Natural oscillations of a gas in an elongated combustion chamber

S V Nesterov, L D Akulenko and V G Baydulov

A. Ishlinsky Institute for Problems in Mechanics of the RAS, 101/1, prospect Vernadskogo, Moscow, 119526, Russia

E-mail: baydulov@gmail.com

Abstract. For the analysis of the frequencies and shapes of the natural oscillations of a gas in an elongated rectilinear combustion chamber, this chamber can be treated as a kind of an organ pipe that has the following specific features:

1. the chamber has an inlet and outlet nozzles;
2. a gas mixture burns in the combustion chamber;
3. the combustion materials flow out from the outlet nozzle;
4. the gas flows in such a way that its velocity in the larger part (closer to the outlet nozzle) of the chamber exceeds the speed of sound (Mach number $M > 1$). There are only separate domains (one or several), where $M < 1$.

The excitation of the natural oscillations of the gas and an increase in the amplitude of such oscillations can lead to instability of the combustion process [1].

1. A simplified model of the natural oscillations of the gas in an elongated chamber

Let the transverse dimensions of the chamber be substantially smaller than its longitudinal dimension. We assume that the chamber is of constant cross section, the length of the chamber is l, and both its ends are open. The gas flows into the chamber at constant velocity U and flows out of it with the same velocity U (figure 1). It is required to find the frequencies and shapes of the natural oscillations of the gas. Using the linearized equations of acoustics for a moving medium [2], we obtain the equations for the sound potential

$$\frac{\partial^2\Phi}{\partial t^2} + 2U\frac{\partial^2\Phi}{\partial x\partial t} - \left(c^2 - U^2\right)\frac{\partial^2\Phi}{\partial x^2} = 0 \tag{1}$$

and the boundary conditions

$$\Phi(0,t) = \Phi(l,t) = 0 \tag{2}$$

Boundary conditions (2) provide zero values for the sound pressure at the left-hand and right-hand boundary points. The variable component of the velocity and the sound pressure are expressed in terms of the potential Φ as follows:

$$u(x,t) = -\frac{\partial\Phi}{\partial x} \equiv -\Phi', \quad p(x,t) = \rho\frac{\partial\Phi}{\partial t} \equiv \rho\dot{\Phi} \tag{3}$$

where ρ is the average gas density.

We will seek the solution of the boundary-value problem (1), (2) in the following form ($c^2 > U^2$, c is the speed of sound in the immobile medium)

$$\Phi = \hat{\Phi}(x) e^{i\omega t} \tag{4}$$

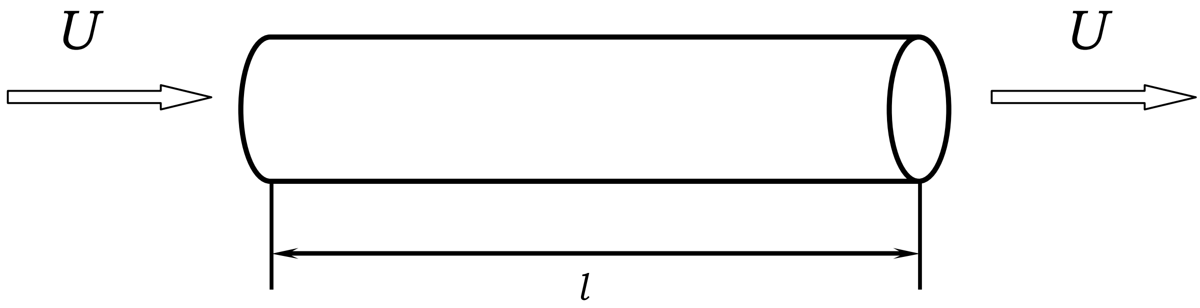

Figure 1. Geometry of the problem.

The conventional calculations yield

$$f_n = \frac{\omega_n}{2\pi} = \frac{n}{2l} c\left(1 - M^2\right), \ \varphi_n = \tilde{A}_n \exp\left(\pm i\pi n M \frac{x}{l} \right) \sin\left(\pi n \frac{x}{l} \right) \tag{5}$$

where n is the number of the oscillation mode, v is the velocity of the gas in the pipe, and $M = U / c$ is the Mach number. From the second relation of (3) it follows that the functions φ_n characterize the natural modes of the acoustic pressure.

Comments to the relations (5):

The natural oscillation frequencies and shapes can exist only when $M < 1$. For $M < 1$, all natural oscillation shapes φ_n are complex-valued. Only for $M = 0$, when the steady-state flow is absent, the natural shapes become real. The real natural shapes can be represented as a superposition of the complex-valued natural shapes. This implies that the sound pressure distribution in the case of acoustic oscillations substantially differs from the sound pressure distribution for the case when the flow is absent.

Table 1. Sound velocity in the methane – air mixture.

$v = \frac{1}{3}$	397.2 m / s
$v = \frac{1}{2}$	380.8 m / s
$v = \frac{1}{5}$	410.3 m / s
$v = \frac{4}{5}$	351.2 m / s

Furthermore, the steady-state flow always leads to a decrease in the natural frequencies.

From the above presented considerations it follows that the speed of sound c is a rather important parameter. If the chamber is filled with a gas at a given temperature T, the speed of sound can be defined rather accurately by

$$c = c_0\sqrt{\frac{T}{T_0}} \tag{6}$$

where $T_0 = 273K$, T is the temperature of the gas in kelvins, and c_0 is the speed of sound for $T = T_0$.

Let the chamber be filled with a mixture of two gases and let ν and $1-\nu$ be the mass fractions of the respective components in the mixture. Then, if the speed of sound is known for each of the components, the speed of sound for the mixture can be defined by one of the following expressions [3]:

$$c = (1-\nu)c_1 + \nu c_2,\; c^2 = (1-\nu)c_1^2 + \nu c_2^2 \tag{7}$$

where c is the speed of sound for the mixture, c_1 is the speed of sound for the first component, c_2 is the speed of sound for the second component, and ν is the mass fraction of the second component.

As an example, we present the data for the methane-air mixture: $T = 273K$, $c_1 = 430\,m/s$ (methane), $c_2 = 331.5\,m/s$ (air) (table. 1).

2. The dependence of the natural frequencies on the change in the speed of sound along the combustion chamber

Let c_0 be the speed of sound in the gas mixture at the inlet of the combustion chamber. Due to a number of causes, the speed of sound will change as the distance from the inlet increases. For example, the mix proportion changes, the combustion materials are formed, and the temperature changes. Abstracting from the causes of the change in the speed of sound, we assume that this speed is a known function of the coordinate x measured along the combustion chamber from the inlet nozzle to the outlet nozzle:

$$c = c_0\left(1 + F(z)\right) \tag{8}$$

where c_0 is the speed of sound at the chamber inlet and $z = x/l$ is the dimensionless length.

In addition we assume the inequality $M \ll 1$ for the Mach number. The determination of the natural frequencies and shapes of the oscillations of the gas is reduced to the Sturm-Liouville boundary-value problem

$$\Phi'' + \frac{\lambda}{1+F(z)}\Phi = 0\,;\; \Phi(0) = \Phi(1) = 0 \tag{9}$$

The natural frequencies ω are related to the eigenvalues λ by $\omega = \sqrt{\lambda}c_0/l$, $f = \omega/2\pi$ (Hz). As an example, we consider the following functions $f(z)$ that characterize the sound speed distributions: 1) $F(z) = z/4$; 2) $F(z) = 8ze^{-5z}$. The results of calculations of natural frequencies for different types of sound speed distribution $F(z)$ are presented in the table 2.

Using the accelerated convergence method [4], we find the eigenvalues λ_n and, thereby, the natural frequencies for three cases: $F(z) = 0$); 1); 2). The calculations for the natural oscillation modes show that the shapes of these oscillations for the sound speed distributions (8) under consideration are close to a sinusoidal form, i.e., they are close to the unperturbed oscillation modes. Using the accelerated convergence method [4], we calculated the natural frequencies up to the sixth digit after the point and then rounded the result. In a similar way, the calculations can be performed for any functions $F(z)$.

IOP Conf. Series: Journal of Physics: Conf. Series **815** (2017) 012021 doi:10.1088/1742-6596/815/1/012021

Table 2. The dependence of natural oscillation frequencies f_n from the form of sound speed distribution

n	$F(z)=0,\ f_n$	$F(z)=0.25z,\ f_n$	$F(z)=8z\exp(-5z),\ f_n$
1	$0.5\,\frac{c_0}{l}$	$0.5298\,\frac{c_0}{l}$	$0.5739\,\frac{c_0}{l}$
2	$1.0\,\frac{c_0}{l}$	$1.0591\,\frac{c_0}{l}$	$1.1490\,\frac{c_0}{l}$
3	$1.5\,\frac{c_0}{l}$	$1.5886\,\frac{c_0}{l}$	$1.7155\,\frac{c_0}{l}$
4	$2.0\,\frac{c_0}{l}$	$2.1181\,\frac{c_0}{l}$	$2.2811\,\frac{c_0}{l}$
5	$2.5\,\frac{c_0}{l}$	$2.6486\,\frac{c_0}{l}$	$2.8470\,\frac{c_0}{l}$

If the Mach number is known as a function of the coordinate x, the natural oscillation frequencies and modes can be determined by using the accelerated convergence method. However, this method needs a modification, since the equation contains a term with the complex unit. We suggest a simplified expression for the natural frequencies f_n, similar to (5), in accordance with [1]:

$$f_n = \frac{n}{2l}\bar{c}\left(1-\hat{M}^2\right), \tag{10}$$

where $\bar{c}$ is the mean value of the speed of sound, and $\hat{M}$ is the generalized Mach number $\hat{M} = \bar{U}/\bar{c}$; the averaging is performed along the chamber length. The conditions for which expression (10) provides an acceptable accuracy can be obtained by means of the accelerated convergence method.

3. Calculations of the acoustic oscillations of the gas in the combustion chamber with taking into account the design features

A simplified scheme for the combustion chamber that is used in engineering applications is shown in figure 2. The chamber has a length of 1 m. The inlet nozzle has a square cross section of the area $S_0 = 400\ cm^2$. The outlet nozzle has a rectangular cross section of the area $S_1 = 800\,cm^2$. Near the middle of the chamber, there is an elevation of the top surface located between the points $x_1 = 0.48\,m$ and $x_2 = 0.58\,m$. If the length of the chamber is taken as the unit of length, the cross-sectional area can be approximated as follows:

$$S(x) = S_0\left(1+a_1x\right)\left[1+a_2\left(\operatorname{th}\left(\frac{x-a_3}{b_1}\right)+\operatorname{th}\left(\frac{a_4-x}{b_2}\right)\right)\right] \tag{11}$$

By varying the parameters a_i and b_i, a rather wide class of design features of the chamber can be covered.

At the first stage, we assume that gas flow is absent from the chamber. It is required to determine the natural oscillation frequencies and shapes for a chamber open at both ends. We arrive at the Sturm-Liouville problem

$$\begin{gathered} \frac{d}{dx}\left(S(x)\frac{du}{dx}\right)+\lambda S(x)u=0 \\ \Phi(0)=\Phi(1)=0 \end{gathered} \tag{12}$$

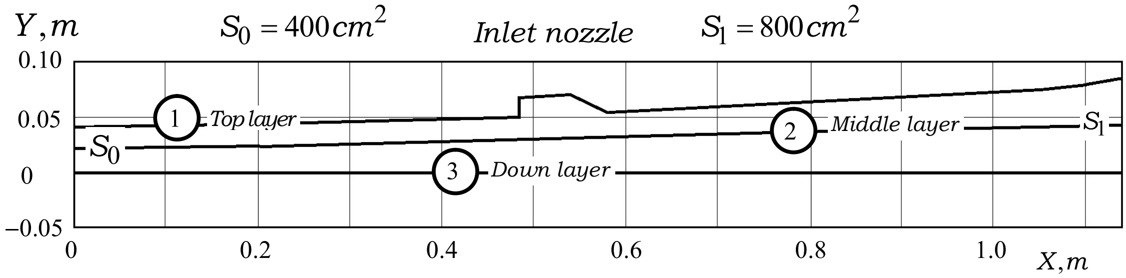

Figure 2. Scheme of the combustion chamber used in technical applications.

After finding the eigenvalues λ_n and the respective eigenfunctions Φ_n, we determine the natural frequencies in the dimensional units:

$$f_n = \sqrt{\lambda_n}\frac{c}{2\pi l}$$

Table 3. Natural oscillation frequencies f_n, Hz, for the chamber without an elevation. $a_1 = 1.875$, $a_2 = 0$.

$f_1 = 0.49339\frac{c}{l}$	$f_1 = 0.5\frac{c}{l}$
$f_2 = 0.99633\frac{c}{l}$	$f_2 = 1.0\frac{c}{l}$
$f_3 = 1.49748\frac{c}{l}$	$f_3 = 1.5\frac{c}{l}$
$f_4 = 1.99809\frac{c}{l}$	$f_4 = 2.0\frac{c}{l}$
$f_5 = 2.49847\frac{c}{l}$	$f_5 = 2.5\frac{c}{l}$

Table 4. Natural oscillation frequencies f_n, Hz, for the chamber with an elevation $a_1 = 1.875; a_2 = 0.2; a_3 = 0.48; a_4 = 0.58; b_1 = 0.001; b_2 = 0.02$

$f_1 = 0.47608\frac{c}{l}Hz\ -2\%$
$f_2 = 1.02134\frac{c}{l}Hz\ +2\%$
$f_3 = 1.46111\frac{c}{l}Hz\ -2.7\%$
$f_4 = 2.02913\frac{c}{l}Hz\ +1.6\%$
$f_5 = 2.47321\frac{c}{l}Hz\ -1\%$

Using the accelerated convergence method [4], we found the solutions of the boundary-value problem (12). The computational results are presented in tables 3,4.

The right-hand column presents the natural frequencies for the uniform chamber, $a_1 = 0$. For the first natural frequency, f_1, the difference as compared to the nonuniform chamber makes up 1.3%; for all other frequencies, this difference is less than 1%. The perturbed frequencies virtually coincide with the relevant values $\tilde{f}_n = \frac{n}{2l}c,\ Hz$ for the natural frequencies of the straight pipe, $S(x) = S_0 = \text{const}$.

Table 4 presents the first five natural frequencies for the chamber that has an elevation, as well as the differences of these frequencies from the respective values obtained for the chamber without an elevation. The elevation located as indicated above leads to a decrease in the odd natural frequencies and an increase in the even frequencies.

For comparison, consider the case where there is a cavity on the top wall of the chamber. The results for this case are presented in table 5.

The cavity on the top wall leads to an increase in the odd natural frequencies and a decrease in the even frequencies. The plots of the shapes of the natural oscillations for all cases considered show that these shapes virtually coincide with sinusoids $\sin\frac{\pi n x}{l}$ for the uniform chamber, $S(x) = S_0$, whereas the derivatives of the velocity have essential features near the elevations or cavities, as was mentioned previously [4].

Table 5. Natural oscillation frequencies f_n for the chamber with a cavity $a_1 = 1.875;\ a_2 = -0.2;\ a_3 = 0.48;\ a_4 = 0.58;\ b_1 = 0.001;\ b_2 = 0.02$

$f_1 = 0.51142\frac{c}{l}:\ \delta = +3.7\%$
$f_2 = 0.94839\frac{c}{l}\ -4.9\%$
$f_3 = 1.53991\frac{c}{l}\ +2.2\%$
$f_4 = 1.94530\frac{c}{l}\ -2.7\%$
$f_5 = 2.54078\frac{c}{l}\ +1.7\%$

For subsonic flows in the chamber, $M < 1$, the natural frequencies for both cases can be calculated according to (5).

4. Chamber through which a supersonic flow is blown

Numerical computations show that for the chamber with positive defect (an elevation on the top wall) a supersonic flow ($M > 1$, air blowing) is observed virtually everywhere. However, $M < 1$ in the neighborhood of the defect for $0.48 \le x \le .58$. In this domain, standing acoustic waves (natural oscillations) may exist. This neighborhood is sketched in figures 3,4.

Table 6. Natural oscillation frequencies f_n for region with defect

$f_n = \frac{(2n+1)}{4l}634(1-0.25)$	
$n = 0$	$f_1 = 1427\,Hz$
$n = 1$	$f_2 = 4461\,Hz$
$n = 2$	$f_3 = 7435\,Hz$
$n = 3$	$f_4 = 10409\,Hz$

While the left-hand boundary is fixed fairly accurately, the right-hand boundary does not coincide with the critical point at which $M = 1$. For this reason, we take the point $x = x_0 = 0.56$ as the left-hand boundary.

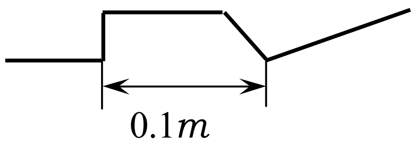

Figure 3. Region of the chamber, satisfying the condition of existence of standing waves.

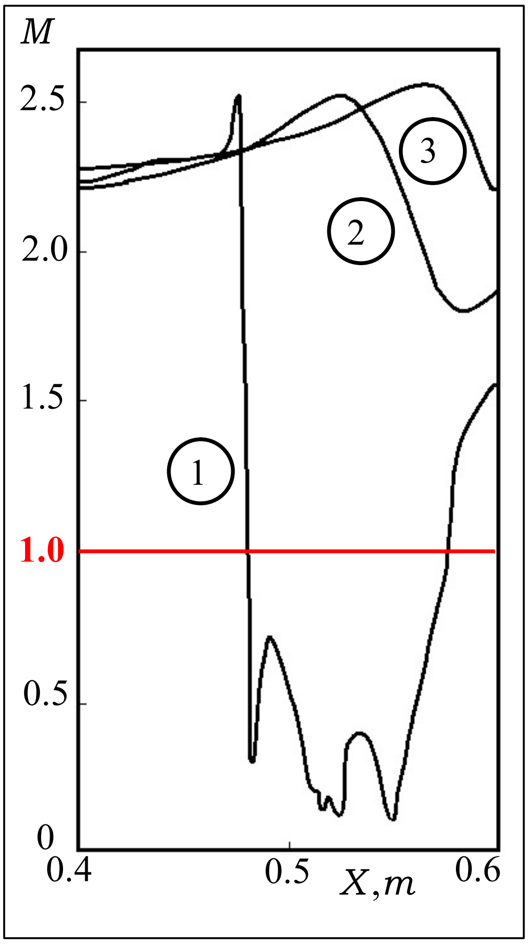

Figure 4. The dependence of the Mach number of the longitudinal coordinate of the Top (1), middle (2) and lower (3) layers of the chamber in the defect area.

The boundary conditions should be taken as follows:

- at the left-hand point of the elevation, the speed of sound is equal to zero;
- at the right-hand point, the pressure is equal to zero.

Then we arrive at the following boundary-value problem for the sound pressure:

$$\frac{d^2u}{dz^2} + \lambda u = 0, \quad \left.\frac{du}{dz}\right|_{z=0} = 0, \quad u\big|_{z=z_0} = 0$$

$$u = C_1 \cos\left[\tfrac{\pi}{2l}(2n+1)z\right], \quad \lambda_n = \left[\tfrac{\pi}{2l}(2n+1)\right]^2$$

$$f_n = c^2 \frac{\sqrt{\lambda_n}}{2\pi} = \frac{c}{4l}(2n+1); \quad l = 0.08\ m$$

The mean speed of sound in this region is defined by $c = 331.5 \cdot \sqrt{\frac{1000}{273}} = 634.5\, m/s$, and the average Mach number is $M = 0.5$. Hence, in accordance with (6), we have results presented in table 6.

The standing waves in the domain where $M < 1$ can be excited due to the vortex shedding or due to a thermomechanical interaction, as it is the case for a heated Helmholtz resonator [5]. If there are several regions in which $M < 1$, several types of standing acoustic waves may occur in each of these regions.

References

[1] D T Harrje and F H Reardon (eds) 1972 *Liquid Propellant Rocket Combustion Instability* (Washington: NASA) p 637.

[2] Blokhintsev D I 1981 *Acoustics of a Nonhomogeneous Moving Medium* (Moscow: Fizmatlit) p 206 (in Russian)

[3] Kudryavtsev B B 1956 *Akusticheskij Zhurnal.* **2** 167 (in Russian)

[4] Akulenko L D and Nesterov S V 2005 *High-Precision Methods in Eigenvalue Problems and Their Applications* (Boca Raton: Chapman and Hall/CRC Press) p 255

[5] Akulenko L D and Nesterov S V 2004 *Fluid Dynamics.* **39** 358

[6] Akulenko L D, Baidulov V G and Nesterov S V 2006 *J. Appl. Math. Mech. (PMM).* **70** 504