A new spectral method using legendre wavelets for shallow water model in limited-area

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Abstract. This paper presents a new spectral method using Legendre wavelets (named LWSTCM), which complete the stepping in spectral space while deal with boundary conditions in grid-point space by collocation method, for the numerical solution of shallow water model in limited-area. In order to deal with the overlapping boundaries, some proper schemes are considered for exchanging the information on the boundaries between sub-domains. 1-D advection equation is used to analysis the exponential convergence property and error characteristics of LWSTCM. Finally, we study LWSTCM on 2-D shallow water equations for a more realistic application. The numerical results are compared with existing numerical solutions found in the literature and demonstrate the validity and applicability of the presented method.

1. Introduction

The most common numerical methods used for atmospheric and oceanic models are finite difference method (FDM), finite elements method (FEM) and spectral method. Moreover, FDM is an approximation to the differential equation while others are an approximation to its solution. Comparing with FDM and FEM, spectral method can eliminate pole problems and has high accuracy due to its “exponential-convergence” property [1]. The spectral methods also offer the discrete conservations of kinetic energy and enstropy, which are very important for the 2-D turbulence modeling [2].

As a relatively new and an emerging area in mathematical research, wavelet theory has been applied in many different fields of science and engineering. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [3]. Therefore, various wavelets [4-13] have been used for studying problems with greater computational complexity and proved to be powerful tools to explore a new direction in solving differential equations. Recently, spectral method using Legendre wavelets is successfully used to obtain the numerical solution of IVP on large intervals [14] and Klein/Sine-Gordon equations [15]. The main attraction of this method is that it can exploit multi-level parallelism by employing the multi-scale analysis and hierarchy structure of Legendre wavelets.

The research of the spectral methods for limited-area model always attracts many attentions. Tatsumi [16] developed a spectral limited-area model by employing sinusoidal-subtracted Fourier sine-cosine series-expansion method. Hoyer [17] divided the basic variables into background field and perturbation components and deal with the perturbation part with spectral methods through double trigonometric series. Fulton, S. R. and Schubert, W. H. [18,19] proposed a Chebyshev spectral method

for the limited-area atmospheric and oceanic modeling. Haugen and Machenhauer [20] used Fourier series with cyclic boundary conditions in high resolution limited-area model; then, harmonic-sine series and cosine series expansion methods for limited-area model were develop by Chen and Kuo in [21] and Chen and Bai in [22] respectively. Hereafter, Juang and Kanamitsu [23] developed NCEP regional spectral model and Chen and Bai (1996) [24] developed a harmonic-Fourier spectral limited-area model with an external wind lateral boundary condition. Kuo [25] discussed the scale-dependent accuracy associated with the regional spectral model variables expanded by sine-cosine series. Katsuyuki V.Ooyama [26] gave some tests in 1-D single domain using cubic-spline spectral transform method for the time integration of nonlinear meteorological equations.
The major issue in large scale problems arising from scientific and engineering computing is that of reducing the computational cost while preserving numerical accuracy. The purpose of this paper is to develop a new effectively and exponential convergent method, which combining the spectral tau method with collocation method using the Legendre wavelets as the basis, for the numerical solution of limited-area models. Section 2 introduces a new spectral method a using Legendre wavelets. A simple linear advection equation is adopted to explore the numerical accuracy of proposed method in section 3. In section 4, LWSTCM is applied to shallow water model. Conclusions and summary are made in section 5.

2. A new spectral method using legendre wavelets

Because of that Legendre wavelets basis doesn't satisfy the boundary conditions and the locality of Legendre wavelets, Legendre wavelets spectral method needs to exchange the information on the boundaries between sub-domains. The overset method is employed at those overlapped boundaries. Let k and M is the scale and order of Legendre wavelets, respectively. Let i denoted the ith sub-domains ($i=1,\ 2,\ \cdots,\ 2^{k-1}-1$). Five simple overset schemes are tested, including (1) $u(iM)=u(iM+1)=\left(u(iM+2)+u(iM-1)\right)/2$, (2) $u(iM+1)=u(iM)$ or $u(iM)=u(iM+1)$ (depending on the propagated direction of wave) (3) spline interpolation (one in each side) (4) spline interpolation (two in each side). The above four schemes is denoted by ib=1, ib=2, ib=3 and ib=4, respectively. Without loss of generality, the method which uses the exact solutions at the overlapped boundaries for comparing is noted as ib=5. The information exchange is show in figure 1 at the overlapped boundary, where the green points denote the points on overlapping boundaries, and the purple points denote the points that participate in the information exchange.

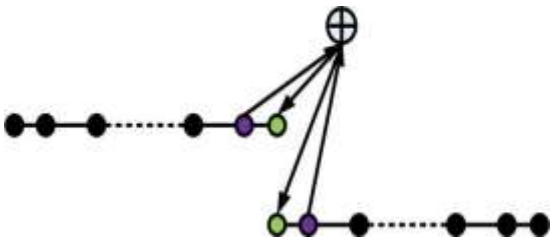

Figure 1. The information exchange at the overlapped boundary.

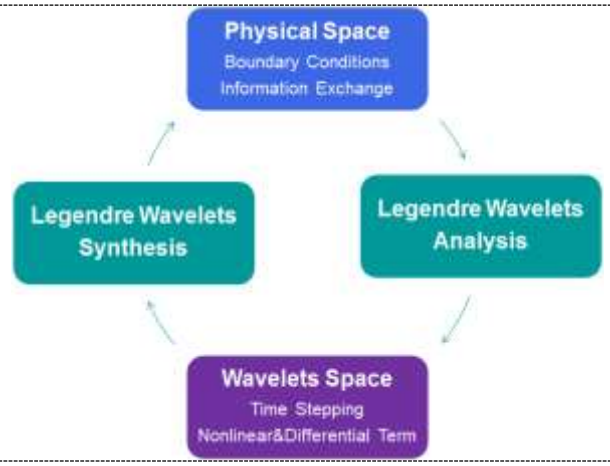

Figure 2. Schematic description of the spectral transform for LWSTCM.

As Fulton and Schubert point out, spectral collocation method is more stable than spectral Tau method. This is duo to the inaccuracy of treatment for the boundary conditions. What's more, it's nearly impossible to conduct the information exchange on the boundaries between sub-domains in spectral

space. So we propose a hybrid method which complete time stepping in spectral space while deal with boundary conditions and information exchange in physical space by using collocation method. It should be noted that the collocation method is employed to evaluate the coefficient of Legendre wavelets expansion. Figure 2 gives the schematic description of the spectral transform for LWSTCM.

3. Test case

In this section, we give some computational results of one-dimensional linear advection equation to validation with the method based on preceding section, to support our theoretical discussion.
Consider the one-dimensional linear advection equation [1, 18, 25]

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial u(x,t)}{\partial x}=0, \tag{1}$$

in the domain $[-1,1]$ .The exact solution of Eq. (1) is

$$u(x,t)=A\exp\left[-\left(\frac{x-x_0-t}{L}\right)^2\right]. \tag{2}$$

This is the simplest model involving wave or advective processes. The incoming boundary condition (1) is specified according to the exact solution at $x=-1$. No boundary condition is needed at right end point. This is an open boundary situation in the sense that any wave should propagated out of the domain without any deformation. In the case of that exact solution is Eq. (2), we assume $A=1.0$ and $x_0=-0.5$. In order to discuss the accuracy of LWSTCM, Chebyshev spectral collocation method (named CSCM) is introduced to do the comparison with it. It should be noted that the CSCM program used here is based on the burgers' code from the home page of Jie Shen (http://lsec.cc.ac.cn/~hyu/teaching/SHONM2013.html).
Figure 3 gives the L_2 errors (in log10 form) of single-scale LWSTCM as a function of the spatial-scale parameter L (Gaussian e-folding distance [1]) at $t=1.0$ for M =48. The single-scale LWSTCM clearly gives a much better approximation than CSCM method as L bigger than 0.2. In particular, the LWSTCM method reaches machine accuracy more slowly than CSCM. Figure 4 shows the corresponding L_2 errors of LWSTCM at $t=1.0$ as a function of M for different Legendre wavelets scale k and overlapped boundaries scheme. The exponential convergence of the two methods is obvious, while rapidly approach their asymptotic rate of decrease for L=2.0. From the results of multiple-scale LWSTCM in figure 4, it can be concluded that (1) the accuracy of overlapped boundaries scheme 2 is nearly the same as scheme 5 which using the exact solution, (2) the accuracy of overlapped boundaries scheme 1 is the same as scheme 3, (3) the overlapped boundaries scheme 2 is the best while the schemes 1 and 3 is the worst, (4) multiple-scale LWSTCM using scheme 2 is convergence fast with the increase of scale k.

Figure 5 presents the L_2 errors of single-scale LWSTCM as a function of M and dt at t =1.0. From the figure 5, it can be found that single-scale LWSTCM has good exponential convergence property and error characteristics.

Like the second test described in [18], we also consider the test case with following analytical solution

$$u(x,t)=\cos\left[\Omega\pi(x-t)\right], \tag{3}$$

where Ω is the wave number in the solution.
Figure 6 shows the L_2 error with dt= 0.0001 at $t=1.0$ as a function of M and Ω for the LWSTCM and CSCM. From figure 6, it can be found that LWSTCM requires less grid points than CSCM to achieve the same error for the same wave number while multiple-scale LWSTCM converges faster than single-scale LWSTCM.
The above results indicate that when the solution is smooth enough, LWSTCM needs less grid points than CSCM for the same accuracy and that this advantage increase dramatically as the desired accuracy increases. Even in the case of not smooth enough, LWSTCM has almost has the same accuracy as CSCM in the same degrees of freedom. Moreover, LWSTCM can save more storage and

computing time profit from the sparseness of the derivative matrix and the previous mentioned advantages.

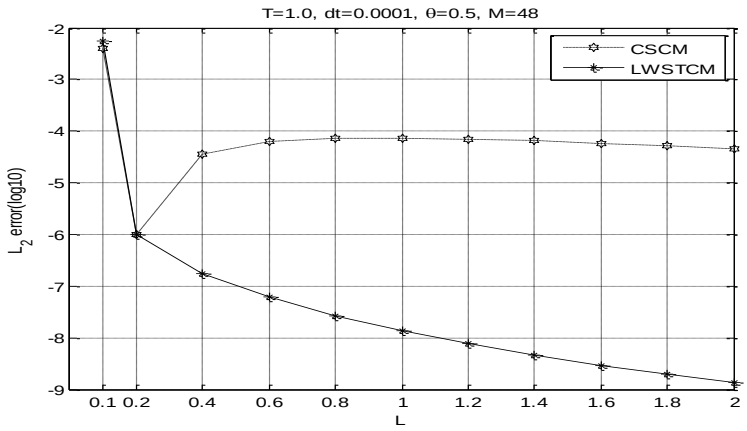

Figure 3. L_2 errors in the numerical solutions of the model problem 1 as a function of L at t=1.0 with dt=0.0001 for M=48 and k=0.

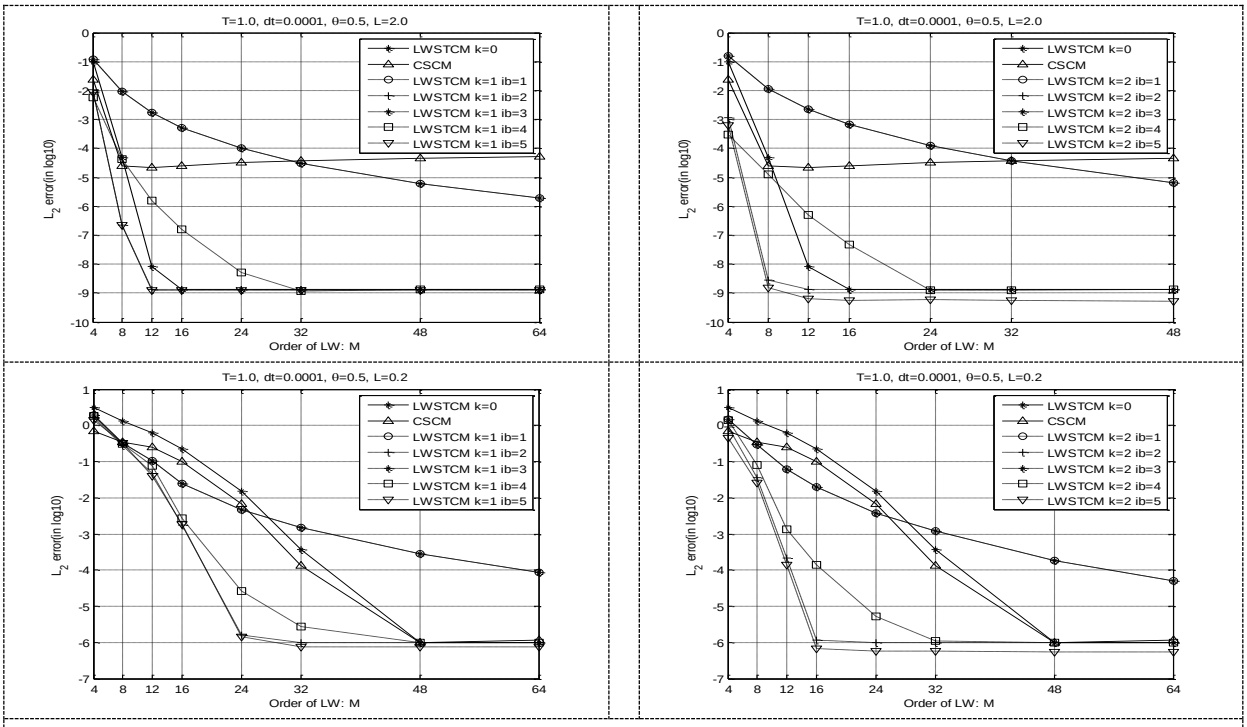

Figure 4. L_2 errors of LWSTCM as a function of M for different Legendre wavelets scale k and overlapped boundaries scheme at t =1.0 with dt=0.0001

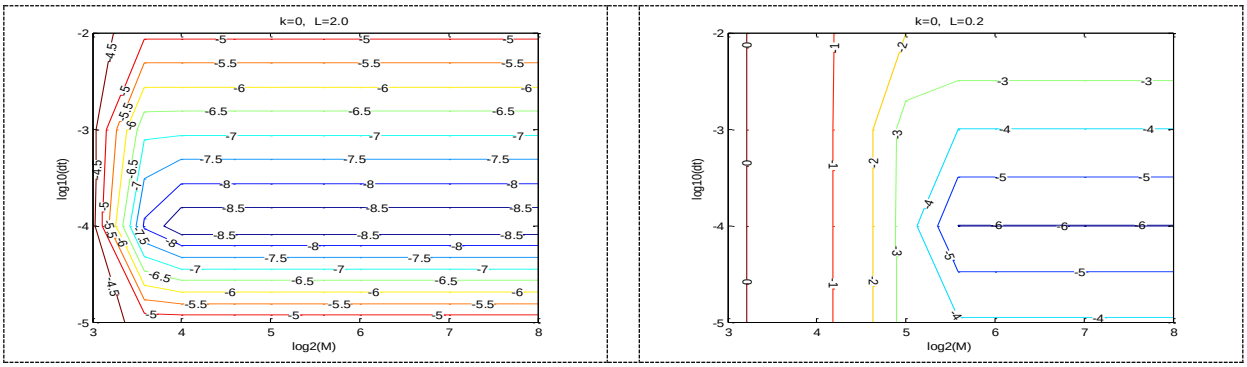

Figure 5. L_2 errors of single-scale LWSTCM as a function of M and dt at t =1.0 for L=2.0 and 0.2

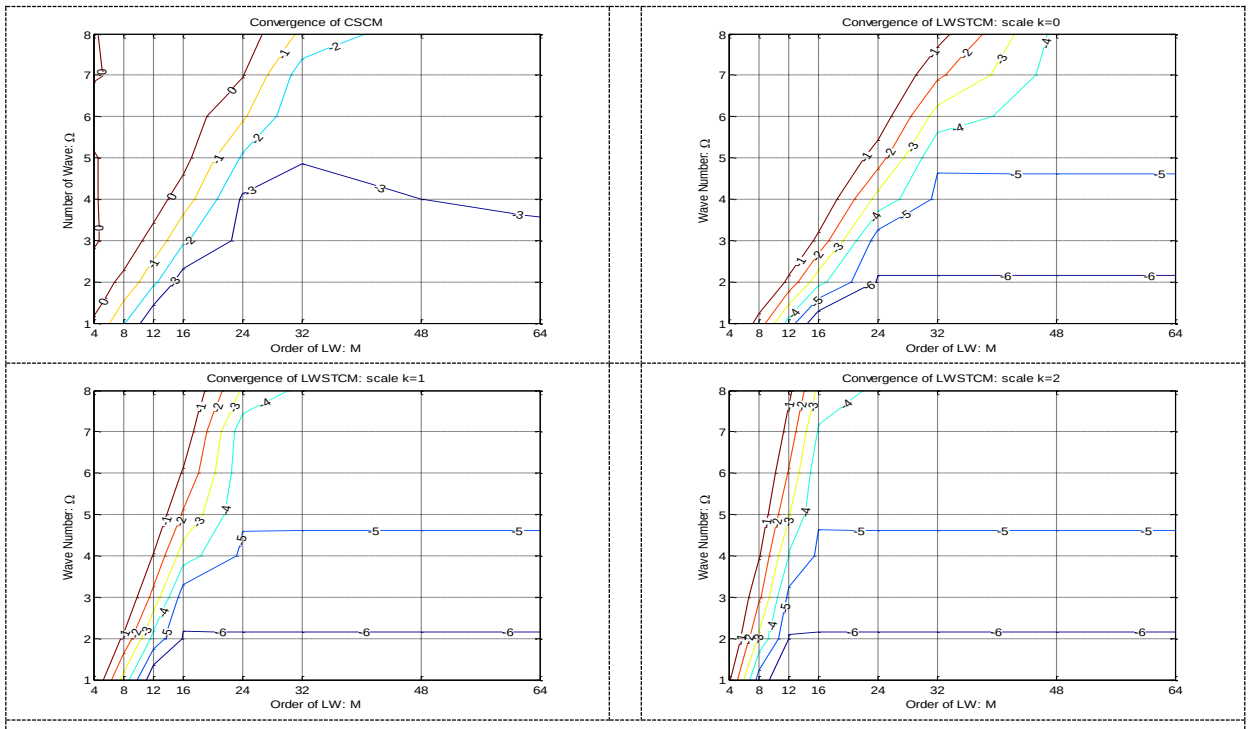

Figure 6. Logarithm of the L_2 errors in the numerical solutions of problem 1 for Eq. (5) as a function of M and Ω.

4. Limited-area shallow water model

In this section, the Houghton's shallow water model [27] is used to validate the feasibility and efficiency of the LWSTCM method. The domain is discretized using a mesh of $N_x \times N_y$ points, and fourth-order Adams-Bashforth scheme is used for time discretion. Let $\bar{u}$, $\bar{v}$ and $\bar{\phi}$ denotes the gird point value of function u, v and ϕ, respectively. The boundary conditions are set by a periodic overset condition along the east/west boundary and a wall-condition is applied at along the south and north boundaries. In other words,

$$\bar{u}\left(1,1:N_y\right)=\bar{u}\left(N_x-1,1:N_y\right),\ \ \bar{v}\left(1,1:N_y\right)=\bar{v}\left(N_x-1,1:N_y\right),\ \ \bar{\phi}\left(1,1:N_y\right)=\bar{\phi}\left(N_x-1,1:N_y\right), \tag{4}$$

while in the y-direction

$$\bar{v}\left(1:N_x,1\right)=\mathbf{O},\quad \bar{v}\left(1:N_x,N_y\right)=\mathbf{O}, \tag{5}$$

Note that no boundary conditions are necessary for u and ϕ in the y-direction. The integration time window was 24h with dt =10s. The time step can be increased by employing some technologies including more stable time discrete scheme and well-posed boundary conditions.

The initial conditions given by Grammeltvedt in reference [28] are

$$h\left(x,y\right)=H_0+H_1\tanh\left(9\frac{D/2-y}{2D}\right)+H_2\,\mathrm{sech}^2\left(9\frac{D/2-y}{2D}\right)\sin\left(\frac{2\pi x}{L}\right), \tag{6}$$

The initial velocity fields can be derived from the following relationship

$$u=-\frac{g}{f}\frac{\partial h}{\partial y},\ \ v=\frac{g}{f}\frac{\partial h}{\partial x}. \tag{7}$$

The constants are listed as

$$L = 4400\text{km},\quad g = 10\text{ms}^{-2},\quad D = 6000\text{km},\quad H_2 = 133\text{m},$$
$$H_0 = 2000\text{m},\quad \bar{f} = 10^{-4}\text{s}^{-1},\quad H_1 = 220\text{m},\quad \beta = 1.5\times10^{-11}\text{s}^{-1}\text{m}^{-1}, \tag{8}$$

Figures 7-9 show the geopotential field *h/c* at *t*=24, 48, and 72 hours for different (*M1*,*M2*,*k1*,*k2*). From figure 7, it can be found that the result of LWSTCM is the same as that obtained by using ADI finite-difference scheme [29]. By comparison with the results of single-scale LWSTCM, multiple-scale LWSTCM for Houghton's shallow water model shows some oscillations in geopotential field. These may be caused by the inaccuracy of the overlapping boundaries. From figures 7-9, it can be found that the results of multi-scale LWSTCM are good agree with those obtained by single scale LWSTCM when use the same number of grid points while have more degree of parallelism.

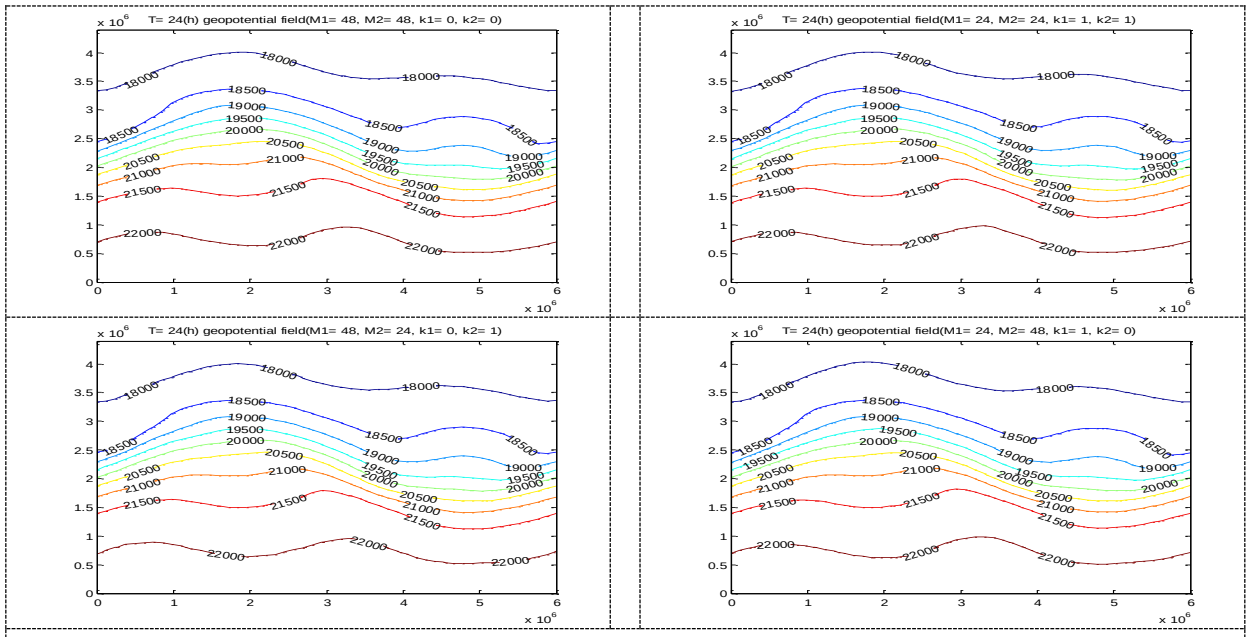

Figure 7. The geopotential field at *T*=24*h* for difference (*M1*,*M2*,*k1*,*k2*).

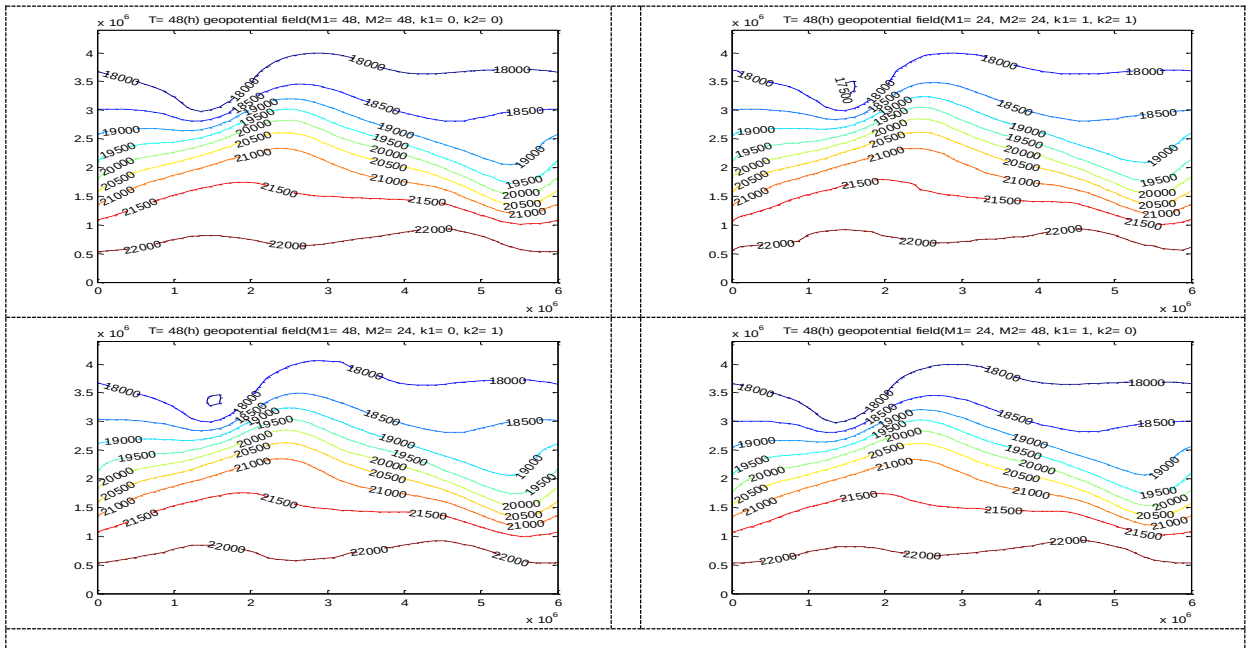

Figure 8. The geopotential field at *T*=48*h* for difference (*M1*,*M2*,*k1*,*k2*).

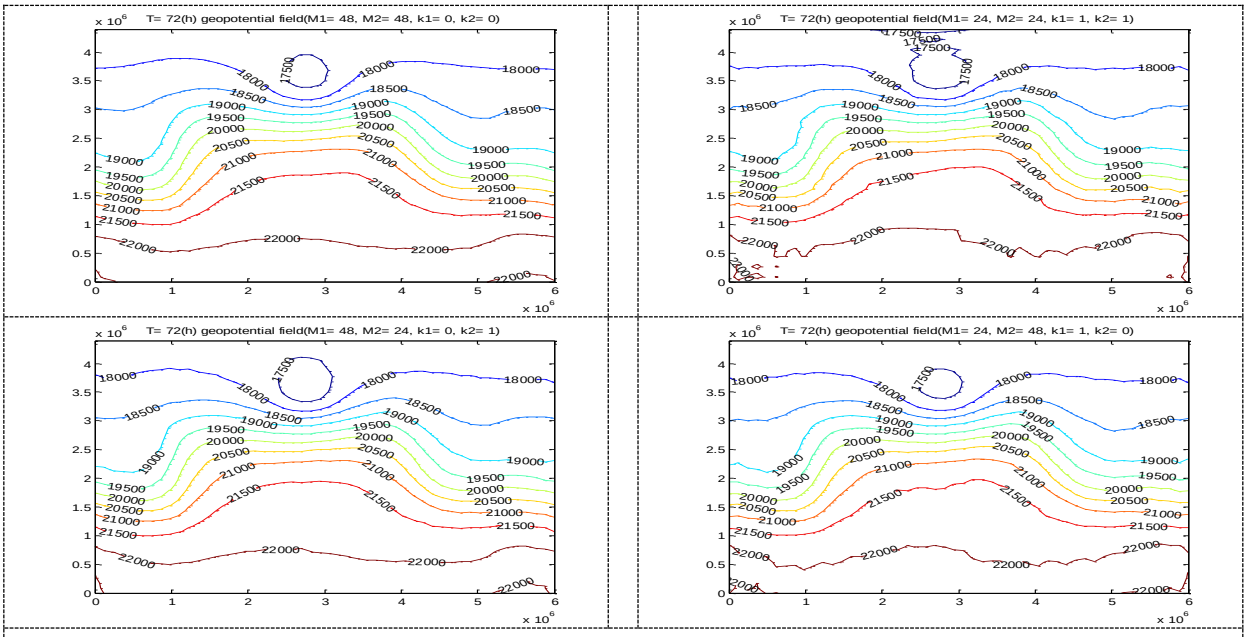

Figure 9. The geopotential field at T=72h for difference ($M1$,$M2$,$k1$,$k2$).

5. Conclusions

In this paper, we presented a hybrid spectral Tau method and collocation method for shallow water model in limited-area. The exponential convergence property and error characteristics are shown in the test of advection equation. LWSTCM clearly gives a much better approximation than CSCM as L (Gaussian e-folding distance) increases. In other words, LWSTCM needs less grid points than CSCM for the same accuracy and that this advantage increase dramatically as the desired accuracy increases. Further, LWSTCM appear to have better stability than CSCM for multiple-waves solutions. Finally, LWSTCM is applied to the solution of 2-D shallow water model for a more realistic application. Numerical results demonstrate that the LWSTCM can reduce the freedom of model and increase the parallelism of model while preserving numerical accuracy.

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