Color groups of colorings of N-planar modules

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Abstract. A submodule of a $\mathbb{Z}$-module determines a coloring of the module where each coset of the submodule is associated to a unique color. Given a submodule coloring of a $\mathbb{Z}$-module, the group formed by the symmetries of the module that induces a permutation of colors is referred to as the color group of the coloring. In this contribution, a method to solve for the color groups of colorings of N-planar modules where $N = 4$ and $N = 6$ are given. Examples of colorings of rectangular lattices and of the vertices of the Ammann-Beenker tiling are given to exhibit how these methods may be extended to the general case.

1. Introduction

Colorings and color symmetries of crystals and quasicrystals can be considered a classical topic in crystallography (see [1, 2] and references therein). Most of the literature on colorings of lattices or $\mathbb{Z}$-modules put emphasis on Bravais colorings (*cf.* [3, 4, 5, 6]). Nevertheless, studies have been made on general colorings of lattices or $\mathbb{Z}$-modules such as in [7, 8, 9]. In this contribution, we give some results on how to identify the color groups of sublattice colorings of lattices or $\mathbb{Z}$-modules, and give examples to illustrate them. In particular, we give nonperfect non-Bravais colorings of the rectangular lattice and the vertices of the Ammann-Beenker tiling.

By a $\mathbb{Z}$-*module $\mathcal{M}$ of rank r and dimension d*, we mean a subset of $\mathbb{R}^d$ that is spanned by r vectors in $\mathbb{R}^d$ that are linearly independent over $\mathbb{Z}$ but span $\mathbb{R}^d$. As a group, $\mathcal{M}$ is isomorphic to the free abelian group of rank r. A $\mathbb{Z}$-module Γ of rank $r = d$ is called a *lattice*. At the outset, we shall talk primarily of lattices in dimension d, but results do hold for $\mathbb{Z}$-modules.

Let Γ_1 be a lattice in $\mathbb{R}^d$. A *coloring* of Γ_1 by m colors is an onto mapping $c : \Gamma_1 \to C$, where C is the set of m colors used in the coloring. Of particular interest are colorings of Γ_1 by m colors wherein two points of Γ_1 are assigned the same color if and only if they belong to the same coset of a sublattice Γ_2 of index m in Γ_1. Such a coloring shall be referred to as a *coloring of Γ_1 determined by its sublattice Γ_2*. Here, the set of colors C can be identified with the quotient group Γ_1/Γ_2 so that the color mapping c is simply the canonical projection of Γ_1 onto Γ_1/Γ_2 with kernel Γ_2. Hence, we may take $C = \{c_0 = \mathbf{0}, c_1, \ldots, c_{m-1}\}$ to be a complete set of coset representatives of Γ_2 in Γ_1, and say that the coset $c_j + \Gamma_2$ has color c_j.

Denote by G the symmetry group of Γ_1 and fix a sublattice coloring c of Γ_1. The *point group* of G, $P(G)$, consists of all symmetries in G that fix the origin. A symmetry in G is called a *color symmetry* of the coloring if all and only those points having the same color are mapped by the symmetry to a fixed color. The set H of all color symmetries of the coloring, that is, $H = \{h \in G : \exists \sigma_h \in S_C \text{ such that } \forall \ell \in \Gamma_1, c(h(\ell)) = \sigma_h(c(\ell))\}$, where S_C is the group of permutations on the set of colors C, forms a group called the *color group* or *color symmetry group* of the coloring. The mapping $P : H \to S_C$ with $h \mapsto \sigma_h$ defines a group homomorphism, and thus the group H acts on C. A coloring where $G = H$ is referred to as a *perfect coloring*.

The color groups of sublattice colorings of the square and hexagonal lattices were computed in [9]. They were obtained by representing sublattices of the square and hexagonal lattices as upper triangular matrices and expressing elements of the point groups of the lattices as matrices relative to the standard basis of $\mathbb{R}^2$. On the other hand, colorings of the square and hexagonal lattices determined by their square and hexagonal sublattices, respectively, (also referred to as *Bravais colorings*) were considered in [6]. In particular, the color groups were computed using number-theoretic properties of the ring of Gaussian and Eisenstein integers.

2. Some general results
Consider a coloring c of the lattice Γ_1 in $\mathbb{R}^d$ determined by a sublattice Γ_2 of index m in Γ_1. The next theorem, which belongs to the folklore and was proven for specific cases (*cf.* [9, Theorems 1 and 2] and [6, Theorem 2]), gives us a method to compute for the color group H of c.

Theorem 1. *Let G be the symmetry group of Γ_1, where $P(G)$ and $T(G)$ denote the point group and translation subgroup of G, respectively. Then $T(G)$ is a subgroup of the color group H. In addition, an element $g \in P(G)$ is a color symmetry of c if and only if g leaves Γ_2 invariant, that is, $g\Gamma_2 = \Gamma_2$.*

Proof. Let $g \in T(G)$ be the translation by $\ell \in \Gamma_1$. For any $c_i + \Gamma_2 \in \Gamma_1/\Gamma_2$, $g(c_i + \Gamma_2) = (\ell + c_i) + \Gamma_2 = c_j + \Gamma_2$ for some $0 \le j \le m-1$. Hence, g sends color c_i to color c_j, and $g \in H$.
  Suppose $g \in P(G)$ is a color symmetry of c. Then g sends color c_0 to some color c_i, $0 \le i \le m-1$, that is, $g\Gamma_2 = c_i + \Gamma_2$. Since $\mathbf{0} \in \Gamma_2$ and $g(\mathbf{0}) = \mathbf{0}$, then $g\Gamma_2 = \Gamma_2$. In the other direction, suppose $g\Gamma_2 = \Gamma_2$. Then for every coset $c_i + \Gamma_2$, $g(c_i + \Gamma_2) = gc_i + g\Gamma_2 = c_j + \Gamma_2$ for some $0 \le j \le m-1$, since $g \in P(G)$. Thus, $g \in H$. □

It follows from above that every color symmetry of c fixes color c_0. Note however that Theorem 1 does not imply that the color group of a sublattice coloring and the symmetry group of the sublattice are the same.
  The following corollary relates color groups of colorings determined by similar sublattices.

Corollary 2. *Let $\Gamma_3 = \alpha R\Gamma_2$ where $\alpha \in \mathbb{R}^+$ and $R \in O(d, \mathbb{R})$. Then $P(H') = R[P(H)]R^{-1}$ where H' is the color group of the coloring of Γ_1 determined by Γ_3.*

Proof. Given $g \in P(G)$, $g\Gamma_3 = \Gamma_3 \Leftrightarrow gR\Gamma_2 = R\Gamma_2 \Leftrightarrow R^{-1}gR\Gamma_2 = \Gamma_2$. The claim now follows from Theorem 1. □

3. Color groups of sublattice colorings of planar lattices
Here, we view planar lattices as discrete subsets of $\mathbb{C}$. In this setting, rotations about the origin by θ in the counterclockwise direction correspond to multiplication by the complex number $e^{i\theta}$. We denote such rotations by $R_{e^{i\theta}}$. Also, the reflection T_r along the real axis is associated to complex conjugation.
  Suppose Γ_2 is a sublattice of the planar lattice Γ_1. Then $R_{(-1)}\Gamma_2 = -\Gamma_2 = \Gamma_2$. By applying Theorem 1, we obtain the following result.

Proposition 3. *The $180°$-rotation about the origin is a color symmetry of a sublattice coloring of any planar lattice.*

3.1. Sublattice Colorings of the Square and Hexagonal Lattices
We identify the square lattice Γ_1 with the ring of Gaussian integers $\mathbb{Z}[i]$. The symmetry group G of Γ_1 is of type $p4m$ (or $*442$ in orbifold notation) and it is symmorphic, that is $G = P(G) \rtimes T(G)$ where $P(G) = \{R_{\pm 1}, R_{\pm i}\} \rtimes \langle T_r \rangle$ and $T(G)$ is the group of translations of G. Given linearly independent vectors $u, v \in \Gamma_1$, let $\Gamma_2 = \langle u, v \rangle_{\mathbb{Z}}$, that is, the sublattice of Γ_1 generated by u and v. Applying Theorem 1 and Proposition 3 in this setting, it suffices to check whether R_i, T_r, and $R_i T_r$ fix the sublattice Γ_2 in order to identify the color group of the coloring of $\mathbb{Z}[i]$ determined by Γ_2. One obtains the following results.

1. Γ_2 is invariant under R_i if and only if $\mathrm{Im}(u\bar{v})$ divides $N(u)$, $N(v)$, and $\mathrm{Re}(u\bar{v})$.

2. Γ_2 is invariant under T_r if and only if $\text{Im}(u\bar{v})$ divides $\text{Im}(u^2)$, $\text{Im}(v^2)$, and $\text{Im}(uv)$.

3. Γ_2 is invariant under $R_i T_r$ if and only if $\text{Im}(u\bar{v})$ divides $\text{Re}(u^2)$, $\text{Re}(v^2)$, and $\text{Re}(uv)$.

These observations yield the following theorem.

Theorem 4. *Let $\Gamma_1 = \mathbb{Z}[i]$ and $\Gamma_2 = \langle u, v \rangle_{\mathbb{Z}} \subseteq \Gamma_1$. Then the color group of the coloring of Γ_1 determined by Γ_2 is of type*

(a) *$p4m$ if $\text{Im}(u\bar{v})$ divides $N(u)$, $N(v)$, $\text{Re}(u\bar{v})$, u^2, v^2, and uv.*

(b) *$p4$ if $\text{Im}(u\bar{v})$ divides $N(u)$, $N(v)$, and $\text{Re}(u\bar{v})$, and does not divide any of the following: $\text{Im}(u^2)$, $\text{Im}(v^2)$, and $\text{Im}(uv)$.*

(c) *pmm if $\text{Im}(u\bar{v})$ divides $\text{Im}(u^2)$, $\text{Im}(v^2)$, and $\text{Im}(uv)$, and does not divide any of the following: $N(u)$, $N(v)$, and $\text{Re}(u\bar{v})$.*

(d) *cmm if $\text{Im}(u\bar{v})$ divides $\text{Re}(u^2)$, $\text{Re}(v^2)$, and $\text{Re}(uv)$, and does not divide any of the following: $N(u)$, $N(v)$, and $\text{Re}(u\bar{v})$.*

(e) *$p2$ if none of the above criteria is satisfied.*

If we write $\Gamma_2 = \langle a, b + ci \rangle_{\mathbb{Z}}$, where $a, b, c \in \mathbb{Z}$, then one readily obtains the results in [9] by applying Theorem 4. Because of Corollary 2, we may even assume that a, b, and c are coprime.

Analogous results also hold for colorings of the hexagonal lattice. We associate the hexagonal lattice to the ring of Eisenstein integers $\mathbb{Z}[\xi]$, where $\xi = e^{2\pi i/3}$. For the following result, given $u = a + b\xi \in \mathbb{Z}[\xi]$, we use the notation $\mathcal{R}e(u) := a$ and $\mathcal{I}m(u) := b$.

Theorem 5. *Let $\Gamma_1 = \mathbb{Z}[\xi]$ and $\Gamma_2 = \langle u, v \rangle_{\mathbb{Z}} \subseteq \Gamma_1$. Then the color group of the coloring of Γ_1 determined by Γ_2 is of type*

(a) *$p6m$ if $\mathcal{I}m(u\bar{v})$ divides $N(u)$, $N(v)$, $\mathcal{R}e(u\bar{v})$, u^2, v^2, and uv.*

(b) *$p6$ if $\mathcal{I}m(u\bar{v})$ divides $N(u)$, $N(v)$, and $\mathcal{R}e(u\bar{v})$, and does not divide any of the following: $\mathcal{I}m(u^2)$, $\mathcal{I}m(v^2)$, and $\mathcal{I}m(uv)$.*

(c) *cmm if $\mathcal{I}m(u\bar{v})$ divides $\mathcal{I}m(u^2)$, $\mathcal{I}m(v^2)$, and $\mathcal{I}m(uv)$, and does not divide any of the following: $N(u)$, $N(v)$, and $\mathcal{R}e(u\bar{v})$.*

(d) *cmm if $\mathcal{I}m(u\bar{v})$ divides $\mathcal{R}e(u^2)$, $\mathcal{R}e(v^2)$, and $\mathcal{R}e(uv)$, and does not divide any of the following: $N(u)$, $N(v)$, and $\mathcal{R}e(u\bar{v})$.*

(e) *cmm if $\mathcal{I}m(u\bar{v})$ divides $\mathcal{R}e(u^2) - \mathcal{I}m(u^2)$, $\mathcal{R}e(v^2) - \mathcal{I}m(v^2)$, and $\mathcal{R}e(uv) - \mathcal{I}m(uv)$, and does not divide any of the following: $N(u)$, $N(v)$, and $\mathcal{R}e(u\bar{v})$.*

(f) *$p2$ if none of the above criteria is satisfied.*

3.2. Example of colorings of a rectangular lattice

Let Γ_1 be the rectangular lattice given by $\Gamma_1 = \langle 1, 2i \rangle_{\mathbb{Z}}$. For Figure 1(a), $\Gamma_2 = \langle 3, 2i \rangle_{\mathbb{Z}}$. By using the above results to check for the invariance of Γ_2 under T_r, the color group of the coloring of Γ_1 determined by Γ_2 is of type pmm. For Figure 1(b), $\Gamma_3 = \langle 4, 1 + 2i \rangle_{\mathbb{Z}}$. Now, by applying the results above, the color group of the coloring of Γ_1 determined by Γ_3 is of type $p2$.

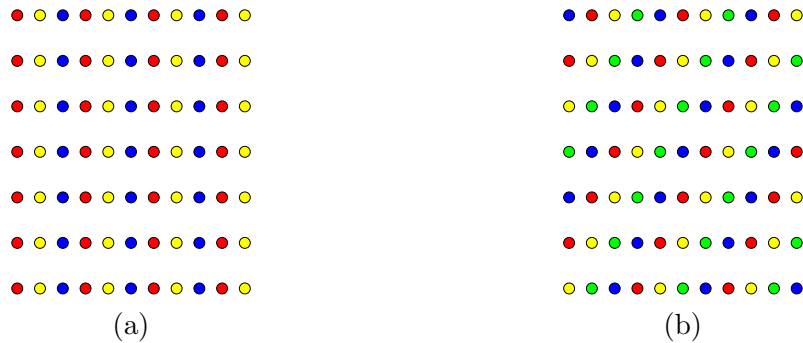(a)   (b)

Figure 1. Colorings of $\Gamma_1 = \langle 1, 2i \rangle_{\mathbb{Z}}$ determined by (a) $\Gamma_2 = \langle 3, 2i \rangle_{\mathbb{Z}}$ and (b) $\Gamma_3 = \langle 4, 1 + 2i \rangle_{\mathbb{Z}}$.

4. Nonperfect colorings of the Ammann-Beenker tiling

The techniques that we have developed in the previous sections may also be used to obtain submodule colorings of $\mathbb{Z}$-modules. We illustrate this by considering the planar $\mathbb{Z}$-module $\mathcal{M}_1 = \mathbb{Z}[\xi]$ of rank 4 with basis $\{1, \xi, \xi^2, \xi^3\}$ where $\xi = e^{\pi i/4}$. The symmetry group of $\mathbb{Z}[\xi]$ is $G = \langle a, b \rangle \cong D_8$, where a and b correspond to the 45°-rotation in the counterclockwise direction about the origin and reflection along the real axis, respectively. These submodule colorings induce colorings of the vertices of the eight-fold symmetric Ammann-Beenker tiling that are not necessarily perfect.

Given a submodule $\mathcal{M}_2$ of $\mathcal{M}_1$, we identify the color group of the coloring determined by $\mathcal{M}_2$ by checking the invariance of $\mathcal{M}_2$ under R_ξ, R_{ξ^2}, R_{ξ^3}, and T_r. Note that $\mathcal{M}_2$ is always fixed by the 180°-rotation about the origin.

As an illustration, to obtain a coloring of the Ammann-Beenker tiling with color group isomorphic to D_4, we choose the basis elements of $\mathcal{M}_2$ such that $\mathcal{M}_2$ is invariant under R_{ξ^2} and T_r but not under R_ξ. For example, if $\mathcal{M}_2 = \langle 1, 2\xi, \xi^2, 2\xi^3 \rangle$ then it can be verified that $\mathcal{M}_2$ satisfies the given conditions. Thus, the color group of the coloring induced by $\mathcal{M}_2$ is $\langle a^2, b \rangle \cong D_4$ (see Figure 2(a)). The submodule $\mathcal{M}_3 = \langle 1, 2\xi, 2\xi^2, \xi^3 \rangle$ is invariant only under the rotation by 180°. The color group of the coloring induced by $\mathcal{M}_3$ is $\langle a^4 \rangle \cong \mathbb{Z}_2$ (see Figure 2(b)).
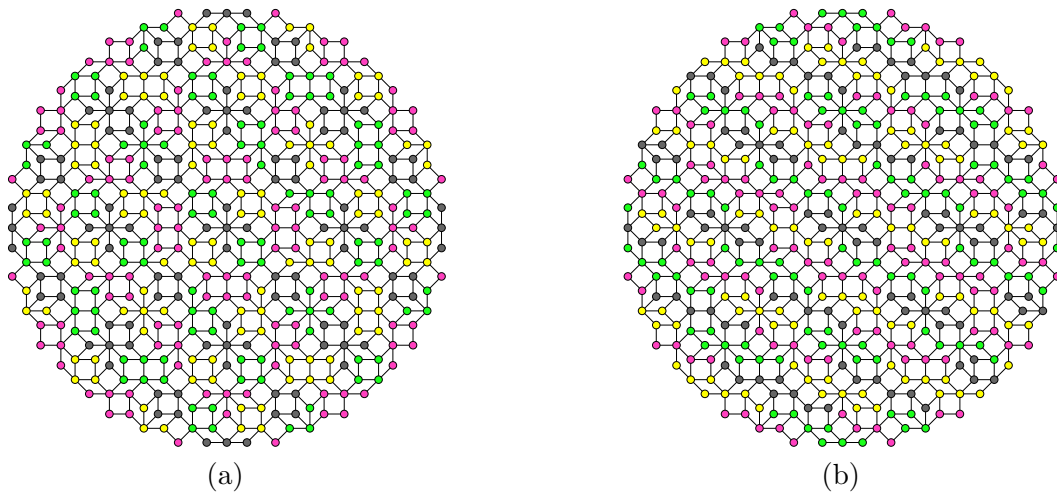

(a)   (b)

Figure 2. Colorings of the Ammann-Beenker tiling induced by (a) $\mathcal{M}_2 = \langle 1, 2\xi, \xi^2, 2\xi^3 \rangle$ and (b) $\mathcal{M}_3 = \langle 1, 2\xi, 2\xi^2, \xi^3 \rangle$

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