Regularization of the matrix Riccati equation in optimal estimation problem with low measurement noise

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Abstract. We explore the possibility of applying the method of order reduction of optimal estimation problem for singularly perturbed systems with low measurement noise. It is shown that matrix Riccati equation for the Kalman-Bucy filter has a periodic solution. An optimal filter is constructed for a dynamic model of a crank mechanism.

1. Introduction and statement of problem

Consider the optimal filtering problem for linear system:

$$\dot{x}(t) = A(t,\mu)x(t) + B(t,\mu)w(t), \tag{1}$$

where $x(t,\mu) \in R^n$ is the system state vector, $t \in R$ is the time, $\mu \ll 1$ is a small positive parameter, $A(t,\mu) \in R^{n\times n}$ is the dynamic coefficient matrix, $B(t,\mu) \in R^{n\times m}$ is the process noise coupling matrix, $w \in R^m$ is the zero-mean white Gaussian process noise with given covariance $Q \in R^{m\times m}$. We assume that the following quantity can be measured:

$$z(t) = C(t,\mu)x(t) + \mu v(t), \tag{2}$$

where $z(t) \in R^k$ is the measurement vector, $C(t,\mu) \in R^{k\times n}$ is the measurement sensitivity matrix, $v \in R^k$ is the zero-mean white Gaussian measurement noise with covariance $R(t,\mu) \in R^{k\times k}$. Let $w(t)$ and $v(t)$ be independent. The presence of the small parameter in (2) means that optimal estimation problem with low measurement noise (see [1] and references therein) is considered.

We are required to obtain an estimate $\hat{x}(t)$ of the state $x(t)$ of system (1) in accordance with the vector function $z(t)$ available for measurement at $t > 0$. The solution to this problem may be obtained by using the Kalman-Bucy filter [2] that involves the solution of the matrix Riccati equation for the covariance matrix of the filter [3]:

$$\mu^2(\dot{P} - AP - PA^T - BQB^T) = -PC^TR^{-1}CP, \quad P(0) = P_0. \tag{3}$$

The asymptotic expansion of solution to the Riccati equation for time-invariant matrices A, B, C, Q, R in the case of optimal control problem was obtained in [4].

The Riccati equation (3) may be considered as a singularly perturbed ODE system [5, 6] in a critical case since the corresponding limiting problem (under $\mu = 0$)

$$PC^T R^{-1} CP = 0 \tag{4}$$

possesses multiple zero solutions [7, 8].

The solution to the differential equation (3) may be separated into the following cases for $t > 0$:

Case 0: matrix BQB^T is positive definite and $rank(C) = n$;

Case 1: matrix $CBQB^TC^T$ is non-singular and $rank(C) = r \leq n$;

Case L ($L \geq 2$) :

$$C_j BQB^T C_j^T = 0, \quad j = \overline{0,\ L-2}, \quad C_{L-1} BQB^T C_{L-1}^T > 0, \tag{5}$$

where $C_0 = C$, $C_j = C_{j-1}A^T - \dot{C}_{j-1}$. The solution to the problem (3) in cases 0 and 1 may be obtained as asymptotic expansion in integer powers of small parameter μ. In case L one may obtain solution of the problem (3) as asymptotic expansion in fractional powers of small parameter $\varepsilon = \mu^{1/L}$ [4]. Thus, neglecting by the initial value condition, we will try to find the partial solution in the following form :

$$P(t, \mu) = \sum_{j=0}^{\infty} P_j(t) \mu^{j/L}.$$

In general, solutions of this kind are the solutions of differential subsystem describing the flow of the original system on the slow integral manifold [7, 9, 10]. The integral manifolds method was used for the investigation of optimal control and filtering problems in [11, 12, 13, 14, 15].

In the paper we use a combination of geometric and asymptotic approaches for the regularization of matrix Riccati equations in some critical cases. Firstly, we try to find a formal solution to the matrix Riccati equation as asymptotic expansions

$$p_k = \sum_{j=0}^{\infty} p_{kj}(t) \varepsilon^j.$$

Here, p_k is some element of matrix P and $\varepsilon = \mu^{1/L}$. Let p_{ks_k} be the first nonzero coefficient in the expansion to p_k, then we introduce a new variable $p_k = \varepsilon^{s_k} y_k$. The corresponding ODE system for new variables possesses the nonzero solution $p_k = p_k(t, \varepsilon)$, $p_k(t, 0) \neq 0$ and we can obtain singularly perturbed ODE system in a non-critical case.

2. Simple example

Consider the optimal estimation problem for the following system:

$$\ddot{x}(t) + ax(t) = w(t), \quad x(0) = x_0, \tag{6}$$

where $x(t)$ is a scalar function, a is a constant coefficient, $w(t)$ is a scalar zero-mean white Gaussian process noise with given dispersion $q = 1 + 2\mu$, where $\mu \ll 1$ is a small positive parameter. The following function is available for measurement:

$$z(t) = x(t) + \mu v(t), \tag{7}$$

where v is a zero-mean scalar white Gaussian measurement noise with given dispersion $r = 1$. Functions $w(t)$ and $v(t)$ are independent. Obviously, this problem is similar to the problem (1) - (2). In this case, we have

$$A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R = \left(\mu^2\right), Q = (q),$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}, P = \begin{pmatrix} p_0 & p_1 \\ p_1 & p_2 \end{pmatrix}.$$

It is easy to see that in the case under consideration, $L = 2$.

We use $a = 1$ for simplicity and construct the matrix Riccati equation (3) for covariance matrix of the Kalman-Bucy filter:

$$\begin{aligned} \dot{p}_0 &= 2p_1 - \frac{p_0^2}{\mu^2}, \\ \dot{p}_1 &= p_2 - p_0 - \frac{p_0 p_1}{\mu^2}, \\ \dot{p}_2 &= 1 + 2\mu - 2p_1 - \frac{p_1^2}{\mu^2}. \end{aligned} \tag{8}$$

The reduced system (4) has the following solution:

$$p_0 = \mu\sqrt{2\mu}, \quad p_1 = \mu, \quad p_2 = \sqrt{2\mu}(1+\mu). \tag{9}$$

System (6) may be integrated explicitly and has the following solution:

$$p_0 = \mu\sqrt{2\mu}\left(1 - \frac{2e^{-\sqrt{\frac{2}{\mu}}t}}{D(t)}\left(\nu^2 \cos\left(\sqrt{\frac{2}{\mu}}\frac{t}{\nu}\right) + \nu \sin\left(\sqrt{\frac{2}{\mu}}\frac{t}{\nu}\right) + 1 + \nu^2 + e^{-\sqrt{\frac{2}{\mu}}t}\right)\right),$$

$$p_1 = \mu\left(1 - \frac{2e^{-\sqrt{\frac{2}{\mu}}t}}{D(t)}\left(\left(\nu^2+1\right)\cos\left(\sqrt{\frac{2}{\mu}}\frac{t}{\nu}\right) + 1 + \nu^2\right)\right),$$

$$\begin{aligned} p_2 = \sqrt{2\mu}(1+\mu)\Bigg(1 + \frac{2e^{-\sqrt{\frac{2}{\mu}}t}}{D(t)}\Bigg(&-\nu^2\cos\left(\sqrt{\frac{2}{\mu}}\frac{t}{\nu}\right) \\ &+ \left(\mu + \frac{\nu}{(1+\mu)}\right)\sin\left(\sqrt{\frac{2}{\mu}}\frac{t}{\nu}\right) - 1 - \nu^2 - e^{-\sqrt{\frac{2}{\mu}}t}\Bigg)\Bigg), \end{aligned}$$

where $\nu = \sqrt{1/(1+2\mu)}$ and

$$D(t) = 1 + e^{-2\sqrt{\frac{2}{\mu}}t} + e^{-\sqrt{\frac{2}{\mu}}t}\left(\frac{1}{1+\mu}\left(\nu^2+1\right)\cos\left(\sqrt{\frac{2}{\mu}}\frac{t}{\nu}\right) + 2\left(1+\nu^2\right)\right).$$

We conclude that the solution of the system (8) tends exponentially to the steady state. Moreover, the difference between two this solutions is of order $O(e^{-\sqrt{\frac{2}{\mu}}t})$ as $\mu \to 0,\ t > 0$. It means that with very high order of accuracy the steady state may be used instead the exact solution.

3. Optimal estimation problem for parametric oscillator

Consider the model of parametric oscillator without damping, but with the presence of process noise on the right side of equation:

$$\ddot{x} + a(t)x = w, \tag{10}$$

where $a(t)$ is a continuous and uniformly bounded together with a sufficient number of derivatives function, w is a zero-mean white Gaussian process noise with a given correlation q. We assume that the following quantity can be measured:

$$z = x + \mu v, \tag{11}$$

where v is a zero-mean white Gaussian measurement noise with a given correlation r, and $\mu \ll 1$ is a small positive parameter. Problem (10)-(11) naturally appears when considering the optimal estimation problem in mechanics. For example, a simple crank mechanism is an arm connected to a rotating shaft by which reciprocating motion is imparted to the shaft [16]. The dynamic model of this system is

$$\ddot{\phi} + c\left(\frac{1}{I_m} + \frac{1}{I_0}\right)\left(1 - \frac{I_M m_2 r^2}{2I_0(I_M + I_0)}\cos 2\pi t\right)\phi = w, \tag{12}$$

where ϕ is the angle of the shaft, I_0 is the average moment of inertia of the crank, I_M is the moment of inertia of the shaft, m_2 is the arm mass and c is the stiffness coefficient of the arm. The system is influenced by an external white Gaussian disturbance w.

The solution of the optimal estimation problem (10)-(11) may be obtained by Kalman-Bucy filter that involves solution of matrix Riccati equation of the following form:

$$\begin{aligned} \mu^2(\dot{p}_0 - 2p_1) &= -\frac{p_0^2}{r}, \\ \mu^2(\dot{p}_1 + a(t)p_0 - p_2) &= -\frac{p_0 p_1}{r}, \\ \mu^2(\dot{p}_2 + 2a(t)p_1 - q) &= -\frac{p_1^2}{r}. \end{aligned} \tag{13}$$

We note that numerical solution of the singularly perturbed system is the most expensive stage in the construction of the filter. One should use sufficiently small time step in the numerical simulations to achieve the desired accuracy of the filter. This issue is of significant importance in real-time algorithms with limited computational resources.

We introduce variables $p_i,\ i = 0..2$:

$$p_0 = \varepsilon^3 y_0, \quad p_1 = \varepsilon^2 y_1, \quad p_2 = \varepsilon y_2, \tag{14}$$

where $\varepsilon^2 = \mu$. Thus, system of equations (13) takes the following form:

$$\begin{aligned} \varepsilon\dot{y}_0 &= 2y_1 - \frac{y_0^2}{r}, \\ \varepsilon\dot{y}_1 &= y_2 - \varepsilon^2 a(t)y_0 - \frac{y_0 y_1}{r}, \\ \varepsilon\dot{y}_2 &= q - 2\varepsilon^2 a(t)y_1 - \frac{y_1^2}{r}. \end{aligned} \tag{15}$$

This singularly perturbed system is not in critical case and usually the integral manifolds method is used in such a situation [15]. It is possible to construct attracting periodic solution for this system. We construct the periodic solution for system (15) with accuracy $O(\varepsilon^3)$. This means that the Riccati matrix equation is replaced by the periodic solution which with accuracy $O(\mu^6)$ has the form

$$\begin{aligned}
p_0 &= \mu^3 \sqrt[4]{4qr^3} - \mu^5 \frac{a(t)r^2}{\sqrt[4]{4qr^3}} + \mu^6 \frac{\dot{a}(t)r^2}{2\sqrt{qr}}, \\
p_1 &= \mu^2 \sqrt{qr} - \mu^4 a(t)r + \mu^5 \frac{\dot{a}(t)r}{2\sqrt{2}} \sqrt[4]{\frac{r}{q}}, \\
p_2 &= \mu \sqrt[4]{4q^3 r} - \mu^3 \frac{a(t)\sqrt[4]{qr^3}}{\sqrt{2}}.
\end{aligned} \tag{16}$$

4. Numerical experiments

Figure 2 shows comparison of the solution of system (13) for Kalman-Bucy filter covariance for the crank mechanism model with the periodic solution (16). When using the latter, we don't need to solve any system of differential equations because the components of the covariance matrix for the filter may be computed explicitly. The following parameters are used: $c = 1$, $I_0 = 50$, $I_m = 1$, $m_2 = 1.2$, $r = 1$, $\mu = 0.05$, $a(t) = \cos(t)$.The RK4 method is used for solving system (13) with time step $dt = 1 \times 10^{-4}$. We see that the solution of the original system tends to the periodic solution exponentially, beyond that, the difference between the solutions for p_0 is of the order 1×10^{-10}, for p_1 is 1×10^{-9} and for p_2 is 1×10^{-7} (see Figure 3).

5. Conclusion

In this paper, we considered the optimal estimation problem with low measurement noise. The matrix Riccati system of differential equations for the Kalman-Bucy filter is singularly perturbed. The method of model order reduction is applicable for this differential system even if the system is in critical case. It was shown that the periodic solution may be used instead the exact initially value problem for the matrix Riccati equation for covariance matrix of the filter. This approach is demonstrated on model of crank mechanism. Our numerical experiment shows that the difference between the solution of the original matrix Riccati differential system and the periodic solution is of the order $O(\mu^6)$. The evaluated example of crank mechanism does not account for dumping, Coulomb type friction, and structure elasticity, but is used as an example to show the approach to the reduction the order of optimal estimation problems with low measurement noise.

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