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Symmetry of String Scattering Amplitudes

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Abstract. We explore the hidden symmetry in string theory by studying string scattering amplitudes. We calculate 4-point open string scattering amplitudes with three tachyons and a massive higher spin string state. The result can be expressed as Type D Lauricella functions which are generalization of Gaussian Hypergeometric functions. In various high energy limits, the string amplitudes reduced to the expected results that we obtained previously. We find exact $SL(K+3,\mathbb{C})$ symmetry for the string amplitudes at general energy.

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1. Introduction

Quantum Field Theory (QFT) is a powerful theory in modern physics. Based on QFT, standard model of particle physics successfully describes our microcosmic world. All important predictions by standard model have been observed in various experiments under rather precise level. However, to solve the UV divergence problem in QFT, the key technical procedure, i.e. renormalization, is complicated and has not been fully understood. More seriously, the renormalization procedure does not work for gravity, which means that it is impossible to construct a consistent quantum gravity theory by using the conventional QFT. Most of people believe that the divergence in QFT comes from the fundamental topological structure of point-like particles, and it cannot be cured without modifying its topological structure. In string theory, one extends a point-like particle to a small piece of a string. This simple extension dramatically changes the topological structure of the theory. The new "Feynman diagram" now is a smooth world-sheet instead of a world-line with singularity at interacting points.

To understand the UV divergence problem better, let us briefly look at the high energy behavior in QFT by a simple power counting. In high energy hard limit, the tree amplitude by interchanging a spin-J particle behaves as $A^{(J)}_{tree} \sim E^{-2(1-J)}$, so that the one-loop amplitude behaves as

$$A^{(J)}_{1-loop} \sim \int d^4p \frac{\left(A^{(J)}_{tree}\right)^2}{\left(p^2\right)^2} \sim \int E^{-4(2-J)}\ d^4E, \tag{1}$$

which is finite for scalar particles ($J=0$) and renormalizable for vector particles ($J=1$), but is nonrenormalizable for particles with $J \geq 2$, including graviton ($J=2$). However, there is a loophole to bypass this simple argument. If we sum over all tree amplitudes by interchanging states with different spins, the final amplitude will be

$$A_{tree} = \sum_J A^{(J)}_{tree} \sim \sum_J a_J E^{-2(1-J)}, \tag{2}$$

which could behave rather soft, so that loop amplitudes would be finite, if the following two conditions are satisfied simultaneously:

(i) there are infinite higher spin J particles
(ii) the coefficients a_J's are precisely related to each other.

In string theory, string scattering amplitudes are exponentially fall-off in high energy hard limit, which leads to string theory being a finite theory without UV divergence. We believe that the reason why the high energy behavior of string theory is so soft is that string theory satisfies the above two conditions.

The first condition is trivially satisfied in string theory because a string has infinite oscillation modes which correspond to infinite higher spin states. The second condition is highly nontrivial. We conjecture that it corresponds to a huge symmetry in string theory, which is complicated and not apparent so that we usually call it hidden symmetries. A useful way to investigate the hidden symmetry is to study the symmetry among the string scattering amplitudes. Gross has conjectured that the string scattering amplitudes are linearly related each other in the high energy, fixed scattering angle limit [1, 2, 3]. Using the methods of Ward identities of zero norm states [4, 5, 6], Virasoro algebra and direct calculation of scattering amplitudes, we are able to prove the Gross conjecture and compute the linear ratios among the different string amplitudes [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. We also extend our study to the high energy, small angle limit, i.e. Regge scattering [18, 19, 20, 21, 22]. Recently, we calculated the four-point string amplitudes at arbitrary energy and found the amplitudes associate to $SL(K+3,\mathbb{C})$ algebra [23, 24]. In the next section, we will calculated the four-point string amplitudes, and study their relations in various high energy limits in section 3. In section 4, we show that how to get the $SL(K+3,\mathbb{C})$ algebra from string amplitudes. We conclude our result in section 5.

2. Four-Point String Amplitudes

To study the symmetry of string scattering amplitudes, we consider four-point open bosonic string scattering In the CM frame, the kinematics are

$$k_1 = \left(\sqrt{M_1^2 + |\vec{k_1}|^2}, -|\vec{k_1}|, 0\right), \tag{3}$$

$$k_2 = \left(\sqrt{M_2^2 + |\vec{k_1}|^2}, +|\vec{k_1}|, 0\right), \tag{4}$$

$$k_3 = \left(-\sqrt{M_3^2 + |\vec{k_3}|^2}, -|\vec{k_3}|\cos\phi, -|\vec{k_3}|\sin\phi\right), \tag{5}$$

$$k_4 = \left(-\sqrt{M_4^2 + |\vec{k_3}|^2}, +|\vec{k_3}|\cos\phi, +|\vec{k_3}|\sin\phi\right), \tag{6}$$

where ϕ is the scattering angle. The Mandelstam variables are defined as usual as

$$s = -\left(k_1 + k_2\right)^2, t = -\left(k_2 + k_3\right)^2, u = -\left(k_1 + k_3\right)^2, \tag{7}$$

with $s + t + u = \sum M_i^2$.

On the 2-dimensional scattering plane, there are three independent polarizations which we can choose to be

$$e^T = (0, 0, 1), \tag{8}$$

$$e^L = \frac{1}{M_2}\left(|\vec{k_1}|, \sqrt{M_2 + |\vec{k_1}|^2}, 0\right), \tag{9}$$

$$e^P = \frac{1}{M_2}\left(\sqrt{M_2 + |\vec{k_1}|^2}, |\vec{k_1}|, 0\right). \tag{10}$$

Note that the string amplitude with polarizations orthogonal to the scattering plane vanish. For later use, we also define

$$k_i^X \equiv e^X \cdot k_i \quad \text{for} \quad X = (T, P, L) . \tag{11}$$

The simplest four-point string amplitude is scattered by four tachyons with $M_i^2 = -2$, i.e. the Veneziano amplitudes. In (s,t) channel, the four-tachyon scattering amplitude can be easily calculated,

$$A_{st}^{(\text{4-tachyon})} = \left\langle e^{ik_1\cdot X(x_1)} e^{ik_2\cdot X(x_2)} e^{ik_3\cdot X(x_3)} e^{ik_4\cdot X(x_4)} \right\rangle = B\left(-\frac{s}{2}-1, -\frac{t}{2}-1\right), \tag{12}$$

where

$$B\left(-\frac{s}{2}-1, -\frac{t}{2}-1\right) = \frac{\Gamma\left(-\frac{s}{2}-1\right)\Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+2\right)}, \tag{13}$$

is Beta function. It is easy to verify that the Veneziano amplitude behaves as exponentially fall-off $A_{st}^{(\text{4-tachyon})} \sim e^{-E}$ in the high energy limit ($s \sim E^2 \to \infty$, $t \to \infty$ and s/t fixed). This property holds for all four-point scattering amplitudes in string theory as we will show later.

Now let us consider a general 4-point string amplitudes with three tachyons and an arbitrary massive higher spin string state of the form,

$$\left|r_n^T, r_m^P, r_l^L\right\rangle = \prod_{n>0} \left(\alpha_{-n}^T\right)^{r_n^T} \prod_{m>0} \left(\alpha_{-m}^P\right)^{r_m^P} \prod_{l>0} \left(\alpha_{-l}^L\right)^{r_l^L} |0,k\rangle, \tag{14}$$

where $M_1^2 = M_3^2 = M_4^2 = -2$ are three tachyons and $M_2^2 \equiv M^2 = 2(N-1)$ is the higher spin string state with the mass level $N = \sum_{n,m,l>0}\left(nr_n^T + mr_m^P + lr_l^L\right)$.

The (s,t) and (t,u) channels string scattering amplitudes of states in Eq.(14) can be calculated to be [23, 24]

$$\begin{aligned} A_{st}^{(r_n^T, r_m^P, r_l^L)} &= B\left(-\frac{t}{2}-1, -\frac{s}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{u}{2}+2-N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right) \\ &\cdot \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}, \end{aligned} \tag{15}$$

$$\begin{aligned} A_{tu}^{(r_n^T, r_m^P, r_l^L)} &= B\left(-\frac{t}{2}-1, -\frac{u}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{s}{2}+2-N; Z_n^T, Z_m^P, Z_l^L\right) \\ &\cdot \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}, \end{aligned} \tag{16}$$

where we have defined

$$R_k^X \equiv \left\{-r_1^X\right\}^1, \cdots, \left\{-r_k^X\right\}^k \text{ with } \{a\}^n = \underbrace{a, a, \cdots, a}_{n}, \tag{17}$$

$$Z_k^X \equiv \left[z_1^X\right], \cdots, \left[z_k^X\right] \text{ with } \left[z_k^X\right] = z_{k0}^X, \cdots, z_{k(k-1)}^X, \tag{18}$$

$$z_{kk'}^X = \left|\frac{k_1^X}{k_3^X}\right|^{\frac{1}{k}} e^{\frac{2\pi i k'}{k}} \text{ and } \tilde{z}_{kk'}^X \equiv 1 - z_{kk'}^X,\ k' = 0, \cdots, k-1, \tag{19}$$

and the integer K is defined to be

$$K = \sum_{\substack{j=1 \\ \{\text{for all } r_j^T \neq 0\}}}^{n} j + \sum_{\substack{j=1 \\ \{\text{for all } r_j^P \neq 0\}}}^{m} j + \sum_{\substack{j=1 \\ \{\text{for all } r_j^L \neq 0\}}}^{l} j, \tag{20}$$

which is usually different from the mass level N.

Finally, the D-type Lauricella function $F_D^{(K)}$ is one of the four extensions of the Gauss hypergeometric function to K variables and is defined as

$$\begin{aligned} & F_D^{(K)}\left(a; b_1, ..., b_K; c; x_1, ..., x_K\right) \\ & = \sum_{k_i} \frac{(a)_{k_1+\cdots+k_K}}{(c)_{k_1+\cdots+k_K}} \frac{(b_1)_{k_1} \cdots (b_n)_{k_K}}{k_1! \cdots k_K!} x_1^{k_1} \cdots x_K^{k_K} \\ & = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt\, t^{a-1}(1-t)^{c-a-1} \cdot (1-x_1 t)^{-b_1}(1-x_2 t)^{-b_2} ... (1-x_K t)^{-b_K}, \end{aligned} \tag{21}$$

where the integral representation of the Lauricella function $F_D^{(K)}$ in the last line was discovered by Appell and Kampe de Feriet (1926) [25].

By using the identity of Lauricella function for $b_i \in Z^-$

$$\begin{aligned} & F_D^{(K)}\left(a; b_1, ..., b_K; c; x_1, ..., x_K\right) \\ & = \frac{\Gamma(c)\Gamma\left(c-a-\sum b_i\right)}{\Gamma(c-a)\Gamma\left(c-\sum b_i\right)} F_D^{(K)}\left(a; b_1, ..., b_K; 1+a+\sum b_i - c; 1-x_1, ..., 1-x_K\right), \end{aligned} \tag{22}$$

we can express the (s,t) channel amplitude (15) in the following form

$$\begin{aligned} A_{st}^{(r_n^T, r_m^P, r_l^L)} = & B\left(-\frac{t}{2}-1, -\frac{s}{2}-1+N\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{s}{2}+2-N; Z_n^T, Z_m^P, Z_l^L\right) \\ & \cdot \prod_{n=1} \left[-(n-1)! k_3^T\right]^{r_n^T} \cdot \prod_{m=1} \left[-(m-1)! k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)! k_3^L\right]^{r_l^L}. \end{aligned} \tag{23}$$

Now it is easy to see the string BCJ relation

$$\frac{A_{st}^{(r_n^T, r_m^P, r_l^L)}}{A_{tu}^{(r_n^T, r_m^P, r_l^L)}} = \frac{(-)^N \Gamma\left(-\frac{s}{2}-1\right)\Gamma\left(\frac{s}{2}+2\right)}{\Gamma\left(\frac{u}{2}+2-N\right)\Gamma\left(-\frac{u}{2}-1+N\right)} = \frac{\sin\left(\frac{\pi u}{2}\right)}{\sin\left(\frac{\pi s}{2}\right)} = \frac{\sin\left(\pi k_2 \cdot k_4\right)}{\sin\left(\pi k_1 \cdot k_2\right)}, \tag{24}$$

which was proved by monodromy of integration of string amplitudes [26, 27] and explicitly proved recently in [28].

3. Symmetry in High Energy Limits

To study the relations among the string scattering amplitudes, we consider two different high energy limits: hard scattering limit and Regge scattering limit. We will briefly describe the results in the following. Readers can find the detail in a current review paper [29].

3.1. Linear Relations in Hard Limit

Hard scattering limit is the fixed angle scattering with $s \sim E^2 \to \infty$ and $\frac{t}{s} \sim \sin^2 \frac{\phi}{2} = $ constant. The linear relations of string amplitudes in the hard scattering limit were conjectured by Gross [1, 2, 3] and proved in [7, 8, 9, 10, 11, 13]. In the hard scattering limit $e^P = e^L$ [7, 8], we can consider only the polarization e^L case. The relevant kinematics are

$$k_1^T = 0, \quad k_3^T \simeq -E \sin\phi, \tag{25}$$

$$k_1^L \simeq -\frac{2p^2}{M_2} \simeq -\frac{2E^2}{M_2}, \tag{26}$$

$$k_3^L \simeq \frac{2E^2}{M_2} \sin^2 \frac{\phi}{2}, \tag{27}$$

with $\tilde{z}^T_{kk'}=1,\ \tilde{z}^L_{kk'}=1-\left(-\frac{s}{t}\right)^{1/k}e^{\frac{i2\pi k'}{k}}\sim O\left(1\right)$.

In the hard limit, the (s,t) channel string amplitude in Eq.(15) reduces to

$$A^{(r^T_n,r^L_l)}_{st}=B\left(-\frac{t}{2}-1,-\frac{s}{2}-1\right)F^{(K)}_D\left(-\frac{t}{2}-1;R^T_n,R^L_l;\frac{u}{2}+2-N;(1)_n,\tilde{Z}^L_l\right)$$
$$\cdot\prod_{n=1}\left[(n-1)!E\sin\phi\right]^{r^T_n}\prod_{l=1}\left[-(l-1)!\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{r^L_l}. \tag{28}$$

Next, we propose the following identity

$$\sum_{k_r=0}^{r^L_1}\frac{\left(-\frac{t}{2}-1\right)_{k_r}}{\left(\frac{u}{2}+2-N\right)_{k_r}}\frac{\left(-r^L_1\right)_{k_r}}{k_r!}\left(1+\frac{s}{t}\right)^{k_r}$$
$$=0\cdot\left(\frac{tu}{s}\right)^0+0\cdot\left(\frac{tu}{s}\right)^{-1}+\cdots+0\cdot\left(\frac{tu}{s}\right)^{-\left[\frac{r^L_1+1}{2}\right]-1}+C_{r^L_l}\left(\frac{tu}{s}\right)^{-\left[\frac{r^L_1+1}{2}\right]}$$
$$+O\left\{\left(\frac{tu}{s}\right)^{-\left[\frac{r^L_1+1}{2}\right]+1}\right\}, \tag{29}$$

where $C_{r^L_1}$ is independent of energy E and depends on r^L_1 and possibly scattering angle ϕ. For $r^L_1=2m$ being an even number, we further propose that $C_{r^L_1}=\frac{(2m)!}{m!}$ and is ϕ independent. We have verified Eq.(29) for $r^L_1=0,1,2,\cdots,10$.

It should be noted that, taking Regge limit ($s\to\infty$ with t fixed) and setting $r^L_1=2m$, Eq.(29) reduces to the Stirling number identity,

$$\sum_{k_r=0}^{2m}\frac{\left(-\frac{t}{2}-1\right)_{k_r}}{\left(-\frac{s}{2}\right)_{k_r}}\frac{(-2m)_{k_r}}{k_r!}\left(\frac{s}{t}\right)^{k_r}\simeq\sum_{k_r=0}^{2m}(-2m)_{k_r}\left(-\frac{t}{2}-1\right)_{k_r}\frac{(-2/t)^{k_r}}{k_r!}$$
$$=0\cdot(-t)^0+0\cdot(-t)^{-1}+\cdots+0\cdot(-t)^{-m+1}+\frac{(2m)!}{m!}(-t)^{-m}+O\left\{\left(\frac{1}{t}\right)^{m+1}\right\}, \tag{30}$$

which was proposed in [30] and proved in [31].

In Eq.(29), the 0 terms correspond to the naive leading energy orders of string amplitudes in the hard scattering limit. The true leading order of string amplitudes in the hard scattering limit can then be identified

$$A^{(r^T_n,r^L_l)}_{st}\simeq B\left(-\frac{t}{2}-1,-\frac{s}{2}-1\right)\cdot C_{r^L_1}\left(E\sin\phi\right)^{-2\left[\frac{r^L_1+1}{2}\right]}\cdot(\cdots)$$
$$\cdot\prod_{n=1}\left[(n-1)!E\sin\phi\right]^{r^T_n}\prod_{l=1}\left[-(l-1)!\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{r^L_l}$$
$$\sim E^{N-\sum_{n\geq 2}nr^T_n-\left(2\left[\frac{r^L_1+1}{2}\right]-r^L_1\right)-\sum_{l\geq 3}lr^L_l}, \tag{31}$$

which means that a string amplitude reaches its highest energy when $r^T_{n\geq 2}=r^L_{l\geq 3}=0$ and $r^L_1=2m$ being an even number. This is first observed in [7, 8, 9, 10, 11, 13]

Finally, the leading order string amplitudes in the hard scattering limit can be calculated to be

$$
\begin{aligned}
A_{st}^{(r_1^T,2m,r_2^L)} &\simeq B\left(-\frac{s}{2}-1,-\frac{t}{2}-1\right)\cdot F_D^{(4)}\left(-\frac{t}{2}-1;R_1^T,R_2^L;\frac{u}{2}+2-N;1,Z_2^L\right) \\
&\cdot\left[E\sin\phi\right]^{r_1^T}\cdot\left[-\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{2m}\cdot\left[-\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{r_2^L} \\
&= B\left(-\frac{s}{2}-1,-\frac{t}{2}-1\right)(E\sin\phi)^N\cdot\left(r_1^L-1\right)!!\left(-\frac{1}{M_2}\right)^{2m+r_2^L}\left(\frac{1}{2}\right)^{m+r_2^L} \\
&= \left(r_1^L-1\right)!!\left(-\frac{1}{M_2}\right)^{2m+r_2^L}\left(\frac{1}{2}\right)^{m+r_2^L}\cdot A^{(N,0,0)},
\end{aligned}
\tag{32}
$$

which reproduces the ratios

$$
\frac{A_{st}^{(r_1^T,2m,r_2^L)}}{A_{st}^{(N,0,0)}}=(2m-1)!!\left(-\frac{1}{M_2}\right)^{2m+r_2^L}\left(\frac{1}{2}\right)^{m+r_2^L}, \tag{33}
$$

which is consistent with the previous result [7, 8, 9, 10, 11, 13].

3.2. Recurrence Relations in Regge Limit

Regge scattering limit is the small angle scattering with $s\sim E^2\to\infty$ and $t\sim E^2\sin^2\frac{\phi}{2}=$ constant. The recurrence relations of string amplitudes in the Regge scattering limit have been studied in [30, 32, 33]. The relevant kinematics in Regge limit are

$$
k_1^T=0,\quad k_3^T\simeq-\sqrt{-t}, \tag{34}
$$

$$
k_1^P\simeq-\frac{s}{2M_2},\ k_3^P\simeq-\frac{\tilde{t}}{2M_2}=-\frac{t-M_2^2-M_3^2}{2M_2}, \tag{35}
$$

$$
k_1^L\simeq-\frac{s}{2M_2},\ k_3^L\simeq-\frac{\tilde{t}'}{2M_2}=-\frac{t+M_2^2-M_3^2}{2M_2}, \tag{36}
$$

with $\tilde{z}_{kk'}^T=1,\ \tilde{z}_{kk'}^P=1-\left(-\frac{s}{\tilde{t}}\right)^{1/k}e^{\frac{i2\pi k'}{k}}\sim s^{1/k}$ and $\tilde{z}_{kk'}^L=1-\left(-\frac{s}{\tilde{t}'}\right)^{1/k}e^{\frac{i2\pi k'}{k}}\sim s^{1/k}$.

In the hard limit, the (s,t) channel string amplitude in Eq.(15) reduces to

$$
\begin{aligned}
A_{st}^{(r_n^T,r_m^P,r_l^L)} &\simeq B\left(-\frac{t}{2}-1,-\frac{s}{2}-1\right)F_1\left(-\frac{t}{2}-1;-q_1,-r_1;-\frac{s}{2};\frac{s}{\tilde{t}},\frac{s}{\tilde{t}'}\right) \\
&\cdot\prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{r_n^T}\cdot\prod_{m=1}\left[(m-1)!\frac{\tilde{t}}{2M_2}\right]^{r_m^P}\prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l^L},
\end{aligned}
\tag{37}
$$

where F_1 is the Appell function. Eq.(37) agrees with the result obtained in [33] previously.

The string amplitudes in Regge limit are much more complicated than that in hard limit and do not have linear relations. However, there are many recurrence relations of Appell functions F_1,

$$
(a-b_1-b_2)F_1-aF_1(a+1)+b_1F_1(b_1+1)+b_2F_1(b_2+1)=0, \tag{38}
$$

$$
cF_1-(c-a)F_1(c+1)-aF_1(a+1;c+1)=0, \tag{39}
$$

$$
cF_1+c(x-1)F_1(b_1+1)-(c-a)xF_1(b_1+1;c+1)=0, \tag{40}
$$

$$
cF_1+c(y-1)F_1(b_2+1)-(c-a)yF_1(b_2+1;c+1)=0. \tag{41}
$$

Using the above recurrence relations, we can obtain a lot of recurrence relations among the string amplitudes in Eq.(37). One can show that by solving the recurrence relations, all the string amplitudes at certain mass level can be expressed in term of a single amplitude [33].

4. Symmetry of Four-Point Amplitudes at General Energy

Let us recall the (s,t) channels string scattering amplitudes with three tachyons and a massive higher spin string state in Eq.(15),

$$A_{st}^{(r_n^T,r_m^P,r_l^L)} = B\left(-\frac{t}{2}-1,-\frac{s}{2}-1\right)F_D^{(K)}\left(-\frac{t}{2}-1;R_n^T,R_m^P,R_l^L;\frac{u}{2}+2-N;\tilde{Z}_n^T,\tilde{Z}_m^P,\tilde{Z}_l^L\right)$$
$$\cdot\prod_{n=1}\left[-(n-1)!k_3^T\right]^{r_n^T}\cdot\prod_{m=1}\left[-(m-1)!k_3^P\right]^{r_m^P}\prod_{l=1}\left[-(l-1)!k_3^L\right]^{r_l^L}. \tag{42}$$

To discover the symmetry of the above amplitudes, we need to understand their mathematical structure in a deeper way. To do it, we follow the mathematical construction in [34] and define the generating functions associated the D-type Lauricella function $F_D^{(K)}$ as

$$f_c^{a,b_j}(s,u_j,t,x_j) \equiv B(a,c-a)\,F_D^{(K)}(a;b_j;c;x_j)\,s^a u_1^{b_1}\cdots u_K^{b_K}t^c,\;\; j=1,\cdots,K, \tag{43}$$

so that the (s,t) channels string scattering amplitudes can be expressed in term of the generating functions as

$$A_{st}^{(r_n^T,r_m^P,r_l^L)} \sim f_{\frac{u}{2}+2-N}^{-\frac{t}{2}-1,R_j^X}\left(1,k_3^X,1,\tilde{Z}_j^X\right),\;\; X=T,P,L. \tag{44}$$

We next define the operators,

$$\begin{aligned}
E^{\alpha} &= s\left(\sum_j x_j\partial_{x_j}+s\partial_s\right),\;\; E^{\alpha\gamma}=st\left(\sum_j(1-x_j)\,\partial_{x_j}-s\partial_s\right),\\
E^{\beta_k} &= u_k\left(x_k\partial_{x_k}+u_k\partial_{u_k}\right),\quad E^{\beta_k\gamma}=u_kt\left(-(1-x_k)\,\partial_{x_k}+u_k\partial_{u_k}\right),\\
E^{\gamma} &= t\left(\sum_j(1-x_j)\,\partial_{x_j}+t\partial_t-s\partial_s-\sum_j u_j\partial_{u_j}\right),\;\; E^{\alpha\beta_k\gamma}=su_kt\partial_{x_k},\\
J_{\alpha} &= s\partial_s,\;\; J_{\beta_k}=u_k\partial_{u_k}-\frac{1}{2}t\partial_t+\frac{1}{2}\sum_{j\neq k}u_j\partial_{u_j},\;\; J_{\gamma}=t\partial_t-\frac{1}{2}\left(s\partial_s+\sum_j u_j\partial_{u_j}+1\right),
\end{aligned} \tag{45}$$

which act on the generating function gives,

$$\begin{aligned}
E^{\alpha}f_c^{a,b_j} &= (c-a-1)\,f_c^{a+1,b_j}, \qquad E^{\alpha\gamma}f_c^{a,b_j}=\left(\sum_j b_j-1\right)f_{c+1}^{a+1,b_j},\\
E^{\beta_k}f_c^{a,b_j} &= b_kf_c^{a,b_k+1}, \qquad E^{\beta_k\gamma}f_c^{a,b_j}=b_kf_{c+1}^{a,b_k+1},\\
E^{\gamma}f_c^{a,b_j} &= \left(c-\sum_j b_j\right)f_{c+1}^{a,b_j},\quad E^{\alpha\beta_k\gamma}f_c^{a,b_j}=b_kf_{c+1}^{a+1,b_k+1},\\
J_{\alpha}f_c^{a,b_j} &= \left(a-\frac{c}{2}\right)f_c^{a,b_j}, \qquad J_{\beta_k}f_c^{a,b_j}=\left(b_k-\frac{c}{2}+\frac{1}{2}\sum_{j\neq k}b_j\right)f_c^{a,b_j},\\
J_{\gamma}f_c^{a,b_j} &= \left[c-\frac{1}{2}\left(a+\sum_j b_j+1\right)\right]f_c^{a,b_j}.
\end{aligned} \tag{46}$$

Finally, we define a set of new operators in the following way,

$$\begin{aligned} E^{\alpha} &= \mathcal{E}_{12}, & E^{\alpha\gamma} &= \mathcal{E}_{32}, & E^{\gamma} &= \mathcal{E}_{31}, \\ E^{\beta_k\gamma} &= -\mathcal{E}_{k+3,1}, & E^{\alpha\beta_k\gamma} &= -\mathcal{E}_{k+3,2}, & E^{\beta_k} &= \mathcal{E}_{k+3,3}, \\ J_{\alpha} &= \frac{1}{2}\left(\mathcal{E}_{11} - \mathcal{E}_{22}\right), & J_{\gamma} &= \frac{1}{2}\left(\mathcal{E}_{33} - \mathcal{E}_{11}\right), & J_{\beta_k} &= \frac{1}{2}\left(\mathcal{E}_{k+3,k+3} - \mathcal{E}_{33}\right). \end{aligned} \tag{47}$$

The algebra satisfied by the new operators becomes

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}, \tag{48}$$

which can be identified as $sl\,(K+3, \mathbb{C})$ algebra.

5. Conclusion

In this work, I briefly reviewed the hidden symmetry in string theory by studying the string scattering amplitudes. The four-point bosonic open string scattering amplitude with three tachyons and an arbitrary massive higher spin string state in both (s,t) and (t,u) channels have been explicitly calculated and expressed in term of D-type Lauricella function in Eqs.(15) and (16). The string BCJ relation can be verified easily. We also considered two high energy limits. In hard limit, the hidden symmetry reduces to the linear relations among the string amplitude. In Regge limit, the hidden symmetry exhibit to be the recurrence relations of the string amplitudes. At general energy, we mathematically showed that the hidden symmetry is associated to $sl\,(K+3, \mathbb{C})$ algebra. In the future, we will study this symmetry in more detail and try to understand its physical picture.

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