr}_{x]}\big(A_{x,(x,1),\dots,(x,k)} \circ \prod_{i=1}^{k} h^{(x,i)} A^*_{x,(x,1),\dots,(x,k)}\big) = h^x, \quad \textit{for every} \ \ x \in L, \tag{15}$$

where as before $A_{x,(x,1),\dots,(x,k)}$ is given by (7). Then the functionals $\{\varphi^{(n)}_{w_0,\mathbf{h}}\}$ satisfy the compatibility condition (13). Moreover, there is a unique forward quantum Markov chain $\varphi_{w_0,\mathbf{h}}$ on $\mathcal{B}_L$ such that $\varphi_{w_0,\mathbf{h}} = w - \lim_{n\to\infty} \varphi^{(n)}_{w_0,\mathbf{h}}$.

Corollary 3.2. *If* (14),(15) *are satisfied then one has* $\varphi^{(n)}_{w_0,\mathbf{h}}(a) = \mathrm{Tr}(\mathcal{W}_{n]}(a))$ *for any* $a \in \mathcal{B}_{\Lambda_n}$.

Our goal in this paper is to establish the existence of translation-invariant QMC for the considered model (see next section).

4. QMC associated with Ising-XY model with competing interactions

In this section, we define the model and formulate the main results of the paper. In what follows we consider a semi-infinite Cayley tree $\Gamma^2_+ = (L, E)$ of order two. Our starting C^*-algebra is the same $\mathcal{B}_L$ but with $\mathcal{B}_x = M_2(\mathbb{C})$ for all $x \in L$. By σ^u_x, σ^u_y, σ^u_z we denote the Pauli spin operators for at site $u \in L$. Here

$$\mathbb{1}^{(u)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^u_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^u_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^u_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For every vertices $(u, (u, 1), (u, 2))$ we put

$$K_{<u,(u,i)>} = \exp\{J_0 \beta H_{<u,(u,i)>}\}, \quad i = 1, 2, \quad J_0 > 0, \ \beta > 0, \tag{16}$$

$$L_{>(u,1),(u,2)<} = \exp\{J\beta H_{>(u,1),(u,2)<}\}, \quad J > 0, \tag{17}$$

where

$$H_{<u,(u,1)>} = \frac{1}{2}\big(\mathbb{1}^{(u)} \otimes \mathbb{1}^{(u,1)} + \sigma^{(u)}_z \otimes \sigma^{(u,1)}_z\big), \tag{18}$$

$$H_{>(u,1),(u,2)<} = \frac{1}{2}\big(\sigma^{(u,1)}_x \otimes \sigma^{(u,2)}_x + \sigma^{(u,1)}_y \otimes \sigma^{(u,2)}_y\big). \tag{19}$$

Furthermore, we assume that $M_{(u,(u,i),(u,i+1))} = \mathbb{1}$ $(i = 1, 2, \ldots, k)$ for all $u \in L$.

The defined model is called the *Ising-XY model with competing interactions* per vertices $(u, (u, 1), (u, 2))$.

Remark 4.1. *Note that if we take* $J = 0$*, then one gets the Ising model on Cayley tree which has been studied in [10] and if we take* $J_0 = 0$ *we get the XY-model on the interactions.*

One can calculate that for $m \in \mathbb{N}$

$$H^m_{<u,v>} = H_{<u,v>} = \frac{1}{2}\big(\mathbb{1}^{(u)}\mathbb{1}^{(v)} + \sigma^{(u)}\sigma^{(v)}\big), \tag{20}$$

$$H^{2m}_{>(u,1),(u,2)<} = H^2_{>(u,1),(u,2)<} = \frac{1}{2}(\mathbb{1}^{(u,1)} \otimes \mathbb{1}^{(u,2)} - \sigma^{(u,1)}_z \otimes \sigma^{(u,2)}_z) \tag{21}$$

$$H^{2m-1}_{>(u,1),(u,2)<} = H_{>(u,1),(u,2)<} \tag{22}$$

Therefore, one finds

$$K_{<u,(u,i)>} = K_0 \mathbb{1}^{(u)} \otimes \mathbb{1}^{(u,i)} + K_3 \sigma^{(u)}_z \otimes \sigma^{(u,i)}_z$$

$$L_{>(u,1),(u,2)<} = \mathbb{1}^{(u,1)} \otimes \mathbb{1}^{(u,2)} + sinh(J\beta) H_{>(u,1),(u,2)<} + (cosh(J\beta) - 1) H^2_{>(u,1),(u,2)<}$$

where

$$K_0 = \frac{\exp J_0\beta + 1}{2}, \quad K_3 = \frac{J_0 \exp \beta - 1}{2},$$

Hence, from (7) for each $x \in L$ we obtain

$$A_{(u,(u,1),(u,2))} = \gamma_1 \mathbb{1}^{(u)} \otimes \mathbb{1}^{(u,1)} \otimes \mathbb{1}^{(u,2)} + \gamma_2 \mathbb{1}^{u} \otimes \sigma_x^{(u,1)} \otimes \sigma_x^{(u,2)} \tag{23}$$

$$+\gamma_2 \mathbb{1}^{(u)} \otimes \sigma_y^{(u,1)} \otimes \sigma_y^{(u,2)} + \gamma_3 \mathbb{1}^{(u)} \otimes \sigma_z^{(u,1)} \otimes \sigma_z^{(u,2)} \tag{24}$$

$$+\delta_1 \sigma_z^{(u)} \otimes \mathbb{1}^{(u,1)} \otimes \sigma_z^{(u,2)} + \delta_1 \sigma_z^{(u)} \otimes \sigma_z^{(u,1)} \otimes \mathbb{1}^{(u,2)} \tag{25}$$

where

$$\begin{cases} \gamma_1 = \frac{1}{4}[\exp(2J_0\beta) + 1 + 2\exp(J_0\beta)\cosh(J\beta)], \quad \gamma_2 = \frac{1}{2} exp(J_0\beta)\sinh(J\beta). \\ \gamma_3 = \frac{1}{4}[\exp(2J_0\beta) + 1 - 2\exp(J_0\beta)\cosh(J\beta)], \quad \delta_1 = \frac{1}{4}(\exp(2J_0\beta) - 1). \end{cases}$$

Recall that a function $\{h^u\}$ is called *translation-invariant* if one has $h^u = h^{\tau_g(u)}$, for all $u, g \in \Gamma_+^2$. Clearly, this is equivalent to $h^u = h^v$ for all $u, v \in L$.

In what follows, we restrict ourselves to the description of translation-invariant solutions of (14),(15). Therefore, we assume that: $h^u = h$ for all $u \in L$, where

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Then we have

$$\begin{aligned} h &= Tr_{u]} A_{(u,(u,1),(u,2))} [\mathbb{1}^{(u)} \otimes h \otimes h] A^*_{(u,(u,1),(u,2))} \\ &= [C_1 \mathrm{Tr}(h)^2 + C_2 \mathrm{Tr}(\sigma_z h)^2] \mathbb{1}^{(u)} + C_3 Tr(h) \mathrm{Tr}(\sigma_z h) \sigma_z^{(u)}. \end{aligned} \tag{26}$$

where

$$\begin{cases} C_1 = \frac{1}{4}(\exp(4J_0\beta) + 1) + \frac{1}{2}\exp(2J_0\beta)\cosh(2J\beta); \\ C_2 = \frac{1}{4}(\exp(4J_0\beta) + 1) - \frac{1}{2}\exp(2J_0\beta)\cosh(2J\beta); \\ C_3 = \frac{1}{2}(\exp(4J_0\beta) - 1). \end{cases}$$

Now taking into account

$$\mathrm{Tr}(h) = \frac{h_{11} + h_{22}}{2}, \quad \mathrm{Tr}(\sigma_z h) = \frac{h_{11} - h_{22}}{2}$$

the equation (26) is reduced to the following one

$$\begin{cases} \mathrm{Tr}(h) = C_1 \mathrm{Tr}(h)^2 + C_2 \mathrm{Tr}(\sigma h)^2, \\ \mathrm{Tr}(\sigma h) = C_3 \mathrm{Tr}(h) \mathrm{Tr}(\sigma h), \\ h_{21} = 0, h_{12} = 0. \end{cases} \tag{27}$$

The obtained equation implies that a solution h is diagonal, and ω_0 could be also chosen diagonal, through the equation. In what follows, we always assume that $h_{21} = 0, h_{12} = 0$. In the next sections we are going to examine (27).

5. Existence of QMC associated with the model.

In this section we are going to solve (27), which yields the existence of QMC associated with the model.

5.1. Case $h_{11} = h_{22}$ and associate QMC
Assume that $h_{1,1} = h_{2,2}$, then (27) is reduced to

$$h_{11} = h_{22} = \frac{1}{C_1}.$$

Then putting $\alpha = \frac{1}{C_1}$ we get

$$h_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \tag{28}$$

Proposition 5.1. *The pair $(\omega_0, \{h^u = h_\alpha | u \in L\})$ with $\omega_0 = \frac{1}{\alpha}\mathbb{1}$, $h^u = h_\alpha, \forall u \in L$, is solution of (14),(15). Moreover the associated QMC can be written on the local algebra $\mathcal{B}_{L,loc}$ by:*

$$\varphi_\alpha(a) = \alpha^{2^n-1}\mathrm{Tr}\Big(a\prod_{i=0}^{n-1} K_{[i,i+1]}K^*_{[i,i+1]}\Big), \quad \forall a \in B_{\Lambda_n}. \tag{29}$$

5.2. Case $h_{11} \neq h_{22}$ and associate QMC
Assume that $h_{11} \neq h_{22}$, put $\theta = \exp(2\beta)$.
the equation (27) is reduced to

$$\begin{cases} \frac{h_{11}+h_{22}}{2} = \frac{1}{C_3}, \\ (\frac{h_{11}-h_{22}}{2})^2 = \frac{C_3-C_1}{C_2.C_3^2}, \end{cases} \tag{30}$$

Lemma 5.2. *Let*

$$\Delta(\theta) = \frac{C_3 - C_1}{C_2} = \frac{\theta^{2J_0} - \theta^{J_0}(\theta^J + \theta^{-J}) - 3}{\theta^{2J_0} - \theta_0(\theta^J + \theta^{-J}) + 1} \tag{31}$$

Then for every $J \in \mathbb{R} \setminus \{-J_0, J_0\}$, there exists θ_0 (depend on J) such that $\Delta(\theta) > 0$, whenever $\theta \geq \theta_0$.

In the sequel let be fixed $J \in \mathbb{R} \setminus \{-J_0, J_0\}$.

Proposition 5.3. *Assume that $\Delta(\theta) > 0$. Then the equation (27) has two solutions given by:*

$$h = \xi_0\mathbb{1} + \xi_3\sigma, \tag{32}$$
$$h' = \xi_0\mathbb{1} - \xi_3\sigma, \tag{33}$$

where

$$\xi_0 = \frac{1}{C_3} = \frac{2}{\theta^{2J_0} - 1}, \quad \xi_3 = \frac{\sqrt{\Delta(\theta)}}{C_3} = \frac{2\sqrt{\Delta(\theta)}}{\theta^{2J_0} - 1} \tag{34}$$

From (14) we find that $\omega_0 = \frac{1}{\xi_0}\mathbb{1} \in \mathcal{B}^+$s. Therefore, the pairs $\big(\omega_0, \ \{h^{(u)} = h, \ u \in L\}\big)$ and $\big(\omega_0, \{h^{(u)} = h', \ u \in L\}\big)$ define two solutions of (14),(15). Hence, they define two QMC φ_1 and φ_2, respectively. Namely, for every $a \in \mathcal{B}_{\Lambda_n}$ one has

$$\varphi_1(a) = \mathrm{Tr}\big(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]}\mathbf{h}_n K^*_{[n-1,n]} \cdots K^*_{[0,1]}a\big) \tag{35}$$

$$\varphi_2(a) = \mathrm{Tr}\big(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]}\mathbf{h}'_n K^*_{[n-1,n]} \cdots K^*_{[0,1]}a\big). \tag{36}$$

Hence, we have the following

Theorem 5.4. *The following statements hold:*

(i) if $\Delta(\theta) \leq 0$, then there is a unique translation invariant QMC φ_α;
(ii) if $\Delta(\theta) > 0$, then there are at least three translation invariant QMC φ_α, φ_1 and φ_2.

6. References

[1] Accardi L 1975 *Funct. Anal. Appl.*, **9** 1–8
[2] Accardi L and Cecchini C 1982 *J. Funct. Anal.* **45** 245–273
[3] Accardi L and Fidaleo F 2005 *Annali di Matematica Pura e Applicata*, **184** 327–346
[4] Accardi L and Fidaleo F 2003 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **6** 123–138
[5] Accardi L and Fidaleo F 2003 *J. Funct. Anal.* **200** 324-347
[6] Accardi L and Fidaleo F 2003 In book: *Proceedings Burg Conference 15–20 March 2001, W. Freudenberg (ed.)* (Singapore: World Scientific) 1–20
[7] Accardi L, Fidaleo F and Mukhamedov F 2007 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **10** 165–183
[8] Accardi L and Liebscher V 1999 *Infin. Dimens. Anal. Quantum Probab. Relat. Topics* **2** 645-661
[9] Accardi L, Mukhamedov F and Saburov M 2011 *Ann. Henri Poincare* **12** 1109–1144
[10] Accardi L, Mukhamedov F and Saburov M 2014 *J. Stat. Phys.* **157** 303-329
[11] Accardi L, Mukhamedov F and Souissi A 2016 *J. Phys.: Conf. Ser.* **697** 012018.
[12] Accardi L, Mukhamedov F and Souissi A 2016 *Adv. Oper. Theory* **1** 206–218
[13] Accardi L, Ohno H and Mukhamedov F 2010 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **13** 165–189
[14] Affleck L, Kennedy E, Lieb E H and Tasaki H 1988 *Commun. Math. Phys.* **115** 477–528
[15] Dobrushin R L 1968 *Probab. Theory Appl.* **13** 201–229
[16] Fannes M, Nachtergaele B amd Werner R F 1992 *J. Stat. Phys.* **66** 939–973
[17] Fannes M, Nachtergaele B and Werner R F 1992 *Commun. Math. Phys.* **144** 443–490
[18] Georgi H-O 1988 *Gibbs measures and phase transitions* (Berlin: Walter de Gruyter)
[19] Liebscher V 2003 In book: *Proceedings Burg Conference 15–20 March 2001, W. Freudenberg (ed.)* (Singapore: World Scientific) 151–159
[20] Mukhamedov F M 2000 *Theor. Math. Phys.* **123** 489–493
[21] Mukhamedov F M 2004 *Rep. Math. Phys.* **53** 1–18
[22] Mukhamedov F, Barhoumi A and Souissi A 2016 *Math. Phys. Anal. Geom.* **19** 21
[23] Mukhamedov F, Barhoumi A and Souissi A 2016 *J. Stat. Phys.* **163** 544–567
[24] Mukhamedov F M and Rozikov U A 2004 *J. Stat. Phys.* **114** 825–848
[25] Mukhamedov F M and Rozikov U A 2005 *J. Stat. Phys.* **119** 427–446
[26] Rozikov U A 2013 *Gibbs measures on Cayley trees*, (Singapore: World Scientific)