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Asymptotic analysis of reaction-diffusion-advection problems: Fronts with periodic motion and blow-up

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Abstract. This is an extended variant of the paper presented at MURPHYS-HSFS 2016 conference in Barcelona. We discuss further development of the asymptotic method of differential inequalities to investigate existence and stability of sharp internal layers (fronts) for nonlinear singularly perturbed periodic parabolic problems and initial boundary value problems with blow-up of fronts for reaction-diffusion-advection equations. In particular, we consider periodic solutions with internal layer in the case of balanced reaction. For the initial boundary value problems we prove the existence of fronts and give their asymptotic approximation including the new case of blowing-up fronts. This case we illustrate by the generalised Burgers equation.

1. Introduction

We present some recent results of investigation of singularly perturbed reaction-advection-diffusion problems, which are based on a further development of the asymptotic comparison principle. We discuss further development of the general scheme of the asymptotic method of differential inequalities (see [1] - [5]) for periodic parabolic problems and apply this scheme to a few new cases. Theorems, which state the existence of periodic solutions with an internal layer, present an asymptotic approximation of such solution and establish Lyapunov stability for them. In particular, we consider periodic solutions with internal layer in the case of balanced reaction for the problems

$$\begin{gathered}
\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)-\varepsilon^2 A(u,x,t)\,\frac{\partial u}{\partial x}-f(u,x,t,\varepsilon)=0 \quad \text{for} \quad x\in(0,1),\ t\in R,\\
\frac{\partial u}{\partial x}(0,t,\varepsilon)=u^{(-)}(t),\quad \frac{\partial u}{\partial x}(1,t,\varepsilon)=u^{(+)}(t) \quad \text{for} \quad t\in R,\\
u(x,t,\varepsilon)=u(x,t+T,\varepsilon) \quad \text{for} \quad t\in R,
\end{gathered}$$

where ε is a small parameter, A, f, $u^{(-)}(t)$ and $u^{(+)}(t)$ are sufficiently smooth T-periodic in t functions.

We also present an extension of these results for the reaction-advection-diffusion equation

$$\varepsilon\frac{\partial^2 u}{\partial x^2}-A(u,x)\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=f(u,x,\varepsilon),\quad x\in(0,1),t>0.$$

For the initial boundary value problems for this equation we prove the existence of fronts and obtain their asymptotic approximation including the new case of the blowing-up fronts. This

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case is illustrated by the generalised Burgers equation. These results can be considered as an extension of the results of [7] - [9].

The paper is structured as follows. In Section 2 we consider the general scheme of our approach, extended to periodic parabolic problems. In Section 3 we apply it to a reaction-advection-diffusion equation with blowing-up fronts.

2. General scheme of asymptotic method of differential inequalities for periodic reaction-advection-diffusion equations

2.1. General scheme

We consider some cases of initial boundary value problem

$$Nu \equiv \varepsilon^2(\Delta u - \tfrac{\partial u}{\partial t}) - f(u, \nabla u, x, t, \varepsilon) = 0, \quad x \in \mathcal{D} \subset R^N,\, t \in R,$$
$$Bu = h(x,t), \quad x \in \partial\mathcal{D},\, t \in R,$$

with periodicity condition $u(x,t,\varepsilon) = u(x, t+T, \varepsilon)$, where ε is a small parameter, f, h, and $\partial\mathcal{D}$ are sufficiently smooth, B is a boundary operator for Dirichlet, Neumann or third type boundary conditions. We also assume that f, h are T-periodic in t functions.

Definition. *The T-periodic functions $\beta(x,t,\varepsilon)$ and $\alpha(x,t,\varepsilon)$ are called asymptotic upper and lower solutions of order $q > 0$ of the problem if for all sufficiently small ε they satisfy the inequalities*

$$N\beta \le -c\varepsilon^q,\; N\alpha \ge c\varepsilon^q, \quad x \in \mathcal{D},\, t \in R,$$
$$B\alpha \le h(x,t) \le B\beta, \quad x \in \partial\mathcal{D},\, t \in R,$$

where c is a positive constant.

We start with the description of our approach to investigate the existence and Lyapunov stability of periodic solutions with boundary and internal layers.

Denote by L the linear operator which we get from N by linearizing f on the periodic solution, by L_f the linearization f on the periodic solution and by H the following characteristic of the nonlinearity

$$H \equiv f(\beta, \nabla\beta, x, t, \varepsilon) - f(\alpha, \nabla\alpha, x, t, \varepsilon) - L_f(\beta - \alpha).$$

The properties of H depend on the asymptotic lower and upper solutions. In what follows we describe the method to construct them by using the formal asymptotic expansion. We assume that

(A_1). *There exist an asymptotic upper solution β and an asymptotetic lower solution α of order q such that $\beta > \alpha$ and $|\beta - \alpha| \le c\varepsilon^r$ for $x \in \mathcal{D}$, $t \in R$ and all sufficiently small ε.*

We also assume that

(A_2). $|H| \le c\varepsilon^p$ for $x \in \mathcal{D}$, $t \in R$ and a sufficiently small ε.

(A_3). $p \ge q$.

Theorem 1. *Suppose that assumptions $(A_1) - (A_3)$ are valid. Then, for any sufficiently small ε there exists a solution $u(x,t,\varepsilon)$ which differs from the upper and lower solutions by the value of order $O(\varepsilon^r)$ and is asymptotically stable in Lyapunov sense with the local domain of stability $[\alpha, \beta]$.*

The existence of the solution follows from assumption (A_1) (see [10]) . The proof of stability is based on a revised maximum principal, which uses Krein-Rutman theorem. The scheme of the proof can be found in [1] (see also [4],[11] and references therein).

In order to obtain the upper and lower solutions satisfying the assumptions of Theorem 1 we use the formal asymptotic expansion, which can be constructed in a lot of cases by our method. Under quite natural assumptions the formal asymptotic expansion of internal layer solution is produced by the boundary layer operators L_B, regular expansion operators L_R and by the operators of order q describing the location of the transition layer. To construct the formal asymptotic expansion we assume that the operators are invertible.

For the construction of asymptotic lower and upper solutions we require that **these operators are monotone (order preserving) when they act in the same classes of functions in which we construct the asymptotic expansions by means of these operators.**

Finally we obtain $\alpha \equiv \alpha_n$, $\beta \equiv \beta_n$, the modified n-th order formal asymptotic expansions.

2.2. Periodic BVP with Balanced Reaction

We consider periodic solutions with internal layer in the case of the balanced reaction for the problem

$$\begin{gathered}\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)-\varepsilon^2 A(u,x,t)\,\frac{\partial u}{\partial x}-F(u,x,t,\varepsilon)=0 \quad \text{for} \quad x\in(0,1),\ t\in R,\\ \frac{\partial u}{\partial x}(0,t,\varepsilon)=u^{(0)}(t), \quad \frac{\partial u}{\partial x}(1,t,\varepsilon)=u^{(1)}(t), \quad t\in R,\\ u(x,t,\varepsilon)=u(x,t+T,\varepsilon) \quad \text{for} \quad t\in R,\end{gathered} \tag{1}$$

where ε is a small parameter, A, F, $u^{(0)}(t)$ and $u^{(1)}(t)$ are sufficiently smooth T-periodic in t functions.

We assume, that

- (A_1). The reduced equation $F(u,x,t,0) = 0$ has exactly three solutions $\phi^{(-)}(x,t), \phi^{(0)}(x,t), \phi^{(+)}(x,t)$, satisfying
$$\phi^{(-)}(x,t) < \phi^{(0)}(x,t) < \phi^{(+)}(x,t);$$
- (A_2). $F_u(\phi^{(\pm)},x,t,0) > 0, F_u(\phi^{(0)},x,t,0) < 0$;

and consider the case of the balanced reaction

- (A_3). $I(x,t) := \int_{\phi^{(-)}(x,t)}^{\phi^{(+)}(x,t)} F(u,x,t,0)du \equiv 0$.

To construct the formal interior layer asymptotic expansion we use the ansatz (a similar treatment one can find in [1, 2, 3, 4, 5])

$$U^{(\pm)}(x,t,\varepsilon,x^*) = \overline{u}^{(\pm)}(x,t,\varepsilon) + Q^{(\pm)}(\tau,x^*,t,\varepsilon) + \Pi^{(\pm)}(\xi,t,),$$

where the terms are the series in ε and the location of the interior layer $x^*(t,\varepsilon)$ is *a priori* unknown. This location is defined by the C^1-matching procedure applied to $U^{(\pm)}(x,t,\varepsilon,x^*)$ with $x^*(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$, $\xi = x/\varepsilon, \quad x \le x^*, \qquad \xi = (x-1)/\varepsilon, \quad x \ge x^*$, and $\tau = (x-x^*)/\varepsilon$ (symbols "-" and "+" are used for the terms of asymptotic expansions in the domains on the left and right sides of the curve $x^*(t)$, respectvely). The terms of the asymptotic expansions are defined in a standard way, see for example [1, 2, 3, 4, 5].

The main term of the layer location is defined by the problem

$$-\frac{dx_0}{dt}\int_{-\infty}^{+\infty}\Phi^2(\tau,x_0,t)d\tau + \int_{-\infty}^{+\infty}\left(\Phi^2(\tau,x_0,t)\tilde{A} + \Phi(\tau,x_0,t)\tilde{F}_x\tau + \Phi(\tau,x_0,t)\tilde{F}_\varepsilon\right)d\tau = 0$$

$$x_0(t) = x_0(t+T),$$

where " $\sim$ " means that the function is evaluated at the point $(\tilde{u}^{(\pm)}(\tau,x_0,t),x_0,t,0)$, and $\Phi^{(\pm)}(\tau,x_0,t) = \frac{\partial Q_0^{(\pm)}}{\partial\tau}(\tau,x_0,t)$.

We assume that

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- (A_4). The problem for $x_0(t)$ has a solution.

For the next terms of the layer location we have

$$m(t)\frac{dx_i}{dt} - d(t)x_i = f_i(t), \quad i = 1, 2, ..., \quad x_i(t) = x_i(t+T),$$

where, at each step, the functions $f_i(t)$ are known and

$$m(t) := \int_{-\infty}^{+\infty} \Phi^2(\tau, x_0, t)d\tau,$$

$$d(t) := -\frac{dx_0}{dt}\frac{\partial\left(\int_{-\infty}^{+\infty}\Phi^2(\tau,x_0,t)d\tau\right)}{\partial x_0} + \frac{\partial\left(\int_{-\infty}^{+\infty}\left(\Phi^2(\tau, x_0, t)\tilde{A} + \Phi(\tau, x_0, t)\tilde{F}_x\tau + \Phi(\tau, x_0, t)\tilde{F}_\varepsilon\right)d\tau\right)}{\partial x_0}.$$

A further assumption

- (A_5). $\frac{1}{T}\int_0^T \frac{d(t)}{m(t)}dt < 0$.

guarantees that the problem for $x_i(t)$ is solvable and the first order periodic operator is an order preserving operator.

According to the previous section we construct an upper solution in the form (similar construction one can find, for example, in [4])

$$\beta_n(x, t, \varepsilon) = \bar{u}_0^{(\pm)}(x, t) + \varepsilon\bar{u}_1^{(\pm)}(x, t) + \ldots + \varepsilon^{n+3}\bar{u}_{n+3}^{(\pm)}(x, t)$$
$$+Q_0^{(\pm)}(\tau_\beta, x_{n\beta}, t) + \varepsilon Q_1^{(\pm)}(\tau_\beta, x_{n\beta}, t) + \ldots + \varepsilon^{n+2}Q_{n+2}^{(\pm)}(\tau_\beta, x_{n\beta}, t)$$
$$+\Pi_\beta(\xi, t, \varepsilon) + \varepsilon^{n+3}(\gamma + Q_{n+3,\beta}^{(\pm)}(\tau_\beta, x_{n\beta}, t)),$$

where $x_{n\beta}(t, \varepsilon) = x_n^*(t, \varepsilon) + \varepsilon^{n+2}(x_{n+2}(t) - \nu(t))$, $\gamma > 0$ is a positive constant and $\nu(t)$ is a positive solution of the the problem in (A_5) with a positive right hand side.

The lower solution is constructed analogously.

We can follow the proof for the unbalanced case of problem (1) considered in [4] to check that $\beta_n(x, t, \varepsilon)$ and $\alpha_n(x, t, \varepsilon)$ are asymptotic upper and lower solutions and that q and p in Theorem 1 are $q = n + 3$, $p = 2n + 2$. Therefore the assumption $p \geq q$ is valid for $n \geq 1$. We have

Theorem 2 *Suppose* $(A_1)-(A_5)$ *are valid. Then there exists a contrast structure type solution* $u(x, t, \varepsilon)$ *of problem (1):*

$$|u(x, t, \varepsilon) - U_n(x, t, \varepsilon)| < C\varepsilon^{n+1},$$

which is asymptotically stable in the sense of Lyapunov with the local domain of stability $[\alpha_1, \beta_1]$, *where* $U_n(x, t, \varepsilon)$ *is the n-th order partial sum of the constructed approximation and* α_1, β_1 *are lower and upper solutions constructed by the modification of* $U_4(x, t, \varepsilon)$.

3. Initial Boundary Value Problems with Fronts: Motion and Blow-Up

We present recent results for some classes of IBVP (initial boundary value problem) where we investigate moving fronts by using the developed comparison technique. For these initial boundary value problems we proved the existence of fronts and give its asymptotic approximation. We proved that the principal term, describing the location of the moving front, is determined by the initial value problem (see [7])

$$\frac{dx_0}{dt} = V(x_0), \quad x_0(0) = x_{00}, \tag{2}$$

where x_{00} is the initial location of the front, $V(x_0)$ is a known function, defined by the input data. We proved that the Lyapunov stability of steady points of equation (2) determine the Lyapunov stability of stationary solutions with interior layer of the IBVP. In the present paper we also have proved that under some conditions the blow-up of the solution problem (2) determine the blow-up of the interior layer solution of the IBVP.

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3.1. Reaction-advection-diffusion equations
We illustrate our results by the problem

$$\varepsilon\frac{\partial^2 u}{\partial x^2} - A(u,x)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = f(u,x,\varepsilon), \quad x \in (0,1), t > 0,$$

$$u(0,t,\varepsilon) = u^0, \quad u(1,t,\varepsilon) = u^1, \quad t \in [0,T],$$
$$u(x,0,\varepsilon) = u_{init}(x,\varepsilon), \quad x \in [0,1].$$

As it was mentioned above $u_{init}(x,\varepsilon)$ is a front type function which is approximated by the formal asymptotics and therefore the initial location x_{00} is considered as the principal term of the location expansion. The generation of the front is not considered.

We assume that

- (H_1). The equation
$$A(u,x)\frac{du}{dx} + B(u,x) = 0$$
with the initial condition $u(0) = u^0$ has a solution $u = \varphi^l(x)$, and the same equation with the initial condition $u(1) = u^1$ has a solution $u = \varphi^r(x)$. Moreover,
$$\varphi^l(x) < \varphi^r(x), \quad x \in [0,1]$$
and
$$A(\varphi^l(x),x) > 0, \quad A(\varphi^r(x),x) < 0, \quad x \in [0,1].$$
- (H_2).
$$I(x) := \int_{\varphi^l(x)}^{\varphi^r(x)} A(u,x)du > 0.$$
- (H_3). The initial value problem
$$\frac{dx_0}{dt} = \frac{I(x_0)}{\varphi^r(x_0) - \varphi^l(x_0)} \equiv V(x_0), \quad x_0(0) = x_{00},$$
where x_{00} is the initial location of the front, has a solution $x_0(t) : [0;T] \to [0,1]$ such that
$$x_0(t) \in (0,1) \quad \text{for} \quad t \in [0,T].$$

The main result for this problem is a theorem on the existence and asymptotic approximation of the moving front with the principal term $x_0(t)$ of the front location. The main result of [7] is that asymptotically stable stationary points of the equation of motion

$$\frac{dx_0}{dt} = V(x_0)$$

produce stable stationary solutions with the fronts of the original parabolic problem located near these points.

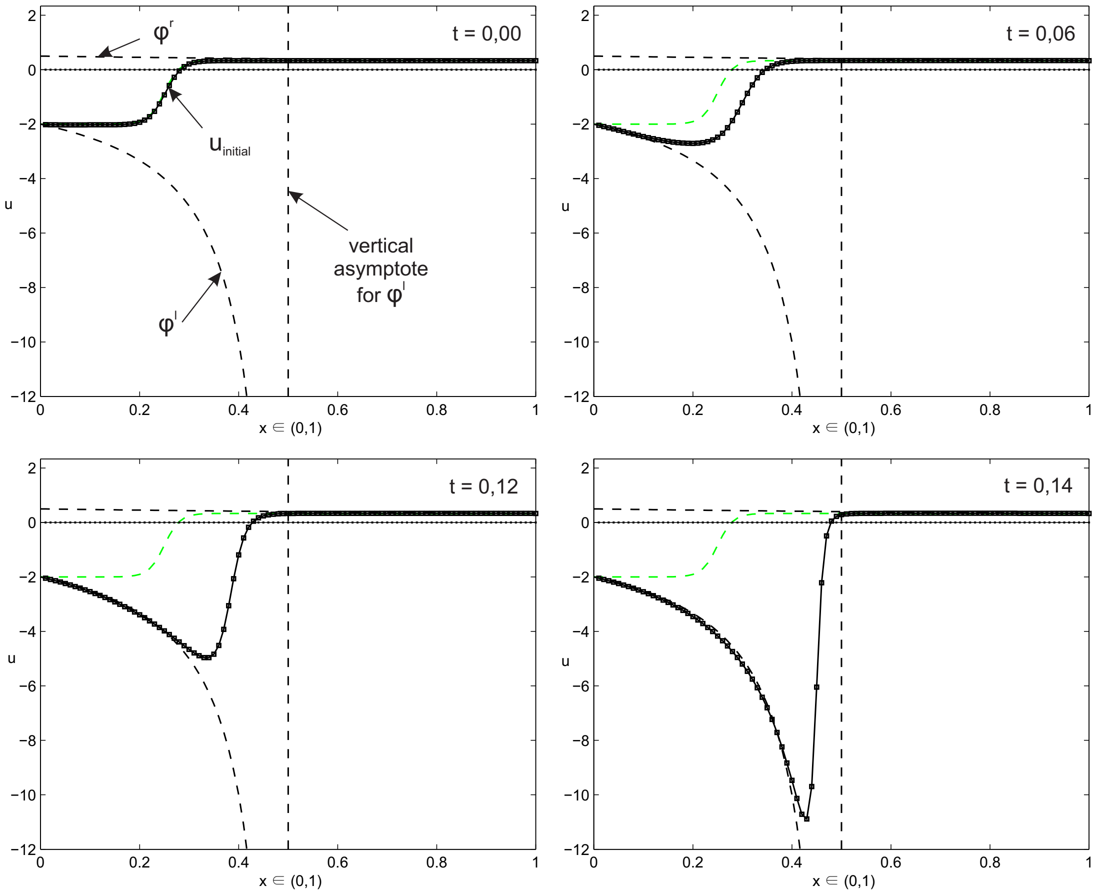

Figure 1. A typical example of a solution $u(x, t)$ to the Burgers type equation (a refinement of the mesh in a neighborhood of the transition point and the bounds has been performed).

3.2. Burgers type equation with blow-up of the solution

Let us replace assumption (H_1) with the following assumption:

- (H_1'). The equation

$$A(u,x)\frac{du}{dx} + B(u,x) = 0$$

with the initial condition $u(0) = u^0$ has a solution $u = \varphi^l(x)$ with blow-up near some point x_c, i.e. $\varphi^l(x) \to -\infty$ for $x \to x_c$, and the same equation with the initial condition $u(1) = u^1$ has a solution $u = \varphi^r(x)$. Moreover,

$$\varphi^l(x) < \varphi^r(x), \quad x \in [0, x_c)$$

and

$$A(\varphi^l(x), x) > 0, \quad x \in [0, x_c) \quad A(\varphi^r(x), x) < 0, \quad x \in [0, 1].$$

We have shown that the Burgers type equation can exhibit a blow-up of the front type solution with the jump at the front tending to infinity. We illustrate this phenomenon by the following

Burgers type equation with cubic forcing:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - u^3, \quad x \in (0,1), \quad t \in (0, 0.3],$$
$$u(0,t) = -2, \quad u(1,t) = \tfrac{1}{3},$$
$$u(x,0) = \tfrac{7}{6}\tanh\frac{x-\frac{1}{4}}{\varepsilon} - \tfrac{5}{6}.$$

An example of calculations for this problem is shown in Figure 1.

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