An ancient rule for constructing dodecagonal quasiperiodic patterns

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Abstract. The discovery of complex dodecagonal patterns in historical Islamic architecture is generating a renewed interest into understanding the mathematical principles of traditional Islamic geometry. By employing a compass and a straightedge, ancient craftsmen utilized consistent design principles that allowed for diverse geometric expressions to be realized throughout the ancient world. Derived from these principles, a global multi-level structural model is proposed that provides a general guiding principle for constructing a wide variety of infinite dodecagon-based quasiperiodic patterns.

1. Introduction

The discovery of complex patterns in ancient Islamic architecture with symmetries that are incompatible with periodicity is providing a unique window into understanding the traditional principles of ancient geometry. To date three types of such symmetries were documented. These include; 10-fold [1-4], 8-fold [5-6] and 12-fold [7]. While many examples of decagonal and octagonal patterns were identified, only one instance of a dodecagonal cartwheel has been found. In 2011 Makovicky and Makovicky recognized the first ancient dodecagonal pattern in the Zaouïa of Moulay Idriss II in Fez, which they analyzed based on a type of Ammann quasilattice [7]. It is worth to note that there is no evidence to suggest that ancient craftsmen were aware of the concept of infinite tilings with long-range quasiperiodic order. Today, much is unknown about the traditional creative process by which these complicated patterns were conceived or applied. The question of which method is the best suited to explain the rich diversity of such patterns in Islamic Architecture remains at the heart of an ongoing debate [8]. While considerable effort has been invested into analyzing these designs based on modern mathematical theories (tiling, matching, grid, substitution, etc.), only few attempts have been made to bring forward a comprehensive hypotheses for explaining the traditional design process based on the nature of the design tools themselves; the compass and the straightedge. Traditionally, the manipulation of circles, straight lines and their intersections provides the underlying structure for guiding the creative development of the different designs. Derived from these simple principles, this paper proposes a global multi-level hierarchical structural model that is able to explain the underlying logic of traditional patterns in ancient Islamic architecture, as well as to provide a general guiding method for constructing a wide variety of infinite dodecagonal quasiperiodic patterns.

2. An ancient dodecagonal quasiperiodic pattern

The only known example of a complex dodecagonal pattern in Islamic architecture was found in the tympanum of the entrance of the Zaouïa Moulay Idriss II in Fez [7]. The central cartwheel portion (figure

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1a) exhibits 12-fold rotational symmetry. Based on my examination of a wide range of complex patterns (5-fold, 10-fold and 8-fold) [4, 6, 9], a multi-level structural model is used to construct the dodecagonal quasiperiodic order. Governed by a tightly controlled proportional system, dodecagonal patterns can be generated based on a combination of two basic design elements; an underlying 'behind the scenes' structural network and the repeating 'seed units'. In this creative system, the network is responsible for defining the type of symmetry by providing the underlying structure for the spatial distribution of the 'seed units', while the internal formations of the 'seed units' derive the different design variations. Constructing the global structure of a dodecagonal quasiperiodic pattern requires building a progression of the structural network in multiple hierarchies, in which every hierarchy is built on the previous one.

To demonstrate this process, Figure 1 shows the construction process of the first hierarchy based on the dodecagonal pattern at the entrance of the Zaouïa Moulay Idriss II in Fez. Using a compass and a straight edge, a framework of nested dodecagrams is constructed (figure 1b). The relationship between each level in this network and the corresponding dodecagon, servers a critical role in maintaining a relational aspect ratio between all parts of the system. In this system, the size of the smallest dodecagon determines the size of the repeating 'seed polygons', allowing them to fit precisely in place without creating gaps or overlapping. Figure 1c shows the structural network after adapting the central portion to match the internal design of the ancient pattern. By mapping the central point of the 'seed polygons' to the intersection points of the structural network, a distribution map of the seed polygons is created (figure 1c). It is important to note that adjustments to some edge locations of the 'seed polygons' are made to match the ancient pattern (figure 1d). Figure 1e shows the process of constructing the internal design of the 'seed polygons' and the connecting formations. Such process follows a design logic that is similar to the process used for constructing the underlying structural network. As mentioned earlier, the internal design of the 'seed polygons' and its connecting units can assume different formations without affecting the dodecagonal symmetry. Such flexibility allows for a wide spectrum of designs to emerge by either manipulating the construction of the structural network or the internal design of the 'seed polygons'. The final formations of the first hierarchy of the dodecagonal pattern is shown in figures 1f and 1g. Figure 1h shows two alternative design options for the internal portion of the pattern. Figure 1i shows the dodecagonal pattern using the structural network in figures 1b and one of the alternative design options in figure 1h.

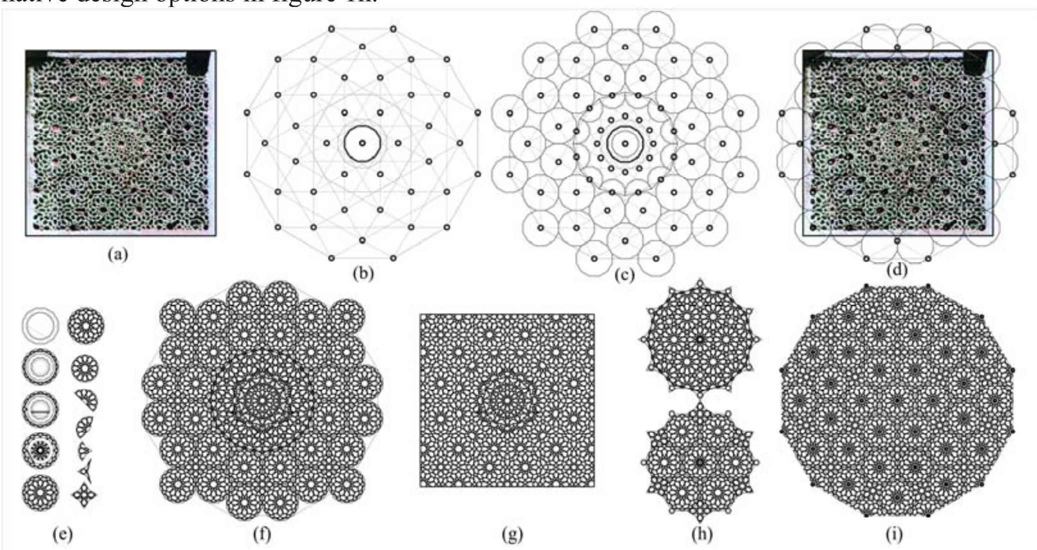

Figure 1. The construction process of the first hierarchy based on the dodecagonal pattern at the entrance of the Zaouïa Moulay Idriss II in Fez. The image used in (a) and (d) is taken from Makovicky and Makovicky [7] "Reproduced with permission of the International Union of Crystallography".

To construct the second hierarchy of the dodecagonal pattern, a new generation of the structural network is constructed (figure 2a). In this level, the final constructed pattern of the first hierarchy (figure 1i) acts as a 'seed unit' for the second hierarchy. The design of the connecting units are flexible and can take different formations including being part of the main 'seed unit' (figure 2b). The final dodecagonal pattern of the second hierarchy is shown in figure 2c. Constructing the infinite global empire of the dodecagonal quasiperiodic pattern requires building an infinite progression of hierarchies, in which, each new level is built on the previous one. This way a dodecagonal quasiperiodic pattern can grow ad infinitum without any gaps or imperfections.

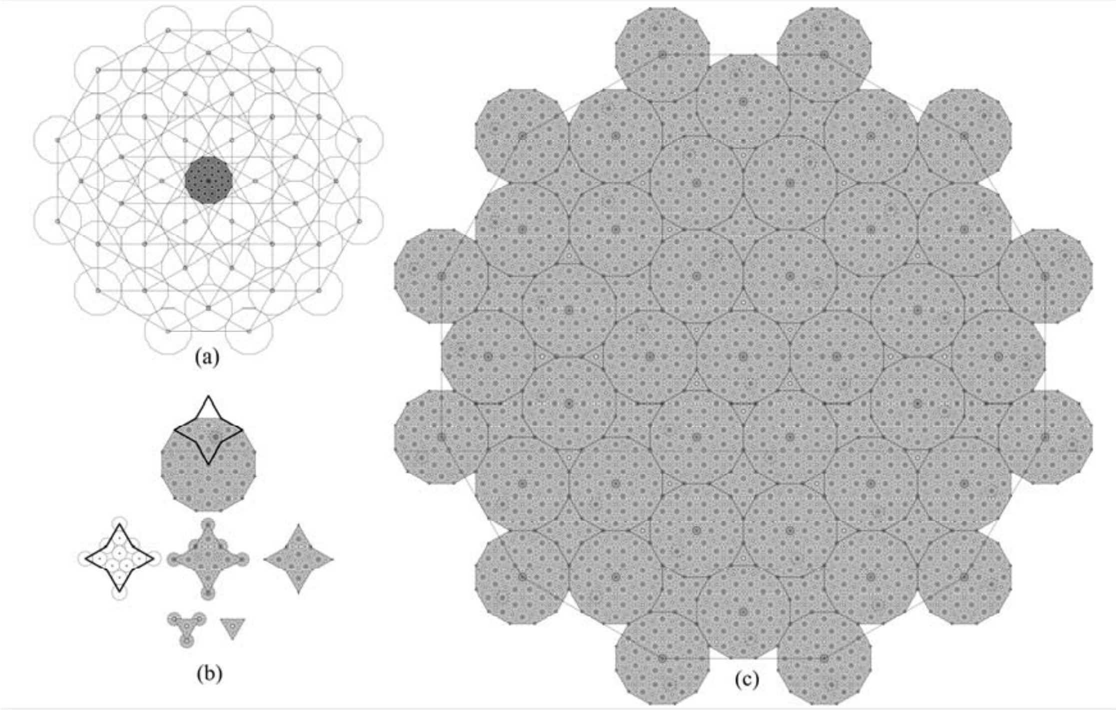

Figure 2. The construction process of the second hierarchy of the dodecagonal quasiperiodic pattern.

3. Constructing new dodecagonal quasiperiodic patterns

This section demonstrates the process of using the multi-level hierarchical method for constructing new variations of dodecagonal quasiperiodic patterns. The process of constructing the first hierarchy of the dodecagonal pattern starts by building a progression of nested dodecagrams. One alternative network design is shown in figure 3a. This network, which can be easily constructed by a using a compass and a straightedge, provides the underlying structural grid for mapping the locations of the 'seed polygons'. In this tightly proportional system, the size of the 'seed polygons' is defined by the generated central dodecagon of the network. This specific size ratio allows for the edges of the seed polygons to fit precisely in place without overlapping. By mapping the locations of the 'seed polygons' to specific intersection points of the structural network, a distribution map of the generalized 'seed polygons' is created (Figure 3b). In addition to the main 'seed polygons', connecting units are needed to fill-in the remaining gaps between the 'seed polygons'. In this specific case, three different connecting units are formed (figure 3c). It is worth noting that by connecting this specific formation of intersection points, a tessellation of three different tiles is created: a square, a thick rhomb and a thin rhomb (figure 3d). While the internal design of the 'seed polygons' can take different forms, using the same three-tile combinations for the internal design of the 'seed polygon' makes it possible to generate dodecagonal patterns that are self-similar (figure 3e). The final dodecagonal pattern of the first hierarchy is shown in figure 3f. Figure 3g shows some alternative design variations of the 'seed polygons'. By manipulating the internal design of the 'seed polygons' and its connecting formations, this flexible method allows for a wide spectrum of pattern variations while maintaining the symmetry of the underlying structural network (figure 3h).

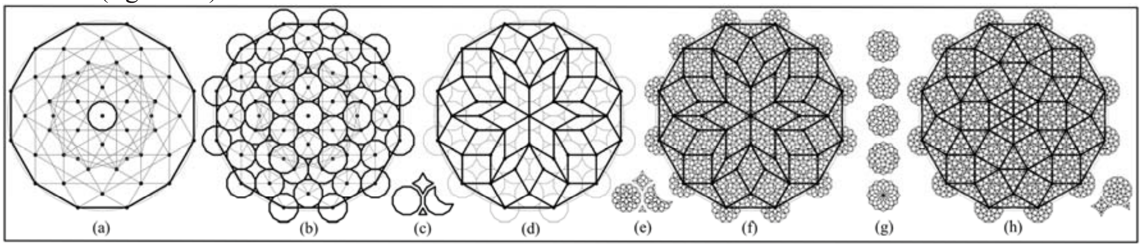

Figure 3. The construction process of the first hierarchy of a new dodecagonal quasiperiodic pattern.

To construct the second hierarchy of the dodecagonal pattern, a new generation of the underlying structural network is constructed, in which, the final constructed pattern of the first hierarchy (figure 3f) acts as a 'seed unit' for the second hierarchy. By mapping the locations of the second generation 'seed units' to the specific intersection points of the structural network, a distribution map of the generalized 'seed polygons' is created (figure 4a). In keeping with a consistent process, three connecting units are needed to fill-in the remaining gaps between the main seed units, all of which, are generated from the main 'seed unit' (figure 4b). The final dodecagonal pattern of the second hierarchy is shown in figure 4c; exhibiting the hierarchical self-similarity attributes.

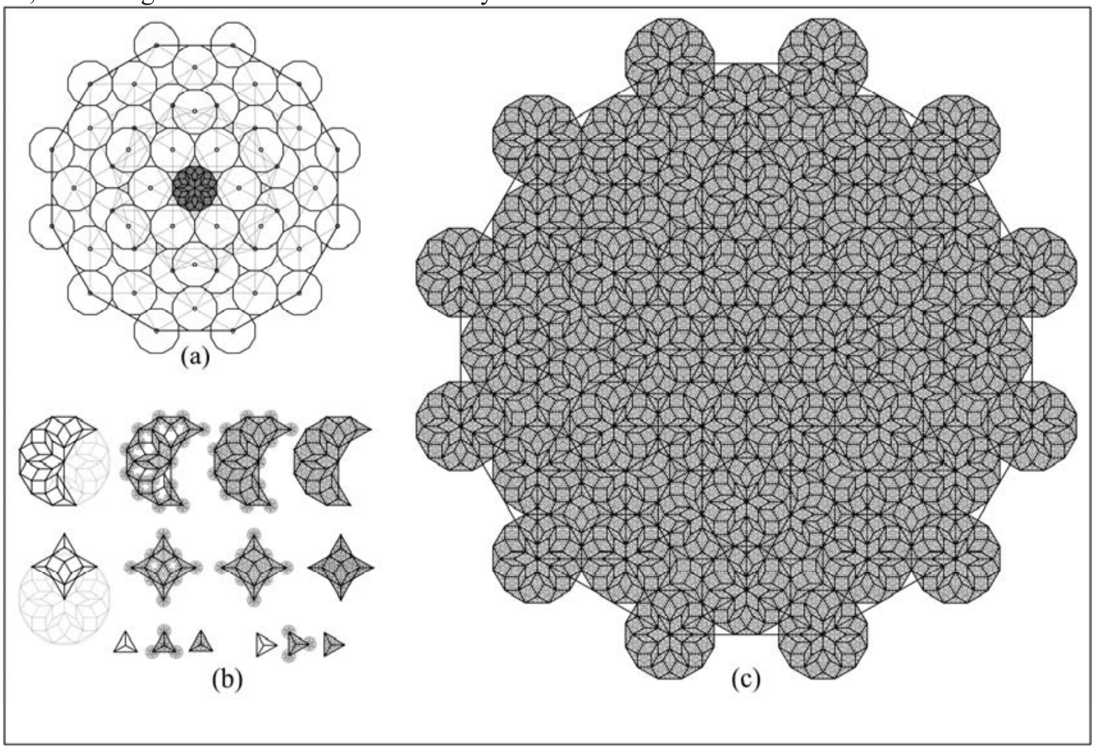

Figure 4. The construction process of the second hierarchy of the dodecagonal pattern.

It is worth noting that this hierarchical quality is a signature characteristic of the natural structure observed in dodecagonal quasicrystals [10]. One of the first dodecagonal quasicrystals ($Ta_{1.6}Te$) was synthesized in 1998 [11, 12]. Figure 5 demonstrates the use of the proposed hierarchical method to describe Conrad et al.'s [11] proposed model for the structure of dodecagonal quasicrystals of $Ta_{1.6}Te$. A framework of nested dodecagrams (figure 5a) forms the structural network for constructing the first hierarchy. The sequence of nested dodecagrams grows based on the square irrational factor $\sqrt{2+\sqrt{3}}$, in which, each larger dodecagram is rotated by 15° degrees. Figure 5c shows the distribution of the seed polygons by this network. The connecting formations, in this case, are derived from the overlapping of the seed polygons (figure 5b). Similarly, the connecting formations of the second hierarchy (figure 5d) are also derived from the main seed cartwheel pattern in Figure 5c. A strong geometrical correlation exists between the structural polygonal map of the second hierarchy and the electron diffractograms of $Ta_{1.6}Te$ quasicrystals [11] (figure 5f). Figure 5g shows that the structural polygonal map of the third hierarchy works in perfect concert with the results of an algorithm for the generation of a self-similar dodecagonal dot pattern produced by Conrad et al. [11].

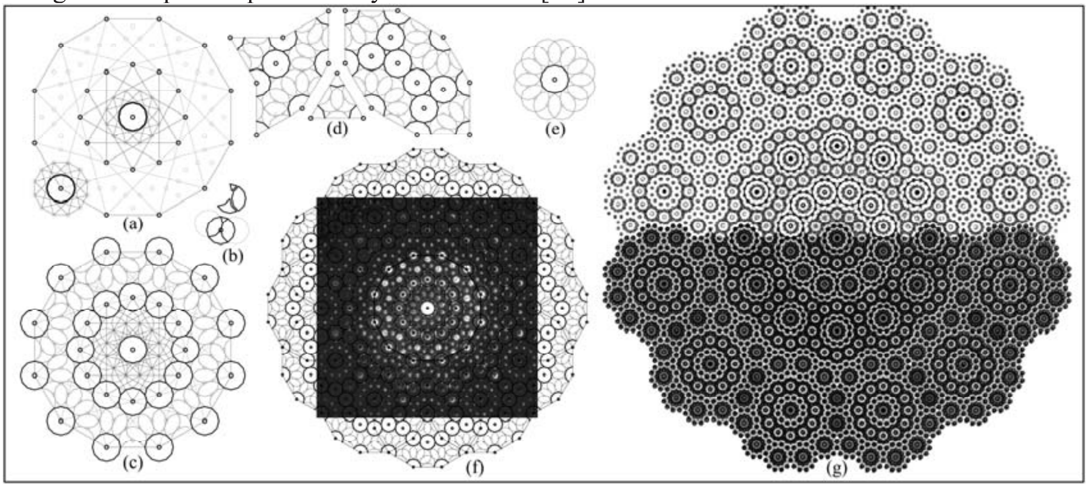

Figure 5. The use of the proposed hierarchical model to describe the structure of dodecagonal quasicrystals of $Ta_{1.6}Te$. The diffractograms in (f) and the algorithm result in (g) are taken from Conrad et al. [11], reproduced with permission from John Wiley and Sons.

4. Conclusion

Derived from the traditional principles of Islamic geometry, this paper presents a global multi-level structural model that is able to describe the underlying structure of dodecagonal patterns in Islamic Architecture. Moreover, this method can be used as a general guiding principle for constructing a wide variety of dodecagonal quasiperiodic patterns without the need for complicated mathematics; providing an easy tool for scientists, mathematicians, teachers, designers and artists, to generate and study a wide range of dodecagonal quasiperiodic patterns.

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