Expert Strategies in Solving Algebraic Structure Sense Problems: The Case of Quadratic Equations

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Abstract. Structure sense, an intuitive ability towards symbolic expressions, including skills to interpret, to manipulate, and to perceive symbols in different roles, is considered as a key success in learning algebra. In this article, we report results of three phases of a case study on solving algebraic structure sense problems aiming at testing the appropriateness of algebraic structure sense tasks and at investigating expert strategies dealing with the tasks. First, we developed three tasks on quadratic equations based on the characteristics of structure sense for high school algebra. Next, we validated the tasks to seven experts. In the validation process, we requested these experts to solve each task using two different strategies. Finally, we analyzing expert solution strategies in the light of structure sense characteristics. We found that even if eventual expert strategies are in line with the characteristics of structure sense; some of their initial solution strategies used standard procedures which might pay less attention to algebraic structures. This finding suggests that experts have reconsidered their procedural work and have provided more efficient solution strategies. For further investigation, we consider to test the tasks to high school algebra students and to see whether they produce similar results as experts.

1. Introduction

Structure sense is considered as an important collection of abilities to deal with algebra ([1], [2], [3]), a core topic in secondary school mathematics that should be mastered by students for advanced study or professional work [4]. Performing this structure sense ability in, for instance, solving algebra problems shows a relational rather than only an instrumental understanding [5].

The lens of structure sense has been used in several studies to investigate student algebraic proficiency in algebra (e.g., [1], [2], [6]). In Indonesia, however, the use of this lens is still scarce where its use is limited only for interpreting student difficulties [7], but not for developing algebra tasks to assess student algebraic proficiency in, for instance, high school level.

Taking this into account, we carried out a small-scale case study on developing and solving algebraic structure sense problems. This case study aims to validate designed algebraic structure sense tasks for high school algebra and to investigate expert strategies dealing with the tasks. To address this aim, the idea of structure sense is used as a main theoretical framework.

The term structure sense, coined by Linchevski and Livneh [8], initially was used to describe students' difficulties with using knowledge of arithmetic structures in the context of learning initial algebra. This term, then was refined and developed by Hoch and Dreyfus [1], refers to a collection of abilities towards symbolic expressions, including skills to interpret, to manipulate and to perceive symbols.

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For the purpose of this study, we distinguish between the use of standard procedures and structure sense strategies when dealing with algebra problems. Someone displays the use of standard procedures if s/he can, for instance, solve equations using certain algebraic procedures without considering the efficiency of the procedures (e.g., [1]). For example, to solve the equation $(3x-2)^2-5(3x-2)-6=0$, rather than using a substitution $(3x-2)$ to obtain a simpler equation that can be solved using a more efficient factorization strategy, someone inefficiently transforms the equation into $9x^2-27x+8=0$ and uses the quadratic formula to solve it.

Someone displays structure sense strategies when dealing with algebra problems if s/he can: (1) recognize a familiar structure in its simplest form; (2) deal with a compound term as a single entity; and (3) choose appropriate manipulations to make best use of a structure ([1], [2], [3]). For example, to solve the equation $(x-5)^4-(x+5)^4=0$ by using structure sense strategies, someone can deal with $(x-5)^2$ and $(x+5)^2$ as single entities, can use these as substitutions to obtain a simpler equation, and can manipulate the equation into $((x-5)^2+(x+5)^2)\,((x-5)^2-(x+5)^2)=0$.

2. Experimental Method

This case study – a part of a larger qualitative study on investigating symbol sense and structure sense abilities – aims to validate the appropriateness of algebraic structure sense tasks and to investigate expert strategies when dealing with the tasks. To do so, first, we developed three algebra tasks on quadratic equations based on the characteristics of structure sense for high school algebra (see Table 1). Next, the tasks were theoretically validated to seven experts, including four mathematics lecturers having mathematics education background, and three mathematics lecturers having no mathematics education background. In this validation process, we requested experts to solve each task using two strategies and to give comments and suggestions whether the tasks are appropriate for senior high school students or not. Finally, we analyzed expert solution strategies dealing with the problems in the light of structure sense characteristics ([1], [2], [3]).

Table 1. Structure sense tasks on solving quadratic equations

No	Structure sense characteristics	Tasks
1.	Recognize a familiar structure in its simplest form	$64-x=8+\sqrt{x}$.
2.	Deal with a compound term as a single entity	$(x^2-2x)^2-(x-2)^2=0$.
3.	Choose appropriate manipulations to make best use of a structure	$(x^2-4x)^2-2x^2+8x-15=0$.

3. Results and Discussion

Table 2 summarizes expert initial strategies dealing with algebraic structure sense tasks on quadratic equations. For example, three experts used standard procedures and four experts used structure sense strategies for solving Task 1. We found that all seven experts, not influenced by having or having no mathematics education background, solved the three tasks correctly using either structure sense strategies or standard algebraic procedures.

Table 2. Expert initial strategies solving algebraic structure sense tasks

Tasks	Expert strategies	
	#Standard procedure	#Structure sense strategy
1	3	4
2	2	5
3	2	5

For the Task 1, all experts considered that this task is an appropriate and a creative task for assessing senior high school skills dealing with equations related to quadratic equations. Concerning solution strategies for solving the task, i.e., solving the equation $64-x=8+\sqrt{x}$, four experts used

structure sense strategies, such as recognizing the structure of $64 - x$ as a $8^2 - \left(\sqrt{x}\right)^2$ and therefore solved it as follows: $8^2 - \left(\sqrt{x}\right)^2 = 8 + \sqrt{x} \Leftrightarrow \left(8 - \sqrt{x}\right)\left(8 + \sqrt{x}\right) = 8 + \sqrt{x} \Leftrightarrow \left(8 + \sqrt{x}\right)\left(7 - \sqrt{x}\right) = 0$, and so $x = 49$ (see Figure 1 left part); whereas three expert used standard procedures, such as by simplifying the equation into $56 - x = \sqrt{x}$ and then squaring both sides (see Figure 1 right part). In the second occasion, six experts used structure sense strategies, and one provided no solution. We consider that the structure sense strategies may include graphical solutions of the equation because perceiving a symbolic equation as a graphical representation shows a part of collection of abilities dealing with symbols ([1]). We found one expert used a graphical solution to the equation, i.e., the solution of the equation is an intersection point between graphs $f(x) = 64 - x$ and $g(x) = 8 + \sqrt{x}$. This suggests that most of the experts had reconsidered their thinking strategies when they were required to provide more than one solution strategy.

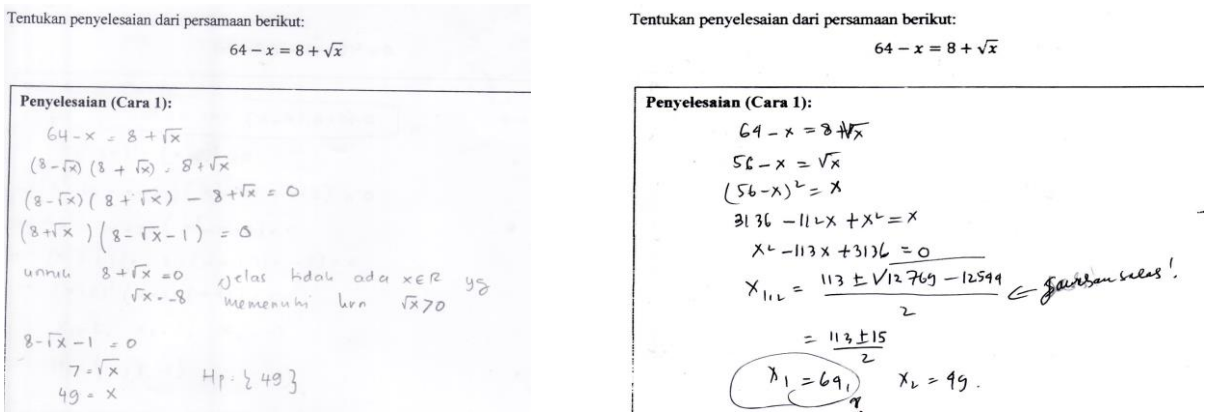


Figure 1. Representative examples of expert solution strategies on Task 1

Task 2, i.e., solving the equation $(x^2 - 2x)^2 - (x - 2)^2 = 0$, is considered by the seven experts as a good and appropriate problem for assessing student skills in dealing with quadratic equations having compound terms. By recognizing the structure of the equation as $a^2 - b^2 = 0$, where $a = (x^2 - 2x)$ and $b = (x - 2)$, someone can solve this equation in the following manner: $(x^2 - 2x)^2 - (x - 2)^2 = 0 \Leftrightarrow (x^2 - x - 2)(x^2 - 3x + 2) = 0 \Leftrightarrow x = -1$, $x = 1$, or $x = 2$. For this task, five experts used structure sense strategies and two experts used standard procedures in the first occasion of solution processes. All experts can eventually produce structure sense strategies in either the first or the second occasion. This finding suggests, again, the experts can change their thinking strategies in a versatile manner. Figure 2, left part, shows an example of structure sense strategies, and the right part shows an example of the use of standard procedures, i.e., using Horner's method to solve, in this case, a polynomial equation of the degree four.

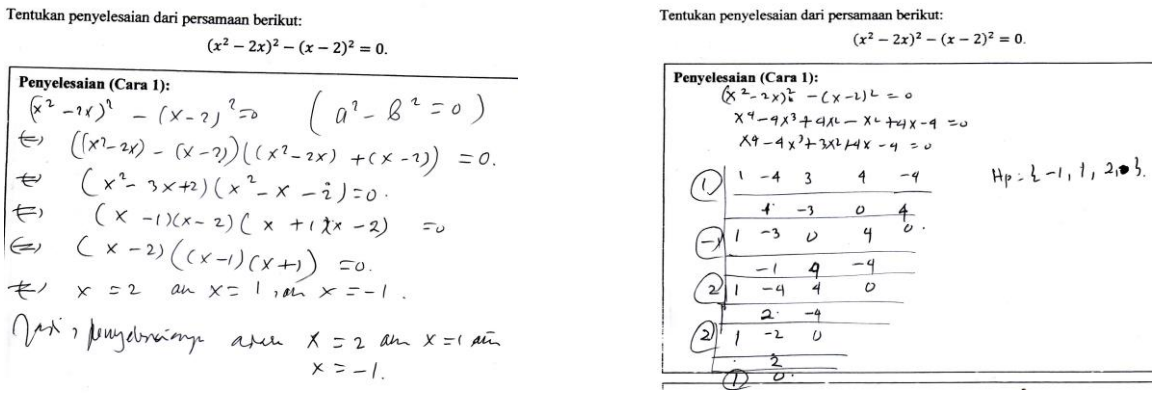


Figure 2. Representative examples of expert solution strategies on Task 2

All seven experts considered the equation in Task 3, i.e., $(x^2-4x)^2-2x^2+8x-15=0$, as a good task for investigating high school students' skills dealing with equations related to quadratic one. We found that five experts used structure sense strategies and two experts used standard procedures in the first occasion of solution processes. The use of structure sense strategies is shown by using the substitution, for instance $y=(x^2-4x)$, to simplify the equation into $y^2-2y-15=0$, to factorize it into $(y-5)(y+3)=0$, and to obtain $y=5$ or $y=-3$. The use of standard procedures is demonstrated by, for instance, expanding the equation, rewriting it into $x^4-8x^3+14x^2+8x-15=0$, and solving it using the Horner's method. Figure 3, left and right parts respectively, shows an example of the use of structure sense strategies and of the standard procedures. Even if six experts eventually used structure sense strategies in the second occasion, still one expert used standard algebraic procedures for solving the equation in both occasions.

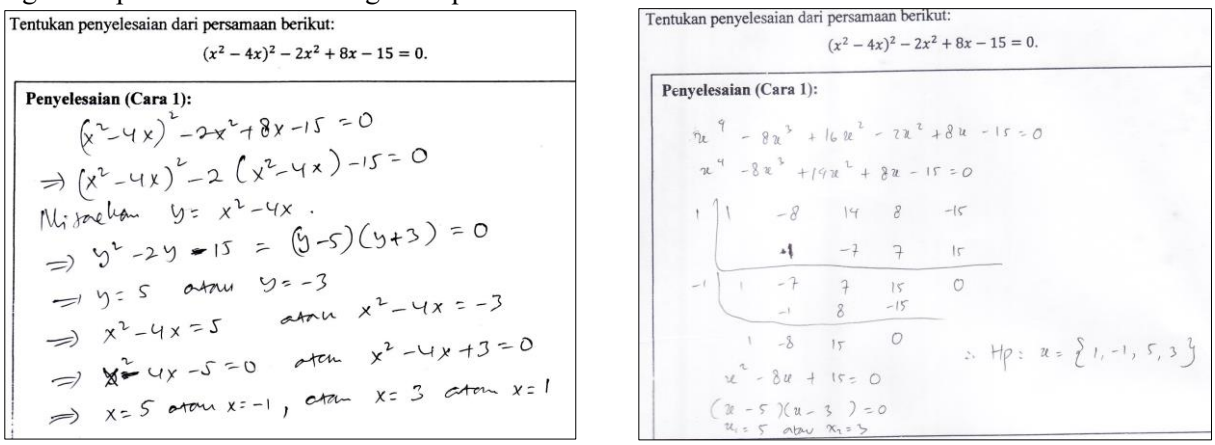


Figure 3. Representative examples of expert solution strategies on Task 3

4. Conclusion

From the results described in the previous section, we draw the following two conclusions. First, the designed algebraic structure sense tasks are considered by experts as appropriate for assessing high school student algebraic proficiency and understanding on the topic of quadratic equations. In other words, the designed tasks are appropriate to structure sense characteristics and are theoretically valid for assessing student skills and understanding on solving quadratic equations. Second, notwithstanding eventual expert strategies in solving equations are in line with the characteristics of structure sense, some of their initial solution strategies, as shown in their written work, used standard procedures. This suggests that initially some experts pay less attention to the structure of the equations, but then they pay more attention after being asked to provide more than one solution strategy. This finding indicates that experts have reconsidered their initial procedural work, and in a flexible manner can provide more efficient solution strategies. For further investigation, we wonder whether high school students will perform similarly as experts when dealing with the designed tasks.

5. References

[1] Hoch M and Dreyfus T 2006 *Proc. of the 30th Conference of the International Group for the Psychology of Mathematics Education,*ed Novotná, H Moraová, M Krátká, and N Stehlíková (Prague: PME) pp 305-312

[2] Hoch M and Dreyfus T 2010 *Proc. of the 6th Congress of the European Society for Research in Mathematics* (Lyon: Institut National de Recherche Pédagogique) pp. 529-538

[3] Novotná J and Hoch M 2008 *Mathematics Education Research Journal* **20** 93-104

[4] Katz V K 2007 *Algebra: Gateway to a Technological Future* (Washington DC: The Mathematical Association of America) pp 1– 45

[5] Skemp R 1976 *Mathematics Teaching* **77** 20-26

[6] Stiphout I M V 2011 *The Development of Algebraic Proficiency* (Eindhoven: Ph. D Thesis Eindhoven University of Technology)
[7] Jupri A, Drijvers A P, and Panhuizen M V D H 2014 *Mathematics Education* **9** 39-55
[8] Linchevski L and Livneh D 1999 *Educational Studies in Mathematics* **40** 173-196

Acknowledgments

This study was funded by Ministry of Research, Technology and Higher Education Indonesia through Fundamental research scheme, year 2016. We thank seven mathematics lecturers, as experts, for their active participation.