Diffusion of a passive impurity in a random velocity field

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Abstract. The method of characteristic functional is used for statistical description of the impurity concentration distribution in the random velocity field. A closed set of equations for a mean concentration response to a change of the external source density and for a pair correlation of the concentration fluctuations is obtained.

1. Statement of the problem

The processes of a diffusion in a random velocity field are described by the equation

$$L^{(0)}C(r,t)+u_{\alpha}(r,t)\partial_{\alpha}C(r,t)-\rho(r,t)\equiv\phi[C,\rho,u_{\alpha}]=0 \tag{1.1}$$

Here $C(r,t)$ is the impurity concentration at the space-time point $\{r,t\}$, $L^{(0)}=\partial_t-\kappa\Delta$; κ is the molecular diffusivity; $\rho(r,t)$ is the density of a impurity sources, and $u_{\alpha}(r,t)$ is the random velocity field given statistically and described by a normal distribution with a given pair correlation function

$$\left\langle u_{\alpha}(r_1,t_1)\cdot u_{\beta}(r_2,t_2)\right\rangle=D_{\alpha\beta}(r_1,t_1;r_2,t_2)$$

Next, let us introduce a digital notation for the space-time point $\{r_1,t_1\}$ according to the condition $\{r_1,t_1\}\equiv 1$, $\mathrm{d}1=\mathrm{d}r_1\cdot\mathrm{d}t_1$ and we will imply an integrating over repeated digital notation and a summing over the components of vector functions (the Einstein rule).

In this notation equation (1.1) takes the form

$$L^{(0)}(1,2)C(2)+V_{\alpha}(1|2,3)u_{\alpha}(2)C(3)-\rho(1)=\phi[C,\rho,u_{\alpha}]=0 \tag{1.2}$$

$$L^{(0)}(1,2)\equiv\left[\partial_t^{(1)}-\kappa\Delta^{(1)}\right]\delta(1-2),\quad V_{\alpha}(1|2,3)\equiv-\delta(1-2)\partial^{(3)}\delta(1-3)$$

2. Characteristic functional and its representation

The concentration $C(1)$ is a random field and the problem of its statistical description in terms of the statistical moments of various orders in various space-time points arises. To do this, one has to know the density distribution of a probability to find a given realization of the concentration field at a given source density $P[C,\rho]$ that can be defined as a mean value of the $\delta-$functional, which is a functional analog of the $\delta-$function generalized to the case of field description.

$$P[C,\rho]=\left\langle\delta\left[C(1)-\tilde{C}(1)\right]\right\rangle \tag{2.1}$$

here $\tilde{C}(1)$ is the realization of the field concentration at the point 1, which is a solution to the equation $\phi[\tilde{C},\rho,u_\alpha]=0$ at given realizations of the velocity field u_α and the source density ρ, the averaging is carried out over the realization of concentration field, which is reduced to averaging over the velocity field realizations u_α.

Using a functional analog of the formula well-known in the theory of generalized functions

$$\delta[\phi(x)] = \delta(x - x_0)\,|\,\phi'(x)\,|,$$

where x_0 is the solution to the equation $\varphi(x) = 0$, one gets

$$\delta\left[C(1)-\tilde{C}(1)\right] = \delta\left\{\phi\left[C,\rho,u\right]\right\}\frac{\delta\phi\left[C,\rho,u\right]}{\delta C(1)} \tag{2.2}$$

The calculation of the Jacobian of functional transformation $\delta\phi[C,\rho,u_\alpha]/\delta C$ shows that the result turns out to be independent of the concentration due to a linear dependence of $\phi[C,\rho,u_\alpha]$ on C as well as on the random quantity u_α. Since, the Jacobian in equation (2.2) may be included in a redefinition of the probabilistic measure and hence one can take the value of Jacobian in equation (2.2) be equal to unity.

The characteristic functional Φ is a functional Fourier-transform of the probability density $P[C,\rho]$

$$\Phi[\eta,\rho] = \int \mathrm{d}[C]P[C,\rho]\exp\{i\eta(1)\cdot C(1)\} = \int \mathrm{d}[C]\langle\delta\{\phi[C,\rho,u_\alpha]\}\rangle\exp\{i\eta(1)\cdot C(1)\} \tag{2.3}$$

Next we apply the formula of the δ – functional representation, which is a functional analog of the δ – function representation in the form of the Fourier-series expansion

$$\delta\{\phi[C,\rho,u_\alpha]\} = \int \mathrm{d}[\lambda]\exp\{i\lambda(1)\phi[C(1),\rho(1),u_\alpha(1)]\} \tag{2.4}$$

As the result we obtain a representation of the characteristic functional $\Phi[\eta,\rho]$ in the form of a functional integral over two fields λ and C.

$$\begin{aligned}\Phi[\eta,\rho] = \int \mathrm{d}[\lambda]\cdot\mathrm{d}[C]\exp\{i[\eta(1)C(1) + \lambda(1)L^{(0)}(1,2)C(2) - \lambda(1)\rho(1)]\}\times \\ \times\left\langle\exp\left\{i\lambda(1)V_\alpha(1\,|\,2,3)u_\alpha(2)C(3)\right\}\right\rangle\end{aligned} \tag{2.5}$$

Using the formula for calculating mean value in the case of a central normal distribution

$$\begin{aligned}&\left\langle\exp\left\{i\lambda(1)V_\alpha(1\,|\,2,3)C(3)u_\alpha(2)\right\}\right\rangle = \\ &= \exp\left\{-\frac{1}{2}\lambda(1)V_\alpha(1\,|\,2,3)C(3)\lambda(1')V_\beta(1'\,|\,2',3')C(3')D_{\alpha\beta}(2,2')\right\}\end{aligned} \tag{2.6}$$

we obtain

$$\begin{aligned}\Phi[\eta,\rho] = \int \mathrm{d}[\lambda]\cdot\mathrm{d}[C]\exp\big\{i[\eta(1)C(1) + \lambda(1)L^{(0)}(1,2)C(2) - \lambda(1)\rho(1)] - \\ -1/2\cdot A(1,1';3,3')\lambda(1)\lambda(1')C(3)C(3')\big\}\end{aligned} \tag{2.7}$$

$$A(1,1';3,3') = V_\alpha(1\,|\,2,3)V_\beta(1'\,|\,2',3')D_{\alpha\beta}(2,2')$$

From the invariance of functional integral in equation (2.7) under a shift of functional variable of integrating $\lambda \to \lambda + \delta\lambda$ it follows the equation in functional derivatives for the characteristic functional $\Phi[\eta,\rho]$

$$\left\{ L^{(0)}(1,2)\frac{\delta}{i\delta\eta(2)} - \rho(1) + A(1,1';3,3')\frac{\delta}{i\delta\eta(3)}\frac{\delta}{\delta\rho(1')}\frac{\delta}{i\delta\eta(3')} \right\}\Phi[\eta,\rho] = 0 \tag{2.8}$$

From equation (2.8) one can obtain the equation for the logarithm of characteristic functional

$$L^{(0)}(1,2)\frac{\delta \ln\Phi}{i\delta\eta(1)} - \rho(1) +$$

$$+A(1,1';3,3')\left\{ \frac{\delta \ln\Phi}{\delta\rho(1')}\frac{\delta \ln\Phi}{i\delta\eta(3)}\frac{\delta \ln\Phi}{i\delta\eta(3')} + \frac{\delta \ln\Phi}{\delta\rho(1')}\frac{\delta^2 \ln\Phi}{i\delta\eta(3)i\delta\eta(3')} + \right.$$

$$\left. +\frac{\delta \ln\Phi}{i\delta\eta(3)}\frac{\delta^2 \ln\Phi}{\delta\rho(1')i\delta\eta(3')} + \frac{\delta \ln\Phi}{i\delta\eta(3')}\frac{\delta^2 \ln\Phi}{\delta\rho(1')i\delta\eta(3)} + \frac{\delta^3 \ln\Phi}{i\delta\rho(1')i\delta\eta(3)i\delta\eta(3')} \right\} = 0 \tag{2.9}$$

Now we perform a transition to new functional variables by putting

$$\hat{\eta}(1) = \frac{\delta \ln\Phi}{i\delta\eta(1)}, \quad \hat{\rho}(1) = \frac{\delta \ln\Phi}{i\delta\rho(1)} \tag{2.10}$$

This transition can be carried out with the help of functional Legendre transformation by the transition to new characteristic functional

$$\Psi[\hat{\eta},\hat{\rho}] = -\ln\Phi[\eta,\rho] + i\eta\hat{\eta} + i\rho\hat{\rho} \tag{2.11}$$

In this case

$$\frac{\delta\Psi}{i\delta\hat{\eta}(1)} = \eta(1), \quad \frac{\delta\Psi}{i\delta\hat{\rho}(1)} = \rho(1) \tag{2.12}$$

In new variables the equation for characteristic functional $\Psi[\hat{\eta},\hat{\rho}]$ can be written in the form

$$L^{(0)}(1,2)\hat{\eta}(2) - \frac{\delta\Psi}{i\delta\hat{\rho}(1)} + A(1,1';3,3')\times$$

$$\times\left\{ i\hat{\rho}(1')\hat{\eta}(3)\hat{\eta}(3') + i\hat{\rho}(1')\frac{\delta\hat{\eta}(3)}{i\delta\eta(3')} + \hat{\eta}(3)\frac{\delta\hat{\eta}(3')}{\delta\rho(1')} + \hat{\eta}(3')\frac{\delta\hat{\eta}(3)}{\delta\rho(1')} + \frac{\delta}{\delta\rho(1')}\frac{\delta\hat{\eta}(3)}{i\delta\eta(3')} \right\} = 0 \tag{2.13}$$

3. Meaning of various functional derivatives

Unlike a traditional approach, when equations are used for their solving, in the functional formulation of a statistical description of stochastic system, the equations for characteristic functional are used for finding relations between various statistical characteristics of the system such as correlation functions and functions of mean response to external actions as well as for constructing various approximations on the basis of the analysis of the equations rather then by using some hypothesis of phenomenological type.

In the context of aforesaid one needs to clarify a meaning of functional derivatives of functionals $\Phi[\eta,\rho]$ and $\Psi[\hat{\eta},\hat{\rho}]$ with respect to fields η,ρ and $\hat{\eta},\hat{\rho}$. Note that, according to the definition of the characteristic functional, when obtaining relations for statistical characteristics of the system under consideration one has to take the limit $\eta \to 0$ after carrying out a functional differentiation of characteristic functional.

Below we will consider how the functional derivatives of the characteristic functionals $\Phi[\eta,\rho]$ and $\Psi[\hat{\eta},\hat{\rho}]$ with respect to fields η,ρ and $\hat{\eta},\hat{\rho}$ relate to statistical characteristics of the system.

By definition

$$\frac{\delta\Phi[\eta,\rho]}{i\delta\eta(1)} = \hat{\eta}(1)\Phi[\eta,\rho]|_{\eta\to 0} \to \langle C(1)\rangle \quad (\Phi[0,\rho]=1) \tag{3.1}$$

here $\langle C(1)\rangle$ is the mean concentration at the space-time point 1 at given source density ρ. And further

$$\frac{\delta \ln\Phi[\eta,\rho]}{i\delta\rho(1)} = \hat{\rho}(1)|_{\eta\to 0} \to 0 \tag{3.2}$$

The quantity $\hat{\rho}(1)$ describes the density distribution of fluctuations of the effective source that is not identical with initial ("bare") distribution of source density $\rho(1)$. The matter is that the turbulent velocity field produces a random flux of impurity which divergence is regarded as a density of effective source added to initial one. It should be particularly emphasized that the concept of effective source density arose as a consequence of statistical description of a stochastic system within the framework of the functional formalism rather then as a certain assumption of phenomenological type.

Higher functional derivatives of the characteristic functional $\Phi[\eta,\rho]$ with respect to the field η in the limit $\eta \to 0$ describe statistic moments of high order whereas the derivatives of the characteristic functional logarithm describe so called irreducible (cumulant) means of a statistical system.

In particular

$$\frac{\delta^2\Phi[\eta,\rho]}{i\delta\eta(1)\cdot i\delta\eta(2)}|_{\eta\to 0} \to \langle C(1)\cdot C(2)\rangle$$

$$\frac{\delta^2 \ln\Phi[\eta,\rho]}{i\delta\eta(1)\cdot i\delta\eta(2)}|_{\eta\to 0} \to \langle C(1)\cdot C(2)\rangle - \langle C(1)\rangle\cdot\langle C(2)\rangle = B(1,2) \tag{3.3}$$

$B(1,2)$ is the irreducible pair mean of the concentration fluctuations at the points 1 and 2.

$$\frac{\delta^2 \ln\Phi[\eta,\rho]}{i\delta\eta(1)\cdot\delta\rho(2)} = \frac{\delta\hat{\eta}(1)}{\delta\rho(2)}|_{\eta\to 0} \to \frac{\delta\langle C(1)\rangle}{\delta\rho(2)} = G(1,2) \tag{3.4}$$

$G(1,2)$ is the function of mean response of the concentration field at the space–time point 1 to the action of instantaneous source localized at the point 2 (Green's function).

4. Diagram technique

The powerful method of further analysis is an application of the Feynman diagram technique that enables to get additional information concerning various statistic characteristics of the system and relations between them. The method of Feynman diagram, firstly proposed in quantum electrodynamics for the analysis of perturbation theory series, consists in putting some graphic symbols in correspondence to various terms of the perturbation series. This method turns out to be very fruitful in many branches of mathematics and mathematical physics for obtaining within the framework of perturbation theory the information about the properties of a total perturbation theory series or its infinite subsequence in the case when a knowledge of some first terms of the series appears to be insufficient, and for understanding the phenomenon one needs to operate with partly or totally summarized series to which so called skeleton diagrams are assigned. The method of characteristic functional just corresponds to a description in terms of summarized perturbation theory series. In the statistical theory of turbulence the diagram methods were introduced in [1] and in the theory of turbulent diffusion this has been done in [2] (see, also [3]).

In a "simple" perturbation theory the solution is constructed by treating the convective term in equation (1.1) as small expansion parameter, and as the result one obtains the solution in the form of power series in the velocity. Next term-by term averaging gives a mean value of concentration. In this approach the elements of diagram technique are follows (see figure 1):

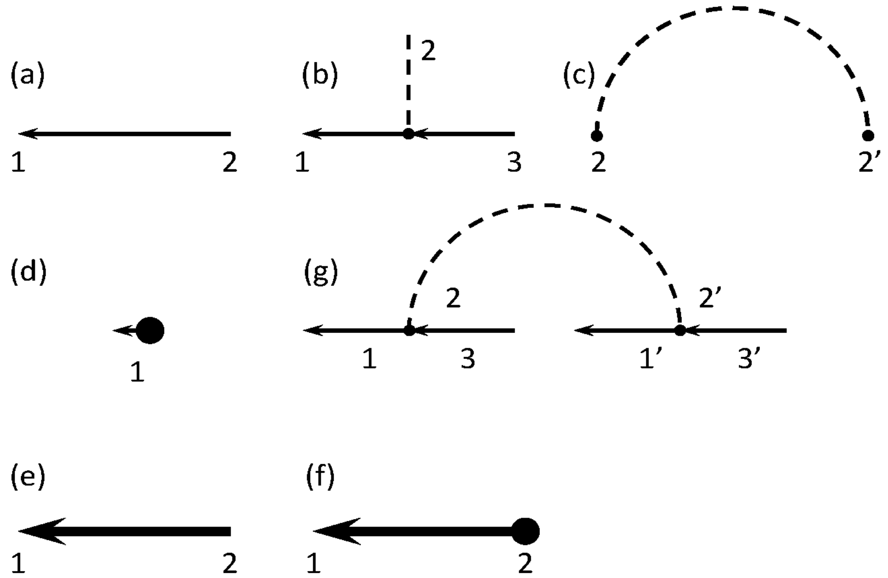

Figure 1. Diagram representation of various elements in "simple" perturbation theory..

(a) Green function $G^{(0)}(1,2)$ of the operator $L^{(0)}(1,2)$ ("bare" Green's function) depicted by a thin line connecting the points 1 and 2 with an arrow incoming into the point 1;

(b) vertex $V_\alpha(1|2,3)$ depicted by a point with outgoing arrow 1, incoming arrow 3 and incoming dashed line 2;

(c) pair correlation function (a variance) of the velocity field $D_{\alpha\beta}(2,2')$ depicted by a dashed line connecting the points 2 and 2';

(d) source density $\rho(1)$ depicted by a fat point 1 with outgoing arrow;

(e) Green function defined by equation (4) is depicted by a fat line connecting the points 1 and 2 with an arrow incoming into the point 1.

Diagram representation of mean concentration $< C(1) >= G(1,2)\cdot\rho(2)$ and those for the quantity $A(1,1';3,3)$ defined by equation (2.7) are shown in figure.1, diagrams (f) and (g).

More detail presentation of building the solution within the framework of a "simple" perturbation theory with using the diagram technique one can find in [3].

When studying the high-order statistical moments of the concentration field an application of a "simple" perturbation theory becomes very complex since one has to multiply infinite perturbation-theory series and next to carry out tern-by term averaging of the result obtained. In this context the use of the functional formulation of the problem turns out to be more suitable since in this formalism the averaging procedure is performed from the very beginning and one deals with summarized perturbation-theory series. Within the framework of a functional formulation of the problem the diagram technique is constructed in terms of so called skeleton diagrams corresponding to summarized diagrams of a "simple" perturbation theory (See, [4]). The technique of skeleton diagrams is constructed by following way (see figure 2).

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To the characteristic functionals Φ, $\ln\Phi$ and Ψ a light circle, light circle with cross inside and dark circle are assigned respectively.

The actions of operators $\delta/i\delta\eta(1)$ and $\delta/\delta\rho(1)$ to the characteristic functional $\Phi[\eta,\rho]$ are putted into correspondence a line connecting the light circle and the point 1 with incoming or outgoing arrow (see figure 3, diagrams (a) and (b)).

According to these convention the action of various products of two operators $\delta/i\delta\eta(1)$ and $\delta/\delta\rho(1)$ to the characteristic functional $\Phi[\eta,\rho]$ and its logarithm defined by the formulas

(c) $\delta^2\Phi/[i\delta\eta(1)\cdot i\delta\eta(2)] = \langle C(1)\cdot C(2)\rangle$,

(d) $\delta^2\ln\Phi/[i\delta\eta(1)\cdot i\delta\eta(2)] = \langle C(1)\cdot C(2)\rangle - \langle C(1)\rangle\cdot\langle C(2)\rangle = B(1,2)$

(e) $\delta^2\Phi/[i\delta\eta(1)\cdot\delta\rho(2)] = G(1,2)$

are depicted in figure 2, diagrams (c), (d) and (e).

The representations of the quantities $B(1,2)$ and $G(1,2)$ defined by equations (3.3) and (3.4) are shown in figure 2, diagrams (d) and (e). Note that under the action of the above pointed operators to the logarithm of characteristic functional the diagram that contains unconnected parts is absent, and this corresponds to an account for only irreducible means (cumulants).

We will depict the action of the operators $\delta/\delta\hat{\eta}(1)$ and $\delta/i\delta\hat{\rho}(1)$ to the characteristic functional $\Psi[\hat{\eta},\hat{\rho}]$ by an insert of incoming and outgoing arrows at the points 1 of dark circle. The characteristic functional $\Psi[\hat{\eta},\hat{\rho}]$ corresponds to a description of so called strongly connected diagrams, i.e. those that cannot become unconnected under breaking one line (one-particle irreducible diagrams).

The analysis of the perturbation theory series with exploiting the method of Feynman diagrams shows that in the limit $\eta\to 0$ all diagrams or parts of diagram in which the number of incoming arrows exceeds the number of outgoing arrows disappear. This means that the functional derivatives of the functional $\ln\Phi[\eta,\rho]$ with respect to only fields ρ equal to zero, as well the derivatives of the functional $\Psi[\hat{\eta},\hat{\rho}]$ with respect to only fields $\hat{\eta}$.

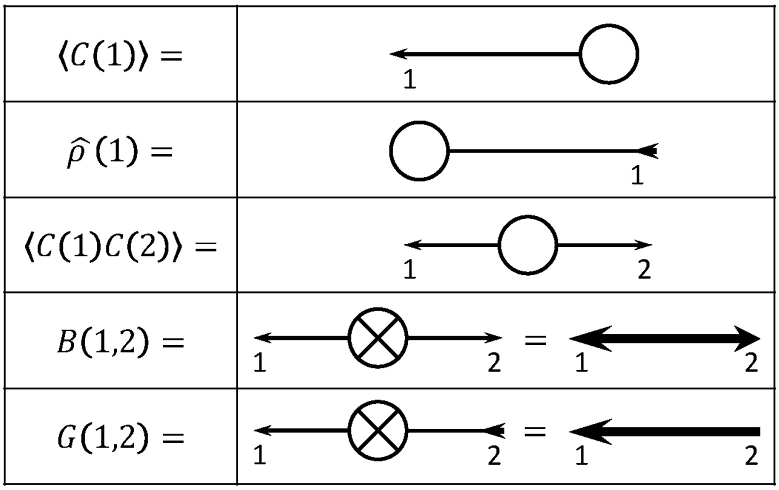

Figure 2. Diagram representation of the action of functional derivatives to characteristic functional.

5. Equations for statistical characteristics of a concentration field

To find equations for the quantities $B(1,2)$ and $G(1,2)$ one has to know some mixed derivatives. Next we consider mixed derivatives

$$\frac{\delta^2\Psi}{\delta\eta(1)\cdot i\delta\hat{\eta}(2)}=\frac{\delta\eta(2)}{\delta\eta(1)}=\delta(1-2)=\frac{\delta\hat{\eta}(1')}{\delta\eta(1)}\cdot\frac{\delta^2\Psi}{\delta\hat{\eta}(1')\cdot i\delta\hat{\eta}(2)}\cdot\frac{\delta\hat{\rho}(1')}{\delta\eta(1)}\cdot\frac{\delta^2\Psi}{\delta\hat{\rho}(1')\cdot i\delta\hat{\eta}(2)} \tag{5.1}$$

As $\eta\to 0$ the first term in the right-hand side of equation (5.1) disappears (two incoming arrows), from what it follows

$$G(1,1')\cdot\delta^2\Psi\delta\hat{\rho}(1')\cdot i\delta\hat{\eta}(2)\equiv G(1,1')G^{-1}(1',2)=\delta(1-2)$$

$$G^{-1}(1,2)=\frac{\delta^2\Psi}{\delta\hat{\rho}(1)\cdot i\delta\hat{\eta}(2)} \tag{5.2}$$

here $G^{-1}(1,2)$ is the reverse Green function.

The other mixed derivative

$$\frac{\delta^2\Psi}{\delta\eta(1)\cdot\delta\rho(2)}=i\frac{\delta\rho(2)}{\delta\eta(1)}=0$$

$$\frac{\delta^2\Psi}{\delta\eta(1)\cdot\delta\hat{\rho}(2)}=\left\{\frac{\delta\hat{\eta}(1')}{\delta\eta(1)}\cdot\frac{\delta}{\delta\hat{\eta}(1')}+\frac{\delta\hat{\rho}(1')}{\delta\eta(1)}\cdot\frac{\delta}{\delta\hat{\rho}(1')}\right\}\frac{\delta\Psi}{\delta\hat{\rho}(2)}$$

Using equations (3.4) and (5.2) in the limit $\eta\to 0$ one obtains

$$\left.\frac{\delta^2\Psi}{\delta\eta(1)\cdot\delta\hat{\rho}(2)}\right|_{\eta\to 0}=-B(1,1')G^{-1}(2,1')+G(1,1')R(1',2)=0 \tag{5.3}$$

here

$$R(1,2)=\left.\frac{\delta^2\Psi}{\delta\hat{\rho}(1)\cdot\delta\hat{\rho}(2)}\right|_{\eta\to 0} \tag{5.4}$$

The quantity $R(1,2)$ can be regarded as a pair correlation function of fluctuations of the effective source density.

Equation (5.3) can be rewritten in the form

$$B(1,2)=G(1,1')G(2,2')R(1',2') \tag{5.5}$$

This equation is an analog to the Wyld equation in the statistical theory of turbulence obtained by summing the perturbation theory series with the help of the technique of Feynman's diagrams [1]. Further we will need for the formulas relating the derivatives with respect to fields η,ρ and the derivatives with respect to fields $\hat{\eta},\hat{\rho}$.

$$\frac{\delta}{\delta\hat{\eta}(1)}=G^{-1}(1',1)\frac{\delta}{\delta\rho(1')},\quad \frac{\delta}{\delta\hat{\rho}(1)}=G^{-1}(1,1')\frac{\delta}{\delta\eta(1')}+R(1,1')\frac{\delta}{i\delta\rho(1')} \tag{5.6}$$

and the reverse formulas

$$\frac{\delta}{i\delta\eta(1)}=B(1,1')\frac{\delta}{\delta\hat{\eta}(1')}+G(1,1')\frac{\delta}{i\delta\hat{\rho}(1')},\quad \frac{\delta}{\delta\rho(1)}=G(1',1)\frac{\delta}{\delta\hat{\eta}(1')} \tag{5.6a}$$

With the aim to get the equations for Green's function, we act to Eq. (2.13) by the operator $\delta/\delta\hat{\eta}(4)$, this gives

$$L^{(0)}(1,2)\delta(1-4)-\frac{\delta\Phi}{\delta\rho(1')\cdot i\delta\hat{\eta}(4)}+$$

$$+A(1,1';3,3')\left\{\frac{\delta\hat{\eta}(3')}{\delta\rho(1')}\delta(3-4)+\frac{\delta\hat{\eta}(3)}{\delta\rho(1')}\delta(3'-4)+\frac{\delta}{\delta\hat{\eta}(4)}\cdot\frac{\delta^3\ln\Phi}{\delta\rho(1')\cdot i\delta\eta(3)\cdot i\delta\eta(3')}\right\}=0 \quad (5.7)$$

In equation (5.7) the terms being proportional to $\hat{\rho}$ and $\hat{\eta}$ are omitted since they include diagrams in which the number of incoming arrows exceeds the number of outgoing arrows that disappear in the limit as $\eta \to 0$.

After performing the limiting transition $\eta \to 0$ in equation(5.7) and usingequations (3.9), (5.2) and (5.6) one gets the expression for inverse Green's function

$$G^{-1}(1,4)=L^{(0)}(1,4)+A(1,1';4,3')G(3',1')+A(1,1';3,4)G(3,1')+$$

$$+A(1,1';3,3')\frac{\delta^4\ln\Phi}{\delta\rho(1')i\delta\eta(3)i\delta\eta(3')i\delta\rho(4')}G^{-1}(4',4) \quad (5.8)$$

The second term in the left-hand side of equation (5.8) contains Green's function at equal times. If in the formula for $V(1|2,3)$ to introduce an infinitely small time of delay in the concentration response to a velocity change, this term vanishes due the causality condition of Green's function (in the Feynman diagram technique a closed loop containing Green's function gives zero contribution). The last term in the right-hand side of equation (5.8) corresponds to an account for "one-particle irreducible diagrams" and we assume the contribution from these diagrams can be neglected (one-loop approximation).

$$G^{-1}(1,2)=L^{(0)}(1,2)-\Sigma(1,2),$$
$$\Sigma(1,2)=-A(1,1';3,2)G(3,1') \quad (5.9)$$

Equation (5.9) may be rewritten in the form of Dyson's equation in the quantum-field theory

$$G(1,2)=G^{(0)}(1,2)+G^{(0)}(1,1')\Sigma(1',2')G(2',2) \quad (5.10)$$

The quantity $\Sigma(1,2)$ is an analog to the "self–energy operator" in the quantum–field theory.

Similarly, under the action of the operator $\delta/\delta\hat{\rho}(4)$ to equation (2.13) in the limit as $\eta \to 0$ one finds

$$R(1,4)+A(1,4;3,3')B(3,3')+A(1,1';3,3')\Big\{\ \frac{\delta^4\ln\Phi}{i\delta\rho(1')i\delta\eta(3)i\delta\eta(3')\delta\eta(4')}G(4',4)-$$

$$-\frac{\delta^4\ln\Phi}{\delta\rho(4')i\delta\rho(1')i\delta\eta(3)i\delta\eta(3')}R(4',4)\ \Big\}=0 \quad (5.11)$$

In the "one-loop approximation" we obtain

$$R(1,2)=-A(1,2;3,3')B(3,3') \quad (5.12)$$

The diagram equations by Wyld and Dyson are shown in figure 3 diagrams (a) and (c), and the quantities $R(1,2)$ and $\Sigma(1,2)$ depicted by a rectangular with two outgoing arrows and by one incoming and one outgoing arrows are presented in figure 3, diagrams (b) and (d). It should be emphasized that the Wyld equation (5.5) and the Dyson equation (5.10) are exact ones since under their obtaining no additional assumptions or any approximation were used. The approximation is defined by a choice of the form for Σ and R. The set of equations (5.9) and (5.12) in combination with equations by Wyld (5.5) and Dyson (5.10) is a closed one and this enables to find the statistic characteristics of the system, namely, the pair correlation function of the concentration fluctuations and the function of mean response of the concentration to instant change of the source density.

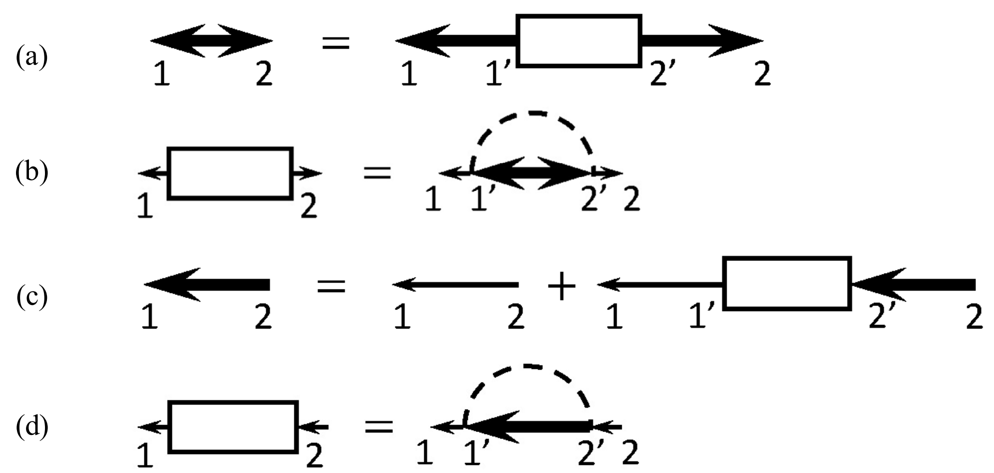

Figure 3. Diagram equations by Wyld and Dyson (a) and (c) and diagram representations for R(1,2) and Σ(1,2).

It is a certain surprise that the equations by Wyld and Dyson obtained in present paper turn out to be identical with those in the statistical theory of turbulence [5].

Note that the equation for Green's function appears to be independent of the statistic characteristics of the concentration field such as $B(1,2)$ and $R(1,2)$, whereas these quantities are related to each other by the Wyld equation into which the Green function enters. Therefore, one must firstly solve the equation for Green's function and then use this solution for obtaining the equation for statistical characteristics of the concentration field.

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