IOP Conf. Series: Journal of Physics: Conf. Series **819** (2017) 012005 doi:10.1088/1742-6596/819/1/012005

Arcwise Connectedness of Solution Sets of Lipschitzian Quantum Stochastic Differential Inclusions

D A Dikko[1] **and E O Ayoola**[2]

[1,2] Department of Mathematics, University of Ibadan , Ibadan , Oyo State , Federal Republic of Nigeria

E-mail: da.dikko@mail.ui.edu.ng[1] , eoayoola@gmail.com[2]

Abstract. In the framework of the Hudson - Parthasarathy quantum stochastic calculus, we employ some recent selection results to prove that the function space of the matrix elements of solutions to quantum stochastic differential inclusion (QSDI) is arcwise connected both locally and globally.

1. Introduction

This paper considers Quantum Stochastic Differential Inclusion(QSDI) in the framework of the Hudson and Parthasarathy formulation [11] of Quantum Stochastic Calculus. It has found applications in the study of quantum stochastic control theory [13] and often occurs as regularization of quantum stochastic differential equations with discontinuous coefficients.

In [3,4,8,9] some topological properties of solution sets of QSDI have been achieved. These were subject to some conditions on the coefficients of their inclusions.

There are some of the interesting motivations [1,2,13,14,15] for studying connectedness, path connectedness and arcwise connectedness of solution sets in the classical differential inclusions with their applications.This provides the possibility of moving from one solution to another. However as established in [1, 14, 15] for the case of differential inclusions on finite dimensional Euclidian spaces, this work concerns the establishment of arcwise connectedness of solution sets of quantum stochastic differential inclusion in the integral form:

$$\begin{aligned} X(t) \;\; &\in \;\; a + \int_0^t \big(E(s,X(s))d\wedge_\pi(s) + F(s,X(s))dA_f(s) \\ &\qquad + G(s,X(s))dA_g^+(s) + H(s,X(s))ds\big), \quad \text{almost all} \quad t \in [0,T]. \quad (1.1) \end{aligned}$$

In equation (1.1), the coefficients $\{E, F, G, H\}$ lie in a certain class of stochastic processes for which quantum stochastic integrals against the guage, creation and annihilation processes

$\bigwedge_\pi, A_f^+, A_g$ and the Lebesgue measure are defined. Equation(1.1) involves unbounded linear operators on a Hilbert space and it is a noncommutative generalization of the classical stochastic integral equations of the form

$$X(t,w) \quad = \quad x_o + \int H(t,X)dt + \int F(t,X)dQ(t), \quad X(t_0) = x_0, \;\; t \in [t_0, T] \quad (1.2)$$

where the driving process $Q(t)$ is a martingale and H, F are sufficiently smooth ordinary functions.

We shall employ the various spaces of quantum stochastic processes introduced in [3, 4, 8].The remaining part of the work shall be arranged as follows; In section 2, some notations and fundamental structures shall be stated which shall be employed in the sequel. In section 3 some results and assumptions shall be stated and in section 4 the main result of this paper shall be established.

2. Notations and Fundamental Structures

In what follows, if N is a topological space, we denote by clos(N), the collection of all non-empty closed subsets of N. To each pair (D, H) which consists of a pre - Hilbert space D with completion H , we associate $\mathcal{L}_w^+(D, H)$ the set of all linear maps x from a pre-Hilbert space D to its completion H. With the property that the domain of the operator adjoint x^* of x contains D.

The members of $\mathcal{L}_w^+(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $\mathcal{L}_w^+(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H also corresponds a Hilbert space $\Gamma(H)$, called the Boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of the linear space generated by the set of exponential vectors in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \quad f \in H$$

where $\otimes^0 f = 1$ and $\otimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $I\!\!D$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert space. We shall write $L^2_\gamma(I\!\!R_+)$ (resp. $L^2_\gamma([0,t])$)' resp. $L^2_\gamma([t,\infty)), t \in I\!\!R_+ \equiv [0,\infty)$ for the Hilbert space of square integrable, γ-valued maps on $I\!\!R_+ = [0,\infty)$ (resp. on $[0,t)$; resp. on $[t,\infty); t \in I\!\!R_+$)

The noncommutative stochastic processes discussed in the sequel are densely-defined linear operators on $\mathcal{R} \otimes \Gamma(L^2_\gamma(I\!\!R_+))$; the inner product of this Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$. For each $t > 0$, the direct sum decomposition

$$L^2_\gamma(I\!\!R_+) = L^2_\gamma([0,t)) \oplus L^2_\gamma([t,\infty))$$

induces a factorization

$$\Gamma(L^2_\gamma(I\!R_+)) = \Gamma(L^2_\gamma([0,t))) \otimes \Gamma(L^2_\gamma([t,\infty)))$$

of Fock space.
Let $I\!E$, $I\!E_t$ and $I\!E^t$, $t > 0$ be the linear spaces generated by the exponential vectors in $\Gamma(L^2_\gamma(I\!R_+))$, $\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$, $t > 0$ respectively.
Then we define

$$\begin{aligned}
\mathcal{A} &\equiv L^+_w(I\!D\underline{\otimes}I\!E,\ \mathcal{R}\otimes\Gamma(L^2_\gamma(I\!R_+))) \\
\mathcal{A}_t &\equiv L^+_w(I\!D\underline{\otimes}I\!E_t,\ \mathcal{R}\otimes\Gamma(L^2_\gamma([0,t)))) \otimes 1^t \\
\mathcal{A}_t &\equiv 1_t \otimes L^+_w(I\!E^t, \Gamma(L^2_\gamma([t,\infty)))), \quad t > 0
\end{aligned}$$

where $\underline{\otimes}$ denotes algebraic tensor product and 1_t (resp. 1^t) denotes the identify map on $\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t)))$ (resp. $\Gamma(L^2_\gamma([t,\infty)))$ $t > 0$. We note that the spaces $\mathcal{A}_t$ and $\mathcal{A}^t$, $t > 0$, may be naturally identified with subspaces of $\mathcal{A}$.

For $\eta,\xi \in I\!D\underline{\otimes}I\!E$, define $\|\cdot\|_{\eta\xi}$ by

$$\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|, \quad x \in \mathcal{A}$$

Then, $\{\|\cdot\|_{\eta\xi},\ \ \eta,\xi \in I\!D\underline{\otimes}I\!E\}$ is a family of locally convex seminorms on $\mathcal{A}$; we write τ_w for the locally convex topology on $\mathcal{A}$ determined by this family.

In the foregoing $\bar{\mathcal{A}}, \bar{\mathcal{A}}_t$ and $\bar{\mathcal{A}}^t$ denote the completions of the locally convex spaces $(\mathcal{A},\tau_w)$, $(\mathcal{A}_t,\tau_w)$, $\mathcal{A}^t,\tau_w)$, $t > 0$ respectively we then note that $\{\tilde{\mathcal{A}}_t,\ t \in I\!R_+\}$ is a filtration of $\tilde{\mathcal{A}}$.

Hausdorff topology: If A is a topological space, then $Clos(A)$ [resp. $Comp(A)$] denotes the collection of all nonvoid closed (resp. Compact) Subsets of A. We shall employ the Hausdorff topology on $Clos(\bar{A})$ which is defined as follows.
For $x \in \tilde{\mathcal{A}}$, $\mathcal{M},\mathcal{N} \in Clos(\tilde{\mathcal{A}})$, and $\eta,\xi \in I\!D\otimes I\!E$, set

$$d_{\eta\xi}(x,\mathcal{N}) = \inf_{y\in\mathcal{N}} \|x-y\|_{\eta\xi}$$

$$\delta_{\eta\xi}(\mathcal{M},\mathcal{N}) = \sup_{x\in\mathcal{M}} d_{\eta\xi}(x,\mathcal{N})$$

and

$$\rho_{\eta\xi}(\mathcal{M},\mathcal{N}) = \max(\delta_{\eta\xi}(\mathcal{M},\mathcal{N}), \delta_{\eta\xi}(\mathcal{N},\mathcal{M}))$$

Then $\{\rho_{\eta\xi}(\cdot) : \eta,\xi \in I\!D\underline{\otimes}I\!E\}$ is a family of pseudometrics which determines a Hausdorff topology on $Clos(\bar{A})$ denoted in the sequel by τ_H. If $\mathcal{M} \in Clos(\bar{A})$, then $\|\mathcal{M}\|_{\eta\xi}$ is defined by

$$\|\mathcal{M}\|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M},\{0\})$$

for arbitrary $\eta,\xi \in I\!D\underline{\otimes}I\!E$.

For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$d(x, B) \equiv \inf_{y \in B} |x - y|$$

,

$$\delta(A, B) \equiv \sup_{x \in A} d(x, B)$$

and

$$\rho(A, B) \equiv max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.

Let $I \subseteq \mathbb{R}_+$. A stochastic process indexed by I is an $\tilde{\mathcal{A}}$-valued map on I. A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$. We write $Ad(\tilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.

Definition: A member X of $Ad(\tilde{\mathcal{A}})$ is called

(i) Weakly absolutely continuous if the map $t \longmapsto \langle \eta, X(t)\xi \rangle, t \in I$, is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(ii) Locally absolutely p-integrable if $\|X(\cdot)\|^p_{\eta\xi}$ is Lebesgue-measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Notation.

We write $Ad(\tilde{\mathcal{A}})_{wac}$ [resp. $L^p_{loc}(\tilde{\mathcal{A}})$] for the set of all weakly absolutely continuous (resp. locally absolutely p-integrable) members of $Ad(\tilde{\mathcal{A}})$.

Stochastic Integrators

Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp. $L^\infty_{B(Y),loc}(\mathbb{R}_+)$] be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to γ [resp. to $B(\gamma)$, the Banach space of Bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t), t \in \mathbb{R}_+$.

For $f \in L^2_\gamma(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, define the operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R}_+)))$ as follows;

$$\begin{aligned}
a(f)e(g) &= \langle f, g \rangle_{L^2_\gamma(\mathbb{R}_+)} e(g) \\
a^+(f)e(g) &= \frac{d}{d\sigma} e(g + \sigma f) \Big|_{\sigma=0} \\
\lambda(\pi)e(g) &= \frac{d}{d\sigma} e(e^{\sigma\pi} f) \Big|_{\sigma=0}
\end{aligned}$$

for $g \in L^2_\gamma(\mathbb{R}_+)$.

These are the annihilation, creation and gauge operators of quantum field theory.

For arbitrary $f \in L^\infty_{\gamma,loc}(I\!R_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(I\!R_+)$, they give rise to the operator-valued maps A_f, A^+_f, and A_π defined by

$$\begin{aligned} A_f(t) &\equiv a(f\chi_{[0,t)}) \\ A^+_f(t) &\equiv a^+(f\chi_{[0,t)}) \\ \wedge_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)}) \end{aligned}$$

$t \in I\!R_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq I\!R_+$.

The maps A_f, A^+_f and $\wedge_\pi$ are stochastic processes, called the annihilation, creation and gauge processes, respectively, when their values are identified with their ampliations on $\mathcal{R}\otimes\Gamma(L^2_\gamma(I\!R_+))$. These are the stochastic integrators in the Hudson and Parthasarathy [11]formulation of Boson quantum stochastic integration, which we shall adopt in the sequel.

Accordingly, if $p, q, u, v \in L^2_{loc}(\tilde{\mathcal{A}})$, $f, g \in L^\infty_{\gamma,loc}(I\!R_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(I\!R_+)$ then we interpret the integral.

$$\int_{t_0}^t p(s)d\wedge_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds;\ t_0, t \in I\!R_+$$

as it is in the Hudson and Parthasarathy [11] formulation.

Stochastic Differential Inclusions

Definition:

(i) By a multivalued stochastic process indexed by $I \subseteq I\!R_+$, we mean a multifunction on I with values in $Clos(\tilde{\mathcal{A}})$.

(ii) If Φ is a multivalued stochastic process indexed by $I \subseteq I\!R_+$, then a selection of Φ is a stochastic process $X : I \to \tilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

(iii) A multivalued stochastic process Φ will be called

  (a) adapted if $\Phi(t) \subseteq \tilde{\mathcal{A}}$, for each $t \in I\!R^+$;

  (b) measurable if $t \longmapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in I\!D\underline{\otimes}I\!E$

(c) locally absolutely p-integrable if $t \mapsto \|\phi(t)\|_{\eta\xi}$, $t \in I\!R_+$, lies in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E$.

We note that

(1) the set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$

(2) For $p \in (0, \infty)$ and $I \subseteq I\!R_+$, $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps

$$\Phi : I \times \tilde{\mathcal{A}} \longrightarrow Clos(\tilde{\mathcal{A}})$$

such that

$$t \longmapsto \Phi(t, X(t)),$$

$t \in I$, lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$,

(3) If $\Phi \in L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, then
$L_p(\Phi) \equiv \{\varphi \in L^p_{loc}(\tilde{\mathcal{A}}) : \varphi$ is a selection of $\Phi\}$.

(4) For $f, g \in L^\infty_{\gamma,loc}(I\!R_+)$, $\pi \in L^\infty_{B(Y),loc}(I\!R_+)$, 1 is the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma(I\!R_+))$, and M is any of the stochastic processes A_f, A^+_g, A_π and $s \longmapsto s1$, $s \in I\!R_+$.

Thus, we introduce stochastic integral (resp. differential) expressions as follows.
If $\Phi \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(\tilde{\mathcal{A}})$, then we make the definition

$$\int_{t_0}^t \Phi(s, X(s))dM(s) \equiv \left\{ \int_{t_0}^t \varphi(s)dM(s) : \varphi \in L_2(\Phi) \right\}$$

This leads to the following notion.

Definition: Let $E, F.G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and (t_0, x_0) be a fixed point of $I \times \tilde{\mathcal{A}}$, then a relation of the form

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^t &(E(s, X(s))d\wedge_\pi + F(s, X(s))dA_f(s) \\ &+ G(s, X(s))dA^+_g(s) + H(s, X(s))ds); t \in I \end{aligned}$$

be called a stochastic integral inclusion with coefficient E, F, G and H initial data (t_0, x_0). We shall sometimes write the foregoing inclusion as follows;

$$\begin{aligned} dX(t) \in\ & E(t, X(t))d\wedge_\pi (t) + F(t, X(t))dA_f(t) \\ & + G(t, X(t))dA^+_g(t) + H(t, X(t))dt \end{aligned} \tag{2.1}$$

for almost all $t \in I$, $X(t_0) = x_0$.
This we refer to as stochastic differential inclusions with coefficients E, F, G and H and initial data (t_0, X_0).

Definition: By a solution of (2.1) we mean a weakly absolutely continuous stochastic process $\varphi \in L^2_{loc}(\tilde{\mathcal{A}})$ such that

$$\begin{aligned} d\varphi(t) \ \in\ & E(t, \varphi(t))d\wedge_\pi (t) + F(t, \varphi(t))dA_f(t) \\ & + G(t, \varphi(t))dA^+_g(t) + H(t, \varphi(t))dt \end{aligned}$$

almost all $t \in I$, $\varphi(t_0) = x_0$.

Remarks

(i) The existence of solution to a stochastic differential inclusion with Lipschitzian coefficients has been proved in [8].

(ii) If $\mathcal{M}$ is a subset of $\tilde{\mathcal{A}}$, we write $co\mathcal{M}$ for the closed convex hull of $\mathcal{M}$ and if
$\Phi : I \times \tilde{\mathcal{A}} \longrightarrow Clos(\tilde{\mathcal{A}})$, we define
$co\ \Phi : I \times \tilde{\mathcal{A}} \longrightarrow Clos(\tilde{\mathcal{A}})$ by

$$(co\ \Phi)(t,x) = co\ \Phi(t,x),\ \ t,x) \in I \times \tilde{\mathcal{A}}$$

(iii) Related to (2.1) is the following stochastic differential inclusion:

$$dX(t) \in co\ E(t,X(t))dA_\pi(t) + co\ F(t,X(t))dA_f(t)$$

$$+co\ G(t,X(t))dA_g^+(t) + co\ H(t,X(t))dt$$

almost all $t \in I$

$$X(t_0) = X_0 \tag{2.2}$$

(iv) In [8], Ekhaguere established equivalent form of (1.1) and (2.1) as follows; $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mus}$ and (t_0, x_0) is some fixed point of $I \times \tilde{\mathcal{A}}$. Taking theorems 4.1 and Theorem 4.4 of Hudson and Parthasarathy which describes the matrix elements of the quantum stochastic integral.
For $\eta, \xi \in I\!\!D \otimes I\!\!E$, with $\eta = c \otimes e(\alpha)$ and $\xi = d \otimes c(\beta)$, define
$\mu_{\alpha\beta},\ V_\beta,\ \sigma_\alpha : I \to I\!\!\!C,\ I \subset I\!\!R_+$, by

$$\begin{aligned} \mu_{\alpha\beta}(t) &= \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma \\ V_\beta(t) &= \langle f(t), \beta(t)\rangle_\gamma \\ \sigma_\alpha(t) &= \langle \alpha(t), g(t)\rangle_\gamma \\ t \in I. \end{aligned}$$

To these functions, are associated the maps $\mu I\!\!E, vF, \sigma G, P$ and coP from $I \times \tilde{\mathcal{A}}$ into the set of multivalued sesquilinear forms on $I\!\!D \underline{\otimes} I\!\!E$ defined by

$$\begin{aligned} (\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\} \\ (vF)(t,x)(\eta,\xi) &= \{\langle \eta, v_\beta(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\} \\ (\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_\alpha(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\} \\ P(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (vF)(t,x)(\eta,\xi) \\ &\quad +(\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi) \\ (coP)(t,x)(\eta,\xi) &= \text{closed convex/hull of } P(t,x)(\eta,\xi) \end{aligned}$$

$$\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E,\ (t,x) \in I \times \tilde{\mathcal{A}}$$

where

$$H(t,x)(\eta,\xi) = \{v(t,x)(\eta,\xi) : v(\cdot, X(\cdot))$$

is a selection of $H(\cdot, X(\cdot))\ \ \forall\, X \in L^2_{loc}(\tilde{\mathcal{A}})$
$\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E,\ (t,x) \in I \times \tilde{\mathcal{A}}$.

As in [3, 4, 5, 6] we shall consider the equivalent form of (1.1) given by

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in P(t, X(t))(\eta,\xi) \\ X(0) &= a, \quad t \in [0,T]. \end{aligned} \tag{2.3}$$

Inclusion (2.3) is a nonclassical ordinary differential inclusion and the map $(\eta,\xi) \to P(t,x)(\eta,\xi)$ is a multivalued sesquilinear form on $(I\!\!D\underline{\otimes} I\!\!E)^2$ for $(t,x) \in [0,T] \times \tilde{\mathcal{A}}$. We refer the reader to [8, 9, 10] for the explicit forms of the map and the existence results for solutions of QSDI (1.1) of Lipschitz, hypermaximal monotone and of evolution types.

3. Preliminary Results and Assumptions

As in [3, 4, 8], we let clos($\mathcal{N}$) denote the family of all nonempty closed subsets of a topological space $\mathcal{N}$. For $\mathcal{N} \in \{\tilde{\mathcal{A}},\ \mathbb{C}\}$, we adopt the Hausdorff topology on clos($\mathcal{N}$) as explained in the references above. We denote by $d(x, A)$, the distance from a point $x \in \mathbb{C}$ to a set $A \subseteq \mathbb{C}$. For $A, B \in$ clos($\mathbb{C}$), $\rho(A,B)$ denote the Hausdorff distance between the sets.

As in the references above,we shall employ the space $\mathbf{wac}(\tilde{\mathcal{A}})$ which is the completion of the locally convex topological space $(Ad(\tilde{\mathcal{A}})_{wac}, \tau)$ of adapted weakly absolutely continuous stochastic processes $\Phi : [0,T] \to \tilde{\mathcal{A}}$ whose topology τ is generated by the family of seminorms given by :

$$|\Phi|_{\eta\xi} := \|\Phi(0)\|_{\eta\xi} + \int_0^T |\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle| dt, \ \text{ for } \ \eta,\ \xi \in I\!\!D\underline{\otimes} I\!\!E. \tag{3.1}$$

Associated with the space $\mathbf{wac}(\tilde{\mathcal{A}})$, we shall employ the space $\mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi)$ consisting of absolutely continuous complex valued functions $\langle \eta, \Phi(\cdot)\xi\rangle$, where $\Phi \in \mathbf{wac}(\tilde{\mathcal{A}})$ for arbitrary pair of points $\eta, \xi \in I\!\!D\underline{\otimes} I\!\!E$. By a solution of QSDI (1.1) or its equivalent form (2.3), we mean a stochastic process $\Phi : [0,T] \to \tilde{\mathcal{A}}$ lying in the space $Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (1.1) or its equivalent form (1.2).

We assume the following conditions in what follows:
$\mathcal{S}_{(1)}$ The coefficients E, F, G, H appearing in QSDI (1.1) are continuous.
$\mathcal{S}_{(2)}$ The multivalued map $(t,x) \to P(t,x)(\eta,\xi)$ has nonempty and closed values as subsets of the field $\mathbb{C}$ of complex numbers.

$\mathcal{S}_{(3)}$ For each $x \in \tilde{\mathcal{A}}$, the map $t \to P(t,x)(\eta,\xi)$ is measurable.

$\mathcal{S}_{(4)}$ There exists a map $K^P_{\eta\xi} : [0,T] \to I\!R_+$ lying in $L^1_{loc}([0,T])$ such that

$$\rho(P(t,x)(\eta,\xi), P(t,y)(\eta,\xi)) \leq K^P_{\eta\xi}(t)\|x-y\|_{\eta\xi} \tag{3.2}$$

for $t \in [0,T]$, and for each pair $x,\ y \in \tilde{\mathcal{A}}$.

$\mathcal{S}_{(5)}$ There exists a stochastic process $Y : [0,T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac}$ such that for each pair $\eta,\xi \in I\!D\underline{\otimes}I\!E$,

$$d\left(\frac{d}{dt}\langle\eta, Y(t)\xi\rangle\ , P(t,Y(t))(\eta,\xi)\right) \leq \rho_{\eta\xi}(t), \tag{3.3}$$

for almost all $t \in [0,T]$ and for some locally integrable map $\rho_{\eta\xi} : [0,T] \to R_+$.

Associated with the space $\tilde{\mathcal{A}}$, we define the space of complex numbers $\tilde{\mathcal{A}}(\eta,\xi) := \{\langle\eta, a\xi\rangle : a \in \tilde{\mathcal{A}}\}$. We shall denote by $S^{(T)}(a)$, the subset of $\mathbf{wac}(\tilde{\mathcal{A}})$ consisting of the set of solutions of QSDI (1.1) corresponding to the initial value $a \in \tilde{\mathcal{A}}$ and write $S^{(T)}(a)(\eta,\xi) = \{\langle\eta, \Phi(\cdot)\xi\rangle : \Phi \in S^{(T)}(a)\}$. Moreover, $S^{(T)}(P)(\eta,\xi) := \bigcup_{a\in\tilde{\mathcal{A}}} S^{(T)}(a)(\eta,\xi)$. In what follows, $a \to S^{(T)}(a)$ is the multivalued solution map of QSDI (1.1) corresponding to the initial value $x = a$. Under the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(5)}$ above, it is well known that the set $S^{(T)}(a)$ is not empty for arbitrary $a \in \tilde{\mathcal{A}}$ (see [8, 9, 10]).

Next, we employ Corollary 3.2 in [3] to establish an auxiliary result needed for the proof of the arcwise connectedness of the entire space $S^{(T)}(P)(\eta,\xi)$. To this end, for any family of linear maps $\{a_\alpha,\ \alpha \in [0,1]\}$ in $\tilde{\mathcal{A}}$, we define $a_{\eta\xi,\alpha} = \langle\eta, a_\alpha\xi\rangle,\ \alpha \in [0,1]$ for arbitrary elements $\eta,\xi \in I\!D\underline{\otimes}I\!E$.

Proposition 3.1: Let $a_0,\ a_1 \in \tilde{\mathcal{A}}$ such that $a_0 \neq a_1$. Let $X_0 \in S^{(T)}(a_0),\ \ X_1 \in S^{(T)}(a_1)$. Then there exists a continuous map $h : [0,1] \to \mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi)$ such that $h(0) = X_{\eta\xi,0},\ \ h(1) = X_{\eta\xi,1}$ and for $\alpha \in [0,1]$, $h(\alpha) \in S^{(T)}(a_\alpha)(\eta,\xi)$ where $a_\alpha = (1-\alpha)a_0 + \alpha a_1$ and $a_{\eta\xi,\alpha} = (1-\alpha)a_{\eta\xi,0} + \alpha a_{\eta\xi,1}$.

Proof: By Corollary (3.2) in [3], there exists a continuous map $\Phi : \tilde{\mathcal{A}}(\eta,\xi) \to wac(\tilde{\mathcal{A}})(\eta,\xi)$ such that for each $a \in \tilde{\mathcal{A}}$, $\Phi(a_{\eta\xi}) \in S^{(T)}(a)(\eta,\xi)$, and $\Phi(a_{\eta\xi,0}) = X_{\eta\xi,0},\ \ \Phi(a_{\eta\xi,1}) = X_{\eta\xi,1}$. Then, the map $h : [0,1] \to wac(\tilde{\mathcal{A}})(\eta,\xi)$ defined by $h(\alpha) = \Phi(a_{\eta\xi,\alpha})$ is the required map.

Definition : A space X is said to be arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval and its image $f([0,1])$

4. Main Result

In order to establish the arcwise connectedness of the space $S^{(T)}(P)(\eta,\xi)$, some idea from [14, 15] were employed in what follows.

Theorem 4.1: Assume that the conditions $\mathcal{S}_{(1)} \;-\; \mathcal{S}_{(5)}$ above are satisfied.
Then, for every $a \in \tilde{\mathcal{A}}$ and arbitrary pair $\eta, \xi \in I\!D \underline{\otimes} I\!E$, the set $S^{(T)}(a)(\eta,\xi)$ is arcwise connected in $C([0,T]; \mathbb{C})$.

Proof: Fix a_0 in $\tilde{\mathcal{A}}$ and let $X, Y \in S^{(T)}(a_0)$. Then the functions $X_{\eta\xi}(\cdot),\ Y_{\eta\xi}(\cdot) \in S^{(T)}(a_0)(\eta,\xi)$. By Corollary (3.2) in [3], there exists a continuous map

$$\Phi : \tilde{\mathcal{A}}(\eta,\xi) \to wac(\tilde{\mathcal{A}})(\eta,\xi)$$

such that $\Phi(a_{\eta\xi,0}) = X_{\eta\xi}(\cdot)$ and

$$\Phi(a_{\eta\xi}) \in S^{(T)}(a)(\eta,\xi), \forall a_{\eta\xi} \in \tilde{\mathcal{A}}(\eta,\xi). \tag{4.1}$$

Since $Y_{\eta\xi}(\cdot)$ is continuous on $[0,T]$, the map $\lambda \to \Phi(Y_{\eta\xi}(\lambda T))$ is continuous from $[0,1]$ to $wac(\tilde{\mathcal{A}})(\eta,\xi)$, being the composition of continuous maps

$$h : [0,1] \to [0,T]; Y_{\eta\xi} : [0,T] \to \tilde{\mathcal{A}}(\eta,\xi); \Phi : \tilde{\mathcal{A}}(\eta,\xi) \to wac(\tilde{\mathcal{A}})(\eta,\xi)$$

where $h(\lambda) = \lambda T$. Moreover,

$$\Phi(Y_{\eta\xi}(\lambda T)) \quad \in S^{(T)}(Y(\lambda T))(\eta,\xi), \tag{4.2}$$

for each $\lambda \in [0,T]$. Thus there exists a stochastic process
$\phi(Y(\lambda T)) \in S^{(T)}(Y(\lambda T))$ such that

$$\Phi(Y_{\eta\xi}(\lambda T)(t) = \langle \eta, \phi(Y(\lambda T))(t)\xi\rangle,\ t \in [0,T]. \tag{4.3}$$

Equation (4.3) implies that

$$\begin{aligned}\frac{d}{dt}\langle \eta, \phi(Y(\lambda T))(t)\xi\rangle &\in P(t, \phi(Y(\lambda T))(t))(\eta,\xi)\\ \phi(Y(\lambda T))(0) &= Y(\lambda T),\ t \in [0,T].\end{aligned} \tag{4.4}$$

Next, we define the following pair of maps. For each $\lambda \in [0,1]$,

$$X_\lambda(t) = \begin{cases} Y(t), & \text{if } 0 \le t \le \lambda T \\ \phi(Y(\lambda T))(t - \lambda T), & \text{if } \lambda T \le t \le T. \end{cases} \tag{4.5}$$

Setting $X_{\eta\xi,\lambda}(\cdot) = \langle \eta, X_\lambda(\cdot)\xi\rangle$, we obtain the inner product form of equation (4.1) given by:

$$X_{\eta\xi,\lambda}(t) = \begin{cases} Y_{\eta\xi}(t), & \text{if } 0 \le t \le \lambda T \\ \Phi(Y_{\eta\xi}(\lambda T))(t - \lambda T), & \text{if } \lambda T \le t \le T. \end{cases} \tag{4.6}$$

Notice that $X_0(\cdot) = X(\cdot)$, and $X_1(\cdot) = Y(\cdot)$, and

$$\begin{aligned} \frac{d}{dt}\langle \eta, X_\lambda(t)\xi\rangle &\in P(t, X_\lambda(t))(\eta, \xi) \\ X_\lambda(0) &= a_0, \text{ almost all } t \in [0, T]. \end{aligned} \tag{4.7}$$

By definition, for each $\lambda \in [0, 1]$,

$$X_\lambda \in Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}}). \tag{4.8}$$

Hence, $X_\lambda \in S^{(T)}(a_0)$. Therefore, $X_{\eta\xi,\lambda} \in S^{(T)}(a_0)(\eta, \xi)$. To complete the proof, it remains to be proved that the map $\lambda \to X_{\eta\xi,\lambda}$ is continuous from $[0, T]$ to the space $\mathbf{wac}(\tilde{\mathcal{A}})(\eta, \xi)$ in the topology of the space $C([0, T]; \mathbb{C})$. To this end, we employ a similar idea from [15]. Let $\epsilon > 0$ be given and let $\lambda_0 \in [0, 1]$ be fixed. We show that there exists $\delta > 0$ such that for any $\lambda \in [0, 1]$ with $|\lambda - \lambda_0| < \delta$, we have

$$\sup_{[0,T]} |X_{\eta\xi,\lambda}(t) - X_{\eta\xi,\lambda_0}(t)| < \epsilon. \tag{4.9}$$

For $t \in [0, T]$, we distinguish three cases as follows:

$$\text{(i) } 0 \le t \le \lambda_0 T \le \lambda T, \text{ (ii) } \lambda_0 T \le t \le \lambda T, \text{ (iii) } \lambda_0 T \le \lambda T \le t \le T. \tag{4.10}$$

In the case of (i) we have

$$|X_{\eta\xi,\lambda}(t) - X_{\eta\xi,\lambda_0}(t)| = |Y_{\eta\xi}(t) - Y_{\eta\xi}(t)| = 0. \tag{4.11}$$

For case (ii) where $\lambda_0 T \le t \le \lambda T$, then

$$|X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| = |\Phi(Y_{\eta\xi}(\lambda_0 T))(t - \lambda_0 T) - Y_{\eta\xi}(t)| \tag{4.12}$$

Since the map $t \to \Phi(Y_{\eta\xi}(\lambda_0 T))(t)$ and $t \to Y_{\eta\xi}(t)$ are uniformly continuous on the interval $I = [0, T]$, there exists $\delta_1 > 0$ such that for any t' and t'' in $[0, T]$ with $|t' - t''| < \delta_1$, we have

$$|\Phi(Y_{\eta\xi}(\lambda_0 T))(t') - \Phi(Y_{\eta\xi}(\lambda_0 T))(t'')| < \frac{\epsilon}{2} \tag{4.13}$$

and

$$|Y_{\eta\xi}(t') - Y_{\eta\xi}(t'')| < \frac{\epsilon}{2}. \tag{4.14}$$

Let $|\lambda_0 - \lambda| < \frac{\delta_1}{T}$. Then $|t - \lambda_0 T| \le |\lambda - \lambda_0| T \le \delta_1$ and since
$|\Phi(Y_{\eta\xi}(\lambda_0 T)(0) = Y_{\eta\xi}(\lambda_0 T)$, we have

$$\begin{aligned} & |X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| \\ \le\; & |\Phi(Y_{\eta\xi}(\lambda_0 T)(t - \lambda_0 T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(0)| + |\Phi(Y_{\eta\xi}(\lambda_0 T))(0) - Y_{\eta\xi}(t)| \\ \le\; & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{4.15}$$

For case (iii), then

$$\begin{aligned} |X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| &= |\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda_0 T) - \Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T)| \\ &\leq |\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda_0 T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T)| \\ &+ |\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T) - \Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T)|. \quad (4.16) \end{aligned}$$

Since the map $\lambda \to \Phi(Y_{\eta\xi}(\lambda T))$ is continuous from $[0,1]$ to $\mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi) \subseteq C([0,T];\mathbb{C})$, there exists $\delta_2 > 0$ such that

$$|\lambda - \lambda_0| < \delta_2 \tag{4.17}$$

implies that

$$\sup_{t\in[0,T]} |\Phi(Y_{\eta\xi}(\lambda T))(t) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t)| < \frac{\epsilon}{2}, \tag{4.18}$$

so that for $|\lambda - \lambda_0| < \delta_2$, we have

$$|\Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T)| < \frac{\epsilon}{2}. \tag{4.19}$$

Furthermore, since the map $t \to \Phi(Y_{\eta\xi}(\lambda_0 T))(t)$ is uniformly continuous on $[0,T]$, there exists $\delta_3 > 0$ such that for any pair of points $t^{'}$, $t^{''}$ in $[0,T]$, $|t^{'} - t^{''}| < \delta_3$ implies that

$$|\Phi(Y_{\eta\xi}(\lambda_0 T))(t^{'}) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t^{''})| < \frac{\epsilon}{2}. \tag{4.20}$$

Then if $|\lambda - \lambda_0| < \frac{\delta_3}{T}$, we have

$$|t - \lambda T - t + \lambda_0 T| \leq |\lambda - \lambda_0| T \leq \delta_3, \tag{4.21}$$

and

$$|\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda_0 T)| < \frac{\epsilon}{2}. \tag{4.22}$$

By Equations (4.16), (4.19) and (4.22), if $|\lambda - \lambda_0| < \min\{\delta_2, \frac{\delta_3}{T}\}$, then

$$|X_{\eta\xi,\lambda}(t) - X_{\eta\xi,0}(t)| < \epsilon. \tag{4.23}$$

Let

$$\delta = \min\{\frac{\delta_1}{T},\ \delta_2,\ \frac{\delta_3}{T}\}, \tag{4.24}$$

then we have proved that if $\lambda_0 \leq \lambda$ and $|\lambda - \lambda_0| < \delta$, then for any $t \in [0,T]$,

$$|X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| < \epsilon. \tag{4.25}$$

This implies that

$$\sup_{t\in[0,T]} |X_{\eta\xi,\lambda}(t) - X_{\eta\xi,\lambda_0}(t)| < \epsilon. \tag{4.26}$$

For $\lambda \leq \lambda_0$, the proof is similar.

Theorem 4.2: Corresponding to an arbitrary pair of points $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$, the function space $S^{(T)}(P)(\eta, \xi)$ is arcwise connected in $C([0,T];\ \mathbb{C})$.

Proof: Let $X,\ Y\ \in S^{(T)}(P) := \bigcup_{a \in \tilde{\mathcal{A}}} S^{(T)}(a)$ such that for any pair of distinct elements $a,\ a_0 \in \tilde{\mathcal{A}}$, $X \in S^{(T)}(a_0)$ and $Y \in S^{(T)}(a)$. Then by Proposition 3.1, there exists a continuous map $h : [0,1] \to wac(\tilde{\mathcal{A}})(\eta, \xi)$ such that $h(0) = X_{\eta\xi}$, $h(1) = Y_{\eta\xi}$ and for each $\alpha \in [0,1]$, $h(\alpha) \in S^{(T)}(a_\alpha)(\eta, \xi)$, where $a_\alpha = (1-\alpha)a_0 + \alpha a$. If $a = a_0$, then $X,\ Y \in S^{(T)}(a_0)$ and the existence of a continuous map $h : [0,1] \to wac(\tilde{\mathcal{A}})(\eta, \xi)$ such that $h(0) = X_{\eta\xi}$ and $h(1) = Y_{\eta\xi}$, $h(\alpha) \in S^{(T)}(a_0)(\eta, \xi)$ follows from Theorem 4.1 above.

Acknowledgments

Research supported by University of Ibadan Senate Research grant 2006

References

[1] Aubin J P and Cellina A 1984 *Differential Inclusions* (Berlin:Springer - Verlag)
[2] Aurelian Cernea 2008 *Miskolc Math. Notes* **9** No.1 pp. 33-39
[3] Ayoola E O 2008 *Acta Appl. Math.* **100** 15 - 37
[4] Ayoola E O 2004 *Internat. Journal Theoret. Physics* **43** 2041 - 2059
[5] Ayoola E O 2003 *Stochastic Anal.Applic.* **21** 1215 - 1230
[6] Ayoola E O 2003 *Stochastic Anal. Applic.***21** 515 - 543
[7] Ayoola E O 2001 *Stochastic Anal. Applic.* **19** 461 - 471
[8] Ekhaguere G O S 1992 *Inter. J. Theor. Phys.* **31** 2003-2034
[9] Ekhaguere G O S 1995 *Inter. J. Theor. Phys.* **34** 323-353
[10] Ekhaguere G O S 1996 *Inter. J. Theor. Phys.* **35** 1909-1946
[11] Hudson R L and Parthasarathy K R 1984 *Comm. Math. Phys.* **93** 301-324
[12] Ogundiran M O and Ayoola E O 2010 *J. Math. Phys.* **51** 023521
[13] Ren-you Zhong, Nan-jing Huang and Mu-ming Wong 2009 *Taiwanese J. Math.* **13** 821–836
[14] Staicu V 1991 *Proc. AMS* **113** 403 - 413
[15] Staicu V and Wu H 1991 *Bollettino U. M. I.* **7** 253-256