

# Substitution-based sequences with absolutely continuous diffraction

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**Abstract.** Modifying Rudin’s original construction of the Rudin–Shapiro sequence, we derive a new substitution-based sequence with purely absolutely continuous diffraction spectrum.

## 1. Introduction

Substitution dynamical systems are widely used as toy models for aperiodic order in one dimension [1]. The binary Rudin–Shapiro sequence [2, 3, 4] is a paradigm for a substitution-based structure with (in its balanced weight case) purely absolutely continuous diffraction spectrum; we refer to [5] for details and background. Indeed, this deterministic sequence has the stronger property that its two-point correlations vanish for *any* non-zero distance, as it would be the case for a random sequence [6]. While currently only very few examples of substitution-based sequences with this property are known, a systematic generalisation in terms of Hadamard matrices [7] allows the construction of one-dimensional as well as higher-dimensional examples.

We start by reviewing Rudin’s original construction [4], and modify it to obtain a new example of a substitution-based structure with absolutely continuous diffraction spectrum. Our approach differs from the construction in [7]. We then use recent work of Bartlett [5] to investigate the properties of the new system in more detail.

## 2. Original construction of the Rudin–Shapiro sequence

In 1958, Salem [4] asked the following question: is it possible to find a sequence  $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$  such that there is a constant  $C > 0$  for which

$$\sup_{\theta \in \mathbb{R}} \left| \sum_{n < N} \varepsilon_n e^{2\pi i n \theta} \right| \leq C\sqrt{N} \quad (1)$$

holds for any positive integer  $N$ ? This is known as the ‘root  $N$ ’ property, which implies absolutely continuous diffraction of the sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  by an application of [8, Prop. 4.9]. Shapiro [2] and Rudin [4], and indeed earlier Golay [3], gave positive answers to the question by constructing what is now known as the Rudin–Shapiro sequence. We briefly review Rudin’s construction to obtain the substitution dynamical system and its balanced weight version (with weights in  $\pm 1$ ).

We start by defining polynomials  $P_k(x)$  and  $Q_k(x)$  of degree  $2^k$  for  $k \in \mathbb{N}_0$  recursively by

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - x^{2^k} Q_k(x), \end{aligned} \quad (2)$$



with  $P_0(x) = Q_0(x) = x$ . Note that from Eq. (2) it is clear that the first  $2^k$  terms of  $P_{k+1}(x)$  and of  $Q_{k+1}(x)$  coincide with those of  $P_k(x)$ , and the remaining terms differ by a sign. By construction,  $P_k(x)$  is of the form

$$P_k(x) = \sum_{n=1}^{2^k} \varepsilon_n x^n, \quad (3)$$

so we can define a binary sequence  $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$  from the corresponding coefficients. This is the binary Rudin–Shapiro sequence. For example,  $P_3(x) = x + x^2 + x^3 - x^4 + x^4(x + x^2 - x^3 + x^4)$ , from which we read off the sequence 111 $\bar{1}$ 11 $\bar{1}$ 1 with  $\bar{1} = -1$ . The main ingredient in the proof of property (1) for this sequence is the parallelogram law for  $\alpha, \beta \in \mathbb{C}$ ,

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2,$$

applied to  $P_k(e^{2\pi i \theta})$ ; see [4] for details.

Often, the Rudin–Shapiro sequence is defined by a four-letter substitution rule and a subsequent reduction map from four to two letters. The underlying four-letter substitution on the alphabet  $\{A, B, C, D\}$  can be read off from the recursion (2), noting that the recursion implies the concatenation of the sequences corresponding to  $P_k$  and  $Q_k$ . Associating letters  $A$  and  $B$  to the coefficients in  $P$  and  $Q$  and the letters  $C$  and  $D$  to those of  $-Q$  and  $-P$ , respectively, this gives rise to the four-letter substitution rule

$$\varrho: \quad A \mapsto AB, \quad B \mapsto AC, \quad C \mapsto DB, \quad D \mapsto DC,$$

which corresponds to the standard four-letter Rudin–Shapiro substitution. Iterating the sequence on the initial seed  $A$  gives

$$A \mapsto AB \mapsto ABAC \mapsto ABACABDB \mapsto \dots$$

which, under the mapping  $A \mapsto 1, B \mapsto 1, C \mapsto -1, D \mapsto -1$  (which corresponds to the choice of signs above) produces the binary Rudin–Shapiro sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$ .

### 3. Generalising Rudin’s construction

Let us now modify Rudin’s original argument by considering the following system

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + (-1)^k x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - (-1)^k x^{2^k} Q_k(x), \end{aligned} \quad (4)$$

starting again from  $P_0(x) = Q_0(x) = x$ . Since from now on we shall only look at this set of recursion relations, we use the same notation as above. What has changed in comparison with Eq. (2) is that we swap the sign in the recursion relation in each step. Since the sign does not affect the argument used in Rudin’s proof [4], it is straightforward to show that the parallelogram law can again be used to prove that the corresponding *new* sequence of coefficients  $(\varepsilon_n)_{n \in \mathbb{N}}$ , defined as above via Eq. (3), also satisfies the root  $N$  property.

Can we read off the corresponding substitution rule as for the Rudin–Shapiro case? Indeed this is possible, but it is slightly more complicated, because the recursion relations in Eq. (4) alternate. One way to deal with that is to look at two consecutive steps together,

$$\begin{aligned} P_{k+2}(x) &= P_k(x) + (-1)^k x^{2^k} Q_k(x) + (-1)^{k+1} x^{2 \cdot 2^k} P_k(x) + x^{3 \cdot 2^k} Q_k(x), \\ Q_{k+2}(x) &= P_k(x) + (-1)^k x^{2^k} Q_k(x) - (-1)^{k+1} x^{2 \cdot 2^k} P_k(x) - x^{3 \cdot 2^k} Q_k(x). \end{aligned} \quad (5)$$

Choosing  $k$  to be even, say, and associating again four letters  $A, B, C, D$  to the sequences corresponding to  $P, Q, -Q, -P$ , we can read off the substitution

$$\sigma: \quad A \mapsto ABDB, \quad B \mapsto ABAC, \quad C \mapsto DCDB, \quad D \mapsto DCAC, \quad (6)$$

which is now a constant length substitution of length four, because we used a double step of the recursion. Therefore Eq. (5) corresponds to concatenation of four sets of coefficients. As before, the binary sequence is obtained from iterating the substitution on the initial letter  $A$ ;

$$A \mapsto ABDB \mapsto ABDBABACDCACABAC \mapsto \dots. \quad (7)$$

By mapping  $A$  and  $B$  to 1 and mapping  $C$  and  $D$  to  $\bar{1} = -1$ , we obtain  $11\bar{1}1111\bar{1}\bar{1}\bar{1}111\bar{1}\dots$  as the initial part of our new binary sequence.

Clearly, we can generate infinitely many such examples by changing the signs in the original recursion relation (2) in more complicated ways. Any finite sequence of signs, when used periodically, will give rise to a substitution-based system. However, the length of the substitution will increase with the length of the sign sequence. There neither appears to be an obvious relation between the different Rudin–Shapiro type sequences obtained from this construction, nor between these sequences and those derived from Frank’s construction [7]. In the latter case, the number of letters in the alphabet increases with the size of the Hadamard matrix, whereas all our sequences will only use four letters. Having said that, it would be possible to use more letters rather than consider multiple recursion steps, so the relation between these two approaches still needs to be analysed in more detail.

#### 4. Spectral properties

Since we have Rudin’s original argument at our disposal, we know that this new binary sequence satisfies the root  $N$  property (1), and hence has (in the balanced weight case with weights  $\pm 1$ ) purely absolutely continuous diffraction. Nevertheless, we would like to use the remainder of this paper to generalise this result by applying Bartlett’s algorithm [9] to our new (four-letter substitution) sequence. This allows us to verify that the four-letter substitution sequence (in the balanced weight case) also has absolutely continuous diffraction spectrum only, whereas Rudin’s argument only covers the binary case. Due to space constraints, we cannot introduce all quantities here, and instead refer to [9] for definitions and further details.

**Theorem 1.** *The balanced weight sequence derived from the substitution rule  $\sigma$  of Eq. (6) has purely absolutely continuous diffraction spectrum.*

*Proof.* The four instruction matrices  $R_i$  and the substitution matrix  $M$  can be read off of the substitution rule (6). They are given by

$$R_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with  $M = R_0 + R_1 + R_2 + R_3$ . As  $M^2$  has positive entries only, the substitution is primitive. The second iterate of the seed  $A$  obtained in Eq. (7) shows that the letter  $A$  can be preceded by either  $B$  or  $C$ . Hence  $A$  has two distinct neighbourhoods and, by Pansiot’s lemma [9, Lem. 3.6], the substitution is aperiodic.

Since we are dealing with a constant-length substitution, the leading (Perron–Frobenius) eigenvalue [5, Thm. 2.2] of  $M$  is  $\lambda_{\text{PF}} = 4$  with (statistically normalised) eigenvector and  $u = \frac{1}{4}(1, 1, 1, 1)$ . By applying [9, Thm. 4.3], we obtain  $\hat{\Sigma}(0) = \frac{1}{4} \sum_{\alpha} e_{\alpha\alpha}$ , where  $e_{\alpha\alpha}$  denotes

the unit vector corresponding to the word  $\alpha\alpha$ . As  $\sigma$  is a length four substitution, we have  $\Delta_1(1) = \{3\}$ . Using [9, Thm. 4.3], we obtain

$$\begin{aligned}\widehat{\Sigma}(1) &= \frac{1}{8} (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0) = \widehat{\Sigma}(3), \\ \widehat{\Sigma}(2) &= \frac{1}{8} (1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1) = \widehat{\Sigma}(4).\end{aligned}$$

One can then verify the following recursive relations

$$\widehat{\Sigma}(4n) = \widehat{\Sigma}(4), \quad \widehat{\Sigma}(4n+1) = \widehat{\Sigma}(1), \quad \widehat{\Sigma}(4n+2) = \widehat{\Sigma}(2), \quad \widehat{\Sigma}(4n+3) = \widehat{\Sigma}(3). \quad (8)$$

Using [9, Prop. 2.2], we calculate the ergodic decomposition of the bi-substitution  $\sigma \otimes \sigma$ . We obtain  $E_1 = \{AA, BB, CC, DD\}$  and  $E_2 = \{AD, DA, BC, CB\}$  as the two ergodic classes and  $T = \{AB, AC, BA, BD, CA, CD, DB, DC\}$  as the transient part. From [9, Lem. 4.7], we get

$$v = w_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + w_2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{w_1 + w_2}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Diagonalising the matrix  $v$ , we obtain

$$v_d = \begin{pmatrix} 2(w_1 + w_2) & 0 & 0 & 0 \\ 0 & w_1 - w_2 & 0 & 0 \\ 0 & 0 & w_1 - w_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Setting  $w_1 = 1$ , strong semi-positivity is equivalent to  $w_2$  satisfying  $-1 \leq w_2 \leq 1$ . The extreme points are then given by  $(w_1, w_2) = (1, 1)$  and  $(w_1, w_2) = (1, -1)$ . Thus the extreme rays are

$$\begin{aligned}v_1 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\ v_2 &= (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1).\end{aligned}$$

As usual,  $\lambda_{v_1} = \delta_0$ , which gives rise to the pure point component of the spectrum. Using Eq. (8), one checks that  $\widehat{\lambda}_{v_2}(k) = 0$  for all  $k \neq 0$ , which gives us the absolutely continuous component. Thus, we have a purely absolute continuous spectrum in the balanced weight case, in which the pure point component is extinguished.  $\square$

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