Impulsive Vaccination for an Epidemiology Model

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Abstract. This paper investigates sufficient conditions of almost periodic sand periodic solutions of an integral model under impulsive controls. Since the model is of generic epidemiological interest, such impulsive controls are either vaccination actions or abrupt variations of the infected population due to infected immigration or lost of infective numbers due to either vaccination or lost of infected population by out-migration.

1. Introduction

This research deals with a nonlinear integral equation with eventual distributed time-varying time delay. Some of its necessary and relevant properties like positivity and boundedness of the solution are investigated. Also, some conditions of almost periodicity and periodicity of the solutions are dealt with under the action of impulsive controls art certain impulsive time instants. The impulsive controls can be interpreted as vaccination effort or infected immigration if the infected population decreases or output migration otherwise. The model has been investigated in [1-9] in a simplified version the impulsive- free case. The impulsive vaccination is a control, although the epidemic model is not controllable, by nature. For state and output measuring-based controller synthesis, see, for instance, [10-13]. The following basic notation is used:

$\boldsymbol{R}_+=\{z\in\boldsymbol{R}:z>0\}$, $\boldsymbol{R}_-=\{z\in\boldsymbol{R}:z<0\}$, $\boldsymbol{R}_{0+}=\boldsymbol{R}_+\cup\{0\}$, $\boldsymbol{R}_{0-}=\boldsymbol{R}_-\cup\{0\}$, $\boldsymbol{R}_{0+}^n$ and $\boldsymbol{R}_+^n$ are the n-th orthants of $\boldsymbol{R}^n$ whose components are in $\boldsymbol{R}_{0+}$ and $\boldsymbol{R}_+$, respectively. $\boldsymbol{R}_{0+}^{n\times n}$ and $\boldsymbol{R}_+^{n\times n}$ denote the sets of real square n-matrices whose entries are in $\boldsymbol{R}_{0+}$ and $\boldsymbol{R}_+$, respectively. $\boldsymbol{R}_{0-}^n$ and $\boldsymbol{R}_-^n$ are the n-th orthants of $\boldsymbol{R}^n$ whose components are in $\boldsymbol{R}_{0-}$ and $\boldsymbol{R}_-$, respectively. $\boldsymbol{R}_{0-}^{n\times n}$ and $\boldsymbol{R}_-^{n\times n}$ denote the sets of real square n-matrices whose entries are in $\boldsymbol{R}_{0-}$ and $\boldsymbol{R}_-$, respectively,

$\boldsymbol{Z}_+=\{z\in\boldsymbol{Z}:z>0\}$, $\boldsymbol{Z}_{0+}=\{z\in\boldsymbol{Z}:z\ge 0\}=\boldsymbol{Z}_+\cup\{0\}$, $\bar{n}=\{1,2,\dots,n\}$,
if $x,y\in\boldsymbol{R}^n$ then $x\le y$ $(x<y)$ means $x_i\le y_i$ $(x_i<y_i)$; $\forall i\in\bar{n}$. If $A,B\in\boldsymbol{R}^{n\times m}$ then $A=(A_{ij})\le B=(B_{ij})$ $(A<B)$ means $A_{ij}\le B_{ij}$ $(A_{ij}<B_{ij})$; $\forall(i,j)\in\bar{n}\times\bar{m}$,
the superscript T stands for the transpose of a real vector or matrix,

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if $x \in \boldsymbol{R}^n$, then $|x| = \left(|x_1|, |x_2|, \dots, |x_n|\right)^T$, if $A \in \boldsymbol{R}^{n\times m}$ then $|A| = \left(|A|_{ij}\right) = \left(\left|A_{ij}\right|\right)$, if a real function $x(t)$ is discontinuous at some t then $x(t^-)$ is its left limit at t and $x(t)$ (abbreviating $x(t^+)$) is its right limit so that if it is continuous at t its value is $x(t) = x(t^-)$.

2. The model and its impulsive controlling actions

Consider the nonlinear integral equation with a finite distributed delay function $\tau : \boldsymbol{R}_{0+} \to \boldsymbol{R}_+$ and eventual bounded discontinuities:

$$x(t^-) = \int_{t-\tau(t)}^{t} F(s, x(s), y(s))ds \ ; \quad x(t) = K(t)x(t^-) \tag{1}$$

, [1-2], where $\tau(0) \ge \sup_{t\in \boldsymbol{R}_{0+}} \tau(t)$, $F : (\boldsymbol{R}_{0+} \cup [-\tau(0), 0)) \times \boldsymbol{R}_{0+}^n \times \boldsymbol{R}_{0+}^n \to \boldsymbol{R}_{0+}^n$, $x, y : [-\tau(0), \infty) \to \boldsymbol{R}^n$, and $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}^{n\times n}$ are a positive vector function and a positive square n-matrix function, respectively. In particular, $x : [-\tau(0), \infty) \to \boldsymbol{R}^n$ is the solution of (1) and $y : [-\tau(0), \infty) \to \boldsymbol{R}^n$ can describe different actions like, for instance, an external disturbance (i.e. a disturbance which is independent of the solution) or either an internal solution-dependent disturbance or a point-delay or distributed delayed or nonlinear contribution to the solution. For instance, the special form $F(t, x, y) = f(t, x) + g(t, y)$ is discussed in [2] for the integral equation (1), where f, g and τ are positive almost periodic. Integral equations of the form (1) are of interest in the descriptions of some epidemic models. For instance, $x(.)$could be the susceptible population which is measurable or partially known and can be treated under vaccination effort, and $y(.)$ some coupled population like, for instance, the infectious one. Note that, for any $t \in \boldsymbol{R}_{0+}$, if $K(t) = I_n$ (the n-th identity matrix) then $x(t) = x(t^-)$ so that $x : [-\tau(0), \infty) \to \boldsymbol{R}^n$ is continuous at time t. If $K(t) \ne I_n$ then $x(t) \ne x(t^-)$ if $x(t^-) \ne 0$. Note that (2) versus (1) can be equivalently viewed for the case when $x \equiv y$ under an integral contribution of an impulsive control gain $K(t)$ as

$$x(t) = x(t^-) + \int_{t^-}^{t} (K(s) - I_n)x(s)\delta(s-t)ds = x(t^-) + \lim_{\varepsilon\to 0^+} \int_{t-\varepsilon}^{t+\varepsilon} (K(s) - I_n)x(s)\delta(s-t)ds = x(t^-) + (K(t) - I_n)x(t^-) \tag{2}$$

where the integrals in (1)-(2) are Lebesgue integrals, $(K(t) - I_n)x(t^-)$ is the left-continuous test vector function which has a compact support and $\delta(t)$ denotes the Dirac distribution such that

$\int_{-\infty}^{\infty} \delta(s)ds = \lim_{\varepsilon\to 0^+} \int_{-\varepsilon}^{\varepsilon} \delta(s)ds = 1$.Then $\delta(t) = +\infty$ if $t = 0$ and $\delta(t) = 0$ if $t \ne 0$. Note that denoting the shifted Dirac delta as $\delta_t(s) = \delta(s-t)$, one has for any $t \in \boldsymbol{R}_{0+}$

$$\int_{-\infty}^{\infty} f(s)\delta(s-t)ds = \lim_{\varepsilon\to 0^+} \int_{t-\varepsilon}^{t+\varepsilon} f(s)\delta(s-t)ds = \int_{t^-}^{t} f(s)\delta(s-t)ds = \int_{t^-}^{t} f(s)\delta_t(s)ds = f(t^-)$$

Note also that

$\int_{-\infty}^{\infty} f(s)\delta(s)ds = \lim_{\varepsilon\to 0^+} \int_{-\varepsilon}^{\varepsilon} f(s)\delta(s-t)ds = f(0^-)$ so that the Dirac distribution delta $\delta(t)$ is also a not absolutely continuous measure with argument being a subset A of the real line with $\delta(A) = 1$ if $0 \in A$ and $\delta(A) = 1$ if $0 \notin A$. The impulsive set of time instants is $Imp = \{t \in \boldsymbol{R}_{0+} : x(t) \ne x(t^-) \quad if \quad x(t^-) \ne 0\}$.

The following positivity result for the solution (1)-(2), in the sense that no component state takes negative values at any time if the initial conditions are non-negative, is immediate:

Proposition 1. Assume some initial condition $x : [-\tau(0), 0) \to \boldsymbol{R}_{0+}^{n}$ for the model with $F : (\boldsymbol{R}_{0+} \cup [-\tau(0), 0)) \times \boldsymbol{R}_{0+}^{n} \times \boldsymbol{R}_{0+}^{n} \to \boldsymbol{R}_{0+}^{n}$ in (1) such that the feedback control gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}^{n \times n}$. Then, the following properties hold:
(i) $x : [-\tau(0), \infty) \to \boldsymbol{R}_{0+}^{n}$,
(ii) if the feedback control is replaced with the non-feedback control $x(t) = K(t)$ if $t \in Imp$ with $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}^{n}$ then $x : [-\tau(0), \infty) \to \boldsymbol{R}_{0+}^{n}$.

Note that the impulsive controlled action can be interpreted as due to a culling or quarantine of the infected population if $\|K(t)\| < 1$. Contrarily, if $K(t)$ causes the inequality $\|x(t)\| > \left\|x\left(t^{-}\right)\right\|$ then there is a positive increase of infected population due to infected immigration into the studied environment.

3. Periodic and almost periodic solutions

Now, consider the impulsive set of time instants $t_i, t_{i+1}(> t_i) \in Imp \subset \boldsymbol{R}_{0+}$, with $T_i = t_{i+1} - t_i$, so that for some sequence $\{K(t_i)\}$ of non-identity matrices, one has:

$$x(t_i) = K(t_i) x\left(t_i^{-}\right) = \int_{t_i - \tau(t_i)}^{t_i} F(s, x(s), y(s)) ds \; ; \forall t_i \in Imp \tag{3}$$

The impulsive set Imp is denumerable and consists of strictly ordered elements and can be finite or infinite. Introduce by convenience the identities:

$$\tau(t_{i+1}) = \tau_0 + \tilde{\tau}(t_{i+1}) = \tau_0 + \tilde{\tau}(t_i) + \tilde{\tilde{\tau}}(t_i) ; \quad K(t_{i+1}) = K_0 + \tilde{K}(t_{i+1}) = K_0 + \tilde{K}(t_i) + \tilde{\tilde{K}}(t_i) = K(t_i) + \tilde{\tilde{K}}(t_i) \tag{4}$$

; $\forall t_i \in Imp$, which allow to refer all delays $\tau(t_{i+1})$ at the impulsive set related to a constant delay τ_0 plus its deviation $\tilde{\tau}(t_{i+1})$. It is assumed that the nominal gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}^{n \times n}$ This deviation can be related to that of its delay $\tilde{\tau}(t_i)$ at the preceding impulsive time instant by its incremental value $\tilde{\tilde{\tau}}(t_i) = \tilde{\tau}(t_{i+1}) - \tilde{\tau}(t_i)$. A similar incremental description can be given for the impulsive gain sequence. The following definitions are then used to characterize almost periodicity of the solutions, [2].

Definition 1. A set $E \subset \boldsymbol{R}$ of real numbers is said to be relatively dense if there exists a number $\ell \in \boldsymbol{R}_{+}$ such that any real interval $(\alpha, \alpha + \ell)$ of length ℓ contains at least one number in E.

Definition 2. A function $f \in BC\left(\boldsymbol{R} \times \boldsymbol{\Omega}, \boldsymbol{R}^{q}\right)$ is said to be almost periodic in $t \in \boldsymbol{R}$ uniformly in $x \in \boldsymbol{\Omega}$ (abbreviated by $f \in AP\left(\boldsymbol{R} \times \boldsymbol{\Omega}, \boldsymbol{R}^{q}\right)$) if the ε-translation set of f

$$T(f, \varepsilon) = \left\{\omega \in \boldsymbol{R} : \| f(t + \omega, z) - f(t, z) \| < \varepsilon ; \forall (t, z) \in \boldsymbol{R} \times \boldsymbol{W}, \forall compact\ set\ \boldsymbol{W} \subset \boldsymbol{\Omega}\right\}$$

is a relatively dense set in $\boldsymbol{R}$ for all $\varepsilon \in \boldsymbol{R}_{+}$. Each $\omega \in T(f, \varepsilon)$ is called an ε- period of f.

Almost periodic functions without uniformity in $\boldsymbol{\Omega}$ are defined closely by removing the argument z in $\boldsymbol{\Omega}$ and the compact subset $\boldsymbol{W}$ from the definition of $T(f, \varepsilon)$. See, for instance, [2].

Definition 3. A function $f \in BC\left(\boldsymbol{R} \times \boldsymbol{\Omega}, \boldsymbol{R}^{q}\right)$ is said to be ρ-almost periodic in $t \in \boldsymbol{R}$ uniformly in $x \in \boldsymbol{\Omega}$ (abbreviated by $f \in AP\left(\mathbf{R} \times \mathbf{\Omega}, \mathbf{R}^{q}, \rho\right)$) for some $\rho \in \mathbf{R}_{+}$ if the ε-translation set of f

$T(f,\varepsilon)=\{\omega\in \boldsymbol{R}: \| f(t+\omega, z)-f(t,z)\|<\varepsilon ; \forall(t,z)\in \boldsymbol{R}\times \boldsymbol{W}, \forall compact\ set\ \boldsymbol{W}\subset \boldsymbol{\Omega}\}$
is a relatively dense set in $\boldsymbol{R}$ for all $\varepsilon\in[\rho,\infty)$. Each $\omega\in T(f,\varepsilon)$ is called an ε- period of f for a given $\varepsilon\in[\rho,\infty)$.

The following main result, whose proof is omitted, follows:

Theorem 1. Assume that the model integral equation, subject to $F:(\boldsymbol{R}_{0+}\cup[-\tau(0),0))\times \boldsymbol{R}^n_{0+}\times \boldsymbol{R}^n_{0+}\to \boldsymbol{R}^n_{0+}$, $x,y:[-\tau(0),\infty)\to \boldsymbol{R}^n$ and $K:\boldsymbol{R}_{0+}\to \boldsymbol{R}^{n\times n}_{0+}$, satisfies the subsequent hypotheses:

H1) $F\in BC\left(\boldsymbol{R}_{0+}\times \boldsymbol{R}^n_{0+}, \boldsymbol{R}^n_{0+}\right)\cap AP\left(\boldsymbol{R}_{0+}\times \boldsymbol{R}^n_{0+}, \boldsymbol{R}^n_{0+},\varepsilon\right)$ for some given $\varepsilon\in \boldsymbol{R}_+$,

H2) $T_i=t_{i+1}-t_i\geq \tau_0+\tilde{\tau}(t_i)$, the i- th inter-impulse time interval $T_i\in P(F,\lambda_{1i}\varepsilon)$ (i.e. it is an $\lambda_{1i}\varepsilon$ - period for F for some given $\varepsilon\in \boldsymbol{R}_+$), $(T_i-\tau_0)\in P(F,\lambda_{2i}\varepsilon)$, $(T_i-\tau_0-\tilde{\tau}(t_i))\in P(F,\lambda_{3i}\varepsilon)$ for some $\lambda_{ji}\in \boldsymbol{R}_+$ and $j\in\overline{3}$; $\forall i\in \boldsymbol{Z}_{0+}$, for $\varepsilon\in \boldsymbol{R}_+$ in H1,

H3) $\tau_0\lambda_{1i}+\tilde{\tau}(t_i)\lambda_{2i}+\tilde{\tilde{\tau}}(t_i)\lambda_{3i}\leq 1$; $\forall i\in \boldsymbol{Z}_{0+}$

Thus, the following properties hold: **(i)** $\left\|\tilde{x}\left(t_i^-\right)\right\|<\varepsilon$ and $\{T_i\}$ is a discrete ε-period sequence for the left limit solution $x\left(t^-\right)$. If $\|K(t_i)\|\leq min\left(\frac{1}{2}, \frac{1-\|K(t_{i-1})\|}{\left\|x\left(t_i^-\right)\right\|}\varepsilon-\|K(t_{i-1})\|\right)$ then $\|\tilde{x}(t_i)\|<\varepsilon$ and $\{T_i\}$ is a discrete ε-period sequence for the right limit solution $x(t)$,

(ii) $\left\|\tilde{x}(t_{i\eta})\right\|<\varepsilon$, where $\tilde{x}(t_{i\eta})=x(t_i+\eta)-x(t_i)$, for any real $\eta\in(0,T_i)$; $\forall i\in \mathbf{Z}_{0+}$ so that $x\in AP\left(\boldsymbol{R}\times \boldsymbol{R}^n_{0+}, \boldsymbol{R}^n_{0+},\varepsilon\right)$, i.e. the solution of (1)-(2) is ε-almost periodic in $t\in \boldsymbol{R}$ uniformly in $\boldsymbol{R}^n_{0+}$.

Under the conditions of Theorem 1, the free-impulsive integrand of the model is assumed almost-periodic according to the Hypothesis H1. However, the impulsive effort at certain impulsive time instants can also achieve the almost periodicity of the solution. The following relations are obtained:

$$\begin{aligned}
x(t+\rho)-x(t)&=\int_{t+\rho-\tau(t)}^{t+\rho}F\left(s,\mu(s)x_{if}(s),V(s,x(s))\right)ds-\int_{t-\tau(t)}^{t}F\left(s,\mu(s)x_{if}(s),V(s,x(s))\right)ds\\
&\geq\lambda\left(x_{if}(t+\rho)-x_{if}(t)\right)=\lambda\left(x_{if}(t+\rho)-x_{if}(t)\right)+\lambda_0(t,\rho)\\
+\sum_{i=1}^{m(t+\rho)}&F\left(t_i(t+\rho),x\left(t_i^-(t+\rho)\right),0\right)V\left(t_i(t+\rho),x\left(t_i^-(t+\rho)\right)\right)-\sum_{i=1}^{m(t)}F\left(t_i(t),x\left(t_i^-(t)\right),0\right)V\left(t_i(t),x\left(t_i^-(t)\right)\right)\\
&\leq\lambda\,\varepsilon_{if}+\lambda_0(t,\rho)-\sum_{i=1}^{m(t+\rho)}F\left(t_i(t+\rho),x\left(t_i^-(t+\rho)\right),0\right)\left|V\left(t_i(t+\rho),x\left(t_i^-(t+\rho)\right)\right)\right|\\
&\quad+\sum_{i=1}^{m(t)}F\left(t_i(t),x\left(t_i^-(t)\right),0\right)\left|V\left(t_i(t),x\left(t_i^-(t)\right)\right)\right|\leq\varepsilon
\end{aligned}\tag{5}$$

provided that under impulsive vaccination (i.e. negative impulses) :

$$\sum_{i=1}^{m(t+\rho)}F\left(t_i(t+\rho),x\left(t_i^-(t+\rho)\right),0\right)\left|V\left(t_i(t+\rho),x\left(t_i^-(t+\rho)\right)\right)\right|$$

$$-\sum_{i=1}^{m(t)} F\left(t_i(t), x\left(t_i^-(t)\right), 0\right)\left|V\left(t_i(t), x\left(t_i^-(t)\right)\right)\right| \geq \lambda\, \varepsilon_{if} + \lambda_0(t, \rho) - \varepsilon \tag{6}$$

and under general impulsive action with non-necessarily negative impulses:

$$\begin{aligned}&\sum_{i=1}^{m(t+\rho)} F\left(t_i(t+\rho), x\left(t_i^-(t+\rho)\right), 0\right) V\left(t_i(t+\rho), x\left(t_i^-(t+\rho)\right)\right)\\ &\quad -\sum_{i=1}^{m(t)} F\left(t_i(t), x\left(t_i^-(t)\right), 0\right) V\left(t_i(t), x\left(t_i^-(t)\right)\right) \leq \varepsilon - \lambda\varepsilon_{if} - \lambda_0(t, \rho)\end{aligned} \tag{7}$$

Let the integrand of the epidemic model be decomposable as $F(t, x, y) = f(t, x) + g(t, y)$ subject to

$$g\left(t, \alpha^{-1}x\right) \geq \varphi(t, \alpha, x) g(t, x) > \alpha g(t, x) \geq \beta g(t, x)$$
$$f(t, \beta x) \geq \varphi\left(t, \beta^{-1}, x\right) f(t, x) > \beta f(t, x)$$

with $min\left(\varphi(t, \alpha, x), \varphi\left(t, \alpha^{-1}, x\right)\right) > \alpha \geq \beta$ and $min\left(\varphi(t, \beta, x), \varphi\left(t, \beta^{-1}, x\right)\right) > \beta$. Then,

$$min\left(\varphi(t, \alpha, x), \varphi\left(t, \beta^{-1}, y\right)\right) > \beta$$

so that

$$F\left(t, \beta x, \alpha^{-1}y\right) = f(t, \beta x) + g\left(t, \alpha^{-1}y\right) > \beta\left(f(t, x) + g(t, y)\right) = \beta F(t, x, y)$$

where $\beta, \alpha \in (0, 1)$ with $\beta \leq \alpha$. Assume that the scalar vaccination gains are generated as follows:

$$k_{di}(t) = -\left|k_{di}(t)\right| sgn\, F_i(t, s, x(s), x(s-\theta))$$

then,

$$x\left(t^-\right) = \int_{t-\tau(t)}^{t^-} F(t, s, x(s), x(s-\theta))\, ds$$
$$x_i(t) = x_i\left(t^-\right) - \int_{t^-}^{t} \left|F_i(t, s, x(s), x(s-\theta(s)))\right| \left|k_i(t)\right| \delta(t-s)\, ds$$
$$= x_i\left(t^-\right) - \left|k_i(t)\right| \left|F_i\left(t, t, x\left(t^-\right), x(t-\theta(t))^-\right)\right|$$

If $x\left(t^-\right) = \alpha^{-1}(t) x(t)$ and $\beta(t)$ are the impulsive infective population increase at time $(t-\theta(t))$ and the corresponding impulsive vaccination correction at the current time t satisfying $\beta(t) \leq \alpha(t-\theta(t)) < 1$

$$x\left(t^-\right) - x(t) = |k(t)| \left|F\left(t, t, x\left(t^-\right), x(t-\theta(t))^-\right)\right| \geq |k(t)| \beta(t) \left|F(t, t, x(t), x(t-\theta(t)))\right| \tag{8}$$

so that

$$\begin{aligned}x(t+\rho) &= \int_{-\tau_0-\tilde{\tau}(t+\rho)}^{0} F\left(s+t+\rho, x(s+t+\rho), V\left(s+t+\rho, x\left(s^-+t+\rho\right)\right)\right) ds\\ &= \int_{-\tau(t+\rho)}^{0} F(s+t+\rho, x(s+t+\rho), 0)\, ds\\ &\quad + \sum_{t_i(t+\rho) \in Imp[t+\rho-\tau_0-\tilde{\tau}(t+\rho), t+\rho]} F\left(t_i(t+\rho), x\left(t_i^-(t+\rho)\right), 0\right) V\left(t_i(t+\rho), x\left(t_i^-(t+\rho)\right)\right)\end{aligned} \tag{9}$$

for $t \ge \tau_0 - \tau(0) + \tilde{\tau}_M$ with $\tau(t) = \tau_0 + \tilde{\tau}(t)$, $\tilde{\tau}(t) \in [\tilde{\tau}_m , \tilde{\tau}_M]$, $\tilde{\tau}_M \ge \tilde{\tau}_m \ge -\tau_0$ and $\rho \in \boldsymbol{R}_{0+}$ .Note that

$$\left\| x(t) - x(t^-) \right\| = \begin{cases} 0 & if \quad t \notin Imp[t - \tau(t), t] \\ \left\| F(t, x(t^-), 0) V(t, x(t^-)) \right\| & if \quad t \in Imp[t - \tau(t), t] \end{cases}$$

$$\left\| x(t^- + \rho) - x(t) \right\| = \left\| x(t^- + \rho) - x(t^-) - (x(t) - x(t^-)) \right\| \le \begin{cases} \left\| x(t^- + \rho) - x(t) \right\| & if \quad t \notin Imp[t - \tau(t), t] \\ \left\| x(t^- + \rho) - x(t) \right\| + \left\| F(t, x(t^-), 0) V(t, x(t^-)) \right\| & if \quad t \in Imp[t - \tau(t), t] \end{cases} \tag{10}$$

On the other hand, if $\tau(t+\rho) = \tau(t)$, $F(t, x(t), 0)$ is Lipschitz with constant K

$$\begin{aligned} \left\| x(t^- + \rho) - x(t^-) \right\| \le{} & K \int_{-\tau(t)}^{0} \left\| x(s + t + \rho) - x(s + t) \right\| ds \\ & + \Big\| \textstyle\sum_{t_i(t+\rho) \in Imp[t+\rho-\tau(t), t+\rho)} F(t_i(t+\rho), x(t_i^-(t+\rho)), 0) V(t_i(t+\rho), x(t_i^-(t+\rho))) \\ & - \textstyle\sum_{t_i(t) \in Imp[t-\tau(t), t)} F(t_i(t), x(t_i^-(t)), 0) V(t_i(t), x(t_i^-(t))) \Big\| \end{aligned} \tag{11}$$

$$\begin{aligned} \left\| x(t^- + \rho) - x(t) \right\| \le{} & K \int_{-\tau(t)}^{0} \left\| x(s + t + \rho) - x(s + t) \right\| ds \\ & + \Big\| \textstyle\sum_{t_i(t+\rho) \in Imp[t+\rho-\tau(t), t+\rho)} F(t_i(t+\rho), x(t_i^-(t+\rho)), 0) V(t_i(t+\rho), x(t_i^-(t+\rho))) \\ & - \textstyle\sum_{t_i(t) \in Imp[t-\tau(t), t]} F(t_i(t), x(t_i^-(t)), 0) V(t_i(t), x(t_i^-(t))) \Big\| \end{aligned} \tag{12}$$

$$\left\| x(t + \rho) - x(t) \right\| \le \begin{cases} \left\| x(t^- + \rho) - x(t) \right\| & if \quad t + \rho \notin Imp[t + \rho - \tau(t + \rho), t + \rho] \\ \left\| x(t^- + \rho) - x(t) \right\| + \left\| F(t + \rho, x(t^- + \rho), 0) V(t + \rho, x(t^- + \rho)) \right\| & if \quad t + \rho \in Imp[t + \rho - \tau(t + \rho), t + \rho] \end{cases} \tag{13}$$

$$\begin{aligned} \left\| x(t + \rho) - x(t) \right\| \le{} & K \int_{-\tau(t)}^{0} \left\| x(s + t + \rho) - x(s + t) \right\| ds \\ & + \Big\| \textstyle\sum_{t_i(t+\rho) \in Imp[t+\rho-\tau(t), t+\rho]} F(t_i(t+\rho), x(t_i^-(t+\rho)), 0) V(t_i(t+\rho), x(t_i^-(t+\rho))) \\ & - \textstyle\sum_{t_i(t) \in Imp[t-\tau(t), t]} F(t_i(t), x(t_i^-(t)), 0) V(t_i(t), x(t_i^-(t))) \Big\| \end{aligned} \tag{14}$$

$$I_{imp}(t, x_t) = \textstyle\sum_{t_i(t) \in Imp[t-\tau(t), t]} V(t_i(t), x(t_i^-(t))) \delta(t - t_i(t)) \tag{15}$$

Thus, the following result holds:

Theorem 2. Assume that $\tau(t+\rho) = \tau(t)$, $F(t, x(t), 0)$ is Lipschitzian with constant *K*. Furthermore,

$$card\, Imp[t - \tau(t), t] = card\, Imp[t + \rho - \tau(t + \rho), t + \rho] = m(t),$$

$t_i(t) \in Imp[t-\tau(t), t] \Rightarrow t_i(t+\rho) \in Imp[t+\rho-\tau(t), t+\rho]$ then $t_i(t), t_i(t+\rho) \in Imp[t-\tau(t), t+\rho]$,

$V(t_i(t + k\rho), x(t_i^-(t + k\rho))) \le V(t_i(t), x(t_i^-(t))) + \nu\,\varepsilon$ for some $\nu \in \boldsymbol{R}^n$; $\forall k \in \boldsymbol{Z}_{0+}$

Then, for all $t \in \boldsymbol{R}_{0+}$, one has that the solution is almost ε-oscillatory with period ρ in the sense [4-5] that:

$$\left\| x(t + \rho) - x(t) \right\| \le K \int_{-\tau(t)}^{0} \left\| x(s + t + \rho) - x(s + t) \right\| ds$$

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$$\leq \varepsilon\left[K\left(\tau(t)+m(t)\max_{t_i(t)\in[t-\tau(t),t]}\left\|V\left(t_i(t),x\left(t_i^-(t)\right)\right)\right\|\right)+\|\nu\|\right] \tag{16}$$

provided that

$$\|\nu\|\leq 1-K\sup_{t\in\boldsymbol{R}_{0+}}\left(\tau(t)+m(t)\max_{t_i(t)\in[t-\tau(t),t]}\left\|V\left(t_i(t),x\left(t_i^-(t)\right)\right)\right\|\right) \tag{17}$$

under the necessary condition that

$$K\leq \sup_{t\in\boldsymbol{R}_{0+}}\left(\tau(t)+m(t)\max_{t_i(t)\in[t-\tau(t),t]}\left\|V\left(t_i(t),x\left(t_i^-(t)\right)\right)\right\|\right)^{-1} \tag{18}$$

Note, on the other hand that $\tilde{x}\left(t^-\right)=x(t)-x\left(t^-\right)=K(t)x\left(t^-\right)$, where $K(t)=0$ implies $\tilde{x}\left(t^-\right)=0$ so that $x(t)\neq x\left(t^-\right)$ provided that $x\left(t^-\right)\neq 0$ if $K(t)\neq 0$. Then, for some $\sigma:\boldsymbol{R}_{0+}\to\boldsymbol{R}_{0+}$, one has

$$\left\|x\left(t^-\right)\right\|=\left\|\int_{-\tau(t)}^{0^-}F(s+t,x(s+t),V(s+t,x(s+t)))ds\right\|\leq\tau(t)\sigma\left(t^-\right)\sup_{s\in[t-\tau(t),t)}\|F(s,x(s),0)\| \tag{19}$$

Note the following features:

a) if $\tilde{x}_t\left(s,x\left(s^-\right)\right)=0$, i.e. $\tilde{x}\left(s,x\left(s^-\right)\right)=0$ for $s\in[t-\tau(t),0)$, equivalently, $Imp[t-\tau(t),t)=\varnothing$ then $\sigma\left(t^-\right)=1$,

b) If $\sigma\left(t^-\right)<1$ then there is a net vaccination impulsive control action on $[t-\tau(t),0)$ with a net reduction of the infective population at the impulsive instants. In this case,

$$\left\|x\left(t^-\right)\right\|\leq\tau(t)\sigma\left(t^-\right)\sup_{s\in[t-\tau(t),t)}\|F(s,x(s),0)\|\leq\tau(t)\sup_{s\in[t-\tau(t),t)}\|F(s,\sigma(t)x(s),0)\| \tag{20}$$

$$\|x(t)\|\leq\tau(t)\sigma(t)\sup_{s\in[t-\tau(t),t]}\|F(s,x(s),0)\|\leq\tau(t)\sup_{s\in[t-\tau(t),t]}\|F(s,\sigma(t)x(s),0)\| \tag{21}$$

c) If $\sigma\left(t^-\right)>1$ then there is a net increase of the infection at impulsive time instants on $[t-\tau(t),0)$ with a net reduction of the infective population at the impulsive instants.

In a similar way, we obtain:

$$\|x(t)\|=\left\|\int_{-\tau(t)}^{0}F(s+t,x(s+t),V(s+t,x(s+t)))ds\right\|\leq\tau(t)\sigma(t)\sup_{s\in[t-\tau(t),t]}\|F(s,x(s),0)\| \tag{22}$$

Assume that $\sigma\left(t^-\right)\leq 1$ and that $F\left(t,x\left(t^-\right),0\right)$ is globally Lipschitz so that for some constant $L\in\boldsymbol{R}^+$, one gets under comparison with the impulsive-free zero solution:

$$\left\|x\left(t^-\right)\right\|\leq\tau(t)\sup_{s\in[t-\tau(t),t)}\left\|\sigma\left(t^-\right)F(s,x(s),0)-F(s,0,0)\right\|\leq\tau(t)\sigma\left(t^-\right)L\sup_{s\in[t-\tau(t),t)}\|x(s)\|, \tag{23}$$

$$\|x(t)\|\leq\tau(t)\sigma(t)L\sup_{s\in[t-\tau(t),t]}\|x(s)\|\leq\tau(t)\sigma\left(t^-\right)L\sup_{s\in[t-\tau(t),t)}\left\|x\left(s^-\right)\right\| \tag{24}$$

If $\sigma(t^-) \le \dfrac{M}{\tau(t)\,L \sup\limits_{s\in[t-\tau(t),t)} \|x(s)\|}$ and $t \notin Imp[t-\tau(t),t]$ then $\|x(t)\| = \|x(t^-)\| \le M$ If instead $\sigma(t^-) > \dfrac{M}{\tau(t)\,L \sup\limits_{s\in[t-\tau(t),t)} \|x(s)\|}$ and $t = t_{m(t)} \in Imp[t-\tau(t),t]$ with $K(t)$ being such that $\sigma(t) \le \dfrac{\lambda M}{\tau(t)\,L \sup\limits_{s\in[t-\tau(t),t]} \|x(s)\|}$ for some $\lambda \in (0, 1-\lambda_0)$ then $\|x(t)\| \le \lambda M < (1-\lambda_0)M$ and since $x(t)$ is continuous in $[t, t+\gamma_0)$ since it is time-differentiable in $(t, t+\gamma_0)$ for some, since it can be concluded that there exists some $\gamma \in [0, \gamma_0]$ such that $\|x(t)\| \le M$ on $\gamma_0 = \gamma_0(t) \in \boldsymbol{R}_+$ $[t, t+\gamma_0)$ provided that $\|x(t)\| \le M$ since $[t, t+\gamma) \cap Imp[t+\gamma-\tau(t+\gamma), t+\gamma) = \varnothing$. Now, proceed in the same way with $\{t+\gamma\} = Imp[t+\gamma-\tau(t+\gamma), t+\gamma]$ if $\sigma(t^- + \gamma) > \dfrac{M}{\tau(t+\gamma)\,L \sup\limits_{s\in[t-\tau(t),t)} \|x(s+\gamma)\|}$ so that $\sigma(t+\gamma) \le \dfrac{M}{\tau(t+\gamma)\,L \sup\limits_{s\in[t-\tau(t),t]} \|x(s+\gamma)\|}$ and $\|x(t+\lambda)\| \le \lambda M$ and $(t+\gamma) \notin Imp[t+\gamma-\tau(t+\gamma), t+\gamma]$ with $Imp[t+\gamma-\tau(t+\gamma), t+\gamma] = \varnothing$, otherwise. We can proceed in the same way along the whole time interval $[0, \infty)$.

4. Concluding remarks

In this paper, the properties of a new proposed integral model which might be related in particular cases to epidemic models has been discussed. This model incorporates delays and two coupled sub-states. The positivity, periodicity and almost-periodicity of the solution have being discussed under impulsive controls.

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