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One Critical Case in Singularly Perturbed Control Problems

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Abstract. The aim of the paper is to describe the special critical case in the theory of singularly perturbed optimal control problems. We reduce the original singularly perturbed problem to a regularized one such that the existence of slow integral manifolds can be established by means of the standard theory. We illustrate our approach by an example of control problem.

1. Introduction

Consider singularly perturbed differential systems of the type

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y,t,\varepsilon), \\ \varepsilon\, \tfrac{dy}{dt} &= g(x,y,t,\varepsilon) \end{aligned} \tag{1}$$

where x and y are vectors, ε is a small positive parameter.

Such systems play an important role as mathematical models of numerous nonlinear phenomena in different fields (see e.g. [1, 2, 3, 4, 5, 6, 7]).

A usual approach in the qualitative study of (1) is to consider first the so called degenerate system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,0), \\ 0 &= g(x,y,t,0) \end{aligned}$$

and then to draw conclusions for the qualitative behavior of the full system (1) for sufficiently small ε. In order to recall a basic result of the geometric theory of singularly perturbed systems we introduce the following notation and assumptions for sufficiently small positive ε_0, $0 \le \varepsilon \le \varepsilon_0$.

(A_1). Functions f and g are sufficiently smooth and uniformly bounded together with all their derivatives.

(A_2). There are some region $\mathcal{G} \in R^m$ and a function $h(x,t,\varepsilon)$ of the same smoothness as g such that

$$g(x,h(x,t),t,0) \equiv 0 \quad \forall (x,t) \in \mathcal{G} \times R.$$

(A_3). The spectrum of the Jacobian matrix $B(x,t) = g_y(x, h(x,t), t, 0)$ is uniformly separated from the imaginary axis for all $(x,t) \in \mathcal{G} \times R$, i.e. the eigenvalues $\lambda_i(x,t)$ $(i = 1, \ldots, n)$ of the matrix $B(x,t)$ satisfy the inequality

$$|Re\lambda_i(x,t)| \geq \gamma \tag{2}$$

for some positive number γ.

Then the following result is valid (see e.g. [8, 9]):

Proposition 1.1. *Under the assumptions $(A_1)-(A_3)$ there is a sufficiently small positive ε_1, $\varepsilon_1 \leq \varepsilon_0$, such that for $\varepsilon \in (0, \varepsilon_1)$ system* (1) *has a smooth integral manifold $\mathcal{M}_\varepsilon$ (*slow *integral manifold) with the representation*

$$\mathcal{M}_\varepsilon := \{(x,y,t) \in R^{n+m+1} : y = \psi(x,t,\varepsilon),\ (x,t) \in \mathcal{G} \times R\}$$

and with the asymptotic expansion

$$\psi(x,t,\varepsilon) = h(x,t) + \varepsilon\psi_1(x,t) + \ldots\ .$$

The motion on this manifold is described by the *slow* differential equation

$$\dot{x} = f(x, \psi(x,t,\varepsilon), t, \varepsilon). \tag{3}$$

Remark 1.1. *The global boundedness assumption in (A_1) with respect to (x,y) can be relaxed by modifying f and g outside some bounded region of $R^n \times R^m$.*

Remark 1.2. *In applications it is usually assumed that the spectrum of the Jacobian matrix $g_y(x, h(x,t), t, 0)$ is located in the left half plane. Under this additional hypothesis the manifold $\mathcal{M}_\varepsilon$ is exponentially attracting for $\varepsilon \in I_1$.*

The case that assumption (A_3) is violated is called critical. We distinguish three subcases:

(i) The Jacobian matrix $g_y(x,y,t,0)$ is singular on some subspace of $R^m \times R^n \times R$. In that case, system (1) is referred to as a singular singularly perturbed system [10].

(ii) The Jacobian matrix $g_y(x,y,t,0)$ has eigenvalues on the imaginary axis with nonvanishing imaginary parts. A similar case has been investigated in [3, 4, 11].

(iii) The Jacobian matrix $g_y(x,y,t,0)$ is singular on the set $\mathcal{M}_0 := \{(x,y,t) \in R^m \times R^n \times R : y = h(x,t), (x,t) \in \mathcal{G} \times R\}$. In that case, $y = h(x,t)$ is generically an isolated root of $g = 0$ but not a simple one.

Other critical cases were considered, for example, in [3, 4, 12].

The critical case (i) has been treated in [3, 4, 6, 10, 13] and it was considered as applied to the high-gain control problem in [3, 4].

The case (ii) was considered as applied to the manipulator control in [3, 4].

The case (iii) was considered as applied to the partially cheap control problem, see, for example, [3, 4].

It is not inconceivable that combinations of other pairs of critical cases and even thrice critical case are of interest as well.

2. Thrice Critical Case

Consider the control system

$$\varepsilon\dot{x} = A(t,\varepsilon)x + \varepsilon B(t,\varepsilon)u,\ \ x \in R^{n+m},\ \ x(0) = x_0 \tag{4}$$

with the cost functional

$$J = \frac{1}{2}x^T(1)Fx(1) + \frac{1}{2}\int_0^1 (x^T(t)Q(t)x(t) + \varepsilon u^T(t)R(t)u(t))dt. \quad (5)$$

where $A,\ F_1,\ Q$ are $(n \times n)$-matrices, B is $(n \times m)$-matrix, and R is $(m \times m)$-matrix. Suppose that all these matrices have the following asymptotic presentations with respect to ε:

$A(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j Aj(t),\ \ B(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j B_j(t),\ \ Q(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j Q_j(t),\quad R(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j R_j(t),$
$F(\varepsilon) = \sum_{j\geq 0} \varepsilon^j F_j,$
with smooth on t matrix coefficients, $t \in [0,1]$.

The solution of this problem is the optimal linear feedback control law

$$u = -\varepsilon^{-1}R^{-1}B^T P(t,\varepsilon)x,$$

where P satisfies the differential matrix Riccati equation

$$\varepsilon\dot{P} = -PA - A^T P + PSP - \varepsilon Q, \quad P(1,\varepsilon) = F. \quad (6)$$

Setting $\varepsilon = 0$ we obtain from (6) the matrix algebraic equation

$$-MA_0 - A_0^T M + MS_0 M - Q_0 = 0,$$

where $S_0 = B_0 R_0^{-1} B_0^T$ and $M = P(t,0)$.

For systems with low energy dissipation the matrices S_0 and Q_0 are equal to zero and the main role plays the linear operator

$$\mathbf{L}X = XA_0 + A_0^T X.$$

For this class of systems the eigenvalues of A_0 are pure imaginary and the spectrum of the linear operator $\mathbf{L}$ has a nontrivial kernel, since sums $(\lambda_i(t) + \lambda_j(t))$, $i,j = 1,\ldots,n$, form its spectrum. This means that the equation (6) is singular singularly perturbed. Thus, the dimension of the slow integral manifold of (6) is greater zero and the problem under consideration is critical in this sense. Moreover, under taking into account that zero eigenvalues are multiple and all other, nonzero eigenvalues of $\mathbf{L}$, are pure imaginary, it is possible to say that this problem is thrice critical.

2.1. Example

Let

$$A = \begin{pmatrix} -\varepsilon & 1 \\ -1 & -\varepsilon \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R = (1), Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}.$$

Consider the corresponding differential system

$$\begin{aligned} \varepsilon\dot{p}_1 &= 2p_2 + 2\varepsilon p_1 + p_2^2 - \varepsilon, \quad p_1(1) = 0, \\ \varepsilon\dot{p}_2 &= 2\varepsilon p_2 - p_1 + p_3 + p_2 p_3, \quad p_2(1) = 0, \\ \varepsilon\dot{p}_3 &= -2p_2 + 2\varepsilon p_3 + p_3^2, \quad p_3(1) = 0. \end{aligned} \quad (7)$$

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First, we need to separate it into a slow and a fast subsystem. At first glance, all three equations are singularly perturbed. However, setting $\varepsilon = 0$ we obtain $p_1 = p_2 = p_3 = 0$, and we should consider the matrix of leading terms on the right hand side of the system, which has the form

$$\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$$

Obviously, this matrix has a zero eigenvalue and two pure imaginary eigenvalues, i.e. the problem under consideration is twice critical. Moreover, the trivial solution is multiple. This means that we have thrice critical case.

Let $\varepsilon = \mu^2$. Introducing the new variables

$$p_1 = \mu^2 q_1 + \mu, \quad p_2 = \mu^2 q_2 + \mu^2/2, \quad p_3 = \mu^2 q_3 + \mu,$$

and then $s = q_1 + q_3$, we obtain the differential system

$$\begin{aligned} \mu\dot{s} &= 2q_3 + \mu q_2 + 2\mu s + \mu q_2^2 + \mu q_3^2 + 4 + \mu/4, \\ \mu^2\dot{q}_2 &= -s + 2\mu^2 q_2 + 2q_3 + \mu q_2 + \mu^2 q_2 q_3 + \mu/2 + \mu^2, \\ \mu^2\dot{q}_3 &= -2q_2 + 2\mu q_3 + 2\mu^2 q_3 + \mu^2 q_3^2 + 2\mu \end{aligned} \tag{8}$$

with the slow variable s and two fast variables $q_2,\ q_3$.

The last system possesses one-dimensional slow invariant manifold which is weakly attractive with respect to argument $1 - t$ because the main matrix of the fast subsystem is

$$\begin{pmatrix} \mu & 2 \\ -2 & 2\mu \end{pmatrix}.$$

Thus, the dimension of the system of Riccati differential equations can be reduced from three to one. Let us construct the slow integral manifold using the fact that it can be asymptotically expanded in powers of the small parameter. Setting

$$q_2 = \varphi(s, \mu) = \mu\varphi_1(s) + \mu^2 \dots ,$$

$$q_3 = \psi(s, \mu) = \psi_0(s) + \mu\psi_1(s) + \mu^2 \dots ,$$

we obtain

$$\psi_0(s) = s/2, \quad \varphi_1(s) = s/2, \quad \psi_1(s) = -1/4.$$

Thus we obtain the slow invariant manifold

$$q_2 = \mu s/4 + O(\mu^2), \quad q_3 = s/2 - \mu/4 + O(\mu^2),$$

with the equation on the integral manifold

$$\mu\dot{s} = 4 - \mu/4 + (1 + 2\mu)s + \mu s^2/4 + O(\mu^2).$$

Neglecting by terms of order $O(\mu^2)$ we obtain the scalar solvable equation

$$\mu\dot{s} = 4 - \mu/4 + (1 + 2\mu)s + \mu s^2/4, \quad s(1, \mu) = -2/\mu, \tag{9}$$

and it is possible to obtain the solution of this initial value problem in the analytical form:

$$s = S(t, \mu).$$

Setting for example, $\varepsilon = 0.01$, i.e. $\mu = 0.1$, we obtain

$$S(t) = -24 - \sqrt{417}\tanh(5.105144465t - 5.303589885).$$

For initial variables p_1, p_2, p_3 we obtain the following formulae

$$\begin{array}{l} p_1 = \mu + \mu^2 S(t,\mu) + \mu^3/4 + O(\mu^4), \\ p_2 = \mu^2/2 + \mu^3 S(t,\mu)/4 + O(\mu^4), \\ p_3 = \mu + \mu^2 S(t,\mu) - \mu^3/4 + O(\mu^4) \end{array} \tag{10}$$

which describe the solution of 10 corresponding to the slow invariant manifold.

The Fig. 1 demonstrates the closeness of solutions of the original system and the system on the slow invariant manifolds for $p_1(t)$. The similar situation takes place for p_2 and p_3, see Fig. 2 and Fig. 3.

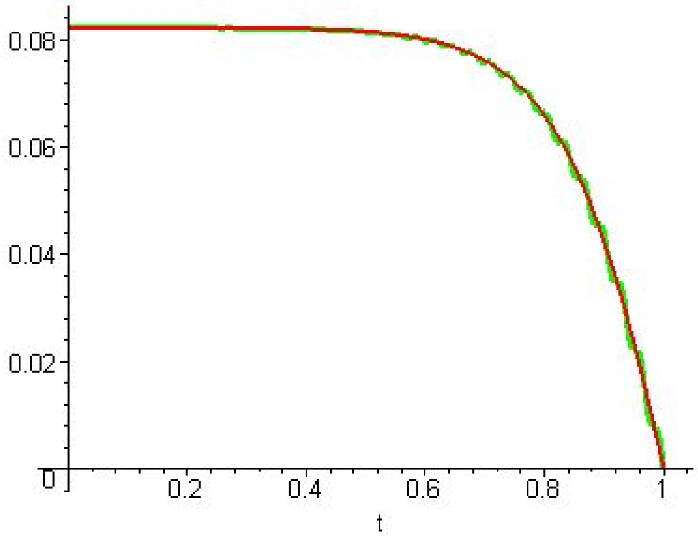

Figure 1. The graphs of p_1 for original differential system (green line) and for the equation on the slow invariant manifold (red line), $\mu = 0.1$.

3. Conclusion

The special critical case for singularly perturbed differential systems are studied in the paper. We have considered singularly perturbed control problem as an application. It has been shown that the reduction of dimensions of these problems can be done by means of the integral manifold method. The slow integral manifolds for the matrix Riccati equation of linear-quadratic control problem are constructed and it is shown that the method of integral manifolds allows us to reduce the dimension of control problems. This approach was used for the investigation of optimal filtering problems in [16, 17].

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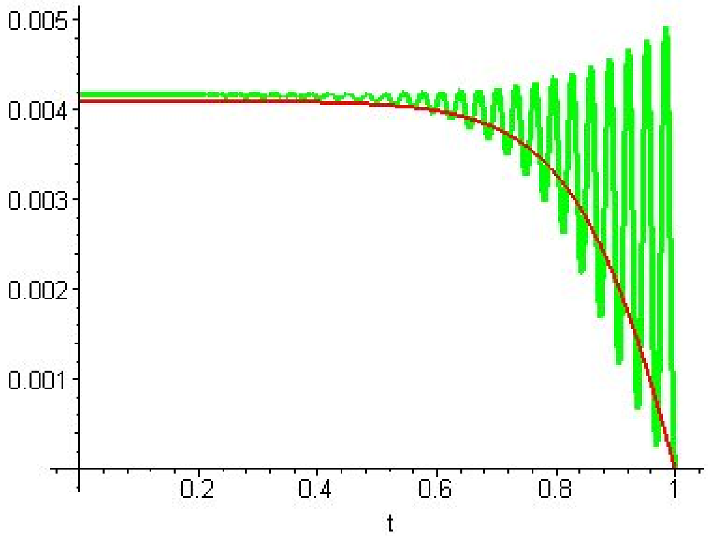

Figure 2. The graphs of p_2 for original differential system (green line) and for the equation on the slow invariant manifold (red line), $\mu = 0.1$.

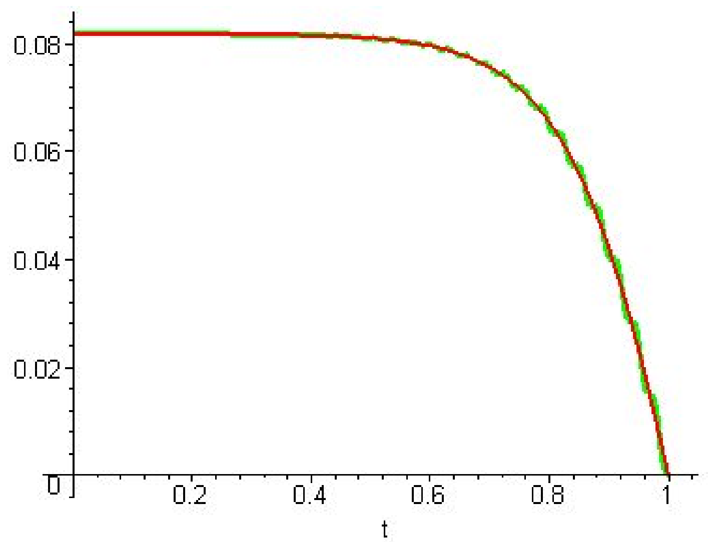

Figure 3. The graphs of p_3 for original differential system (green line) and for the equation on the slow invariant manifold (red line), $\mu = 0.1$.

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