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Generalized Jack and Macdonald polynomials arising from AGT conjecture

Y Ohkubo

Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan

E-mail: m12010t@math.nagoya-u.ac.jp

Abstract. We investigate the existence and the orthogonality of the generalized Jack symmetric functions which play an important role in the AGT relations. We show their orthogonality by deforming them to the generalized Macdonald symmetric functions.

1. Introduction

In [1], Aldey, Gaiotto and Tachikawa discovered surprising relations (called AGT relations) between 2D CFTs and 4D gauge theories. Since then, a number of studies on AGT relations have been carried out by mathematicians and physicists. One of them, a q-analogue of these relations is given by [2, 3], in which connections between the deformed Virasoro/W algebra and 5D gauge theories are proposed. Moreover in [4], it is conjectured with the help of the Ding-Iohara-Miki algebra [5, 6] that the matrix elements of the vertex operators with respect to the generalized Macdonald symmetric functions reproduce the Nekrasov factors of the 5D instanton partition function. In [7], an M-theoretic derivation of it is given. These generalized Macdonald functions are the eigenfunctions of a certain operator in the Ding-Iohara-Miki algebra. At last, the refined topological vertex of Awata-Kanno or Iqbal-Kozcaz-Vafa is deriven from another representation of Ding-Iohara-Miki algebra [8]. This result show that the correlation functions of the Ding-Iohara-Miki algebra coincide with the 5D instanton partition functions.

On the other hand, the 4D $SU(2)$ AGT relation as Hubbard-Stratanovich (HS) duality is proved in [9] with the help of the generalized Jack symmetric functions. (The formula of their selberg averages is still conjecture.) In their previous paper [10], the integrand of the Dotsenko-Fateev (DF) representation of the 4-point conformal block is expanded by the ordinary Jack symmetric polynomials and compared with the Nekrasov partition function. They show that, in the case $\beta = 1$, every term of the DF-integral coincides with the Nekrasov formula parametrized by pairs of Young diagrams. However at $\beta \neq 1$ it is not working. At this, a generalization of the Jack symmetric polynomials is introduced in [9] such that each term of DF-integral expanded by the generalized Jack polynomials corresponds to each term of Nekrasov partition function. However, the existence of the generalized Jack symmetric functions is unproven. Also, it is a little hard to give a mathematical, general proof of the orthogonality [11, Section 2] of these functions, because they have degenerate eigenvalues.

The generalized Macdonald and Jack symmetric functions were introduced independently and their relation had not been known. However, we found that in the limit $q \to 1$, the generalized Macdonald symmetric functions are reduced to the generalized Jack symmetric

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functions naturally. This leads the orthogonality and the existence of the generalized Jack symmetric functions. Since the eigenvalues of the generalized Macdonald symmetric functions are non-degenerate, we can prove their orthogonality without any other commutative operators, i.e., higher rank hamiltonians. Moreover we can describe 5D AGT relations as HS duality by using the generalized Macdonald symmetric functions [12] in the same way of [9].

This letter is organized as follows. In section 2, we give a short summary of the generalized Macdonald symmetric functions. In section 3, we calculate the limit $q \to 1$ of the operator whose eigenfunctions are the generalized Macdonald functions and prove the existence and the orthogonality of the generalized Jack symmetric functions. In section 4 we present several examples.

2. Generalized Macdonald symmetric function

The generalized Macdonald symmetric functions are described by N kinds of independent variables $\{x_n^{(i)} \mid n \in \mathbb{N}, i = 1, \dots, N\}$. Hence we use N kinds of power sum symmetric functions $p_n^{(i)} = \sum_{k\geq 1} \left(x_k^{(i)}\right)^n$. Using this notation, set

$$\eta^{(i)}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} z^n p_n^{(i)}\right) \exp\left(-\sum_{n=1}^{\infty}(1-q^n) z^{-n} \frac{\partial}{\partial p_n^{(i)}}\right), \tag{1}$$

$$\xi^{(i)}(z) := \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (t/q)^{\frac{n}{2}} z^n p_n^{(i)}\right) \exp\left(\sum_{n=1}^{\infty}(1-q^n)(t/q)^{\frac{n}{2}} z^{-n} \frac{\partial}{\partial p_n^{(i)}}\right), \tag{2}$$

$$\varphi_+^{(i)}(z) := \exp\left(-\sum_{n=1}^{\infty}(1-q^n)(1-t^n q^{-n})(t/q)^{-\frac{n}{4}} z^{-n} \frac{\partial}{\partial p_n^{(i)}}\right), \tag{3}$$

$$\varphi_-^{(i)}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n}(1-t^n q^{-n})(t/q)^{-\frac{n}{4}} z^n p_n^{(i)}\right), \tag{4}$$

where q and t are independent parameters. Then these operators represent the Ding-Iohara-Miki algebra [13]. Moreover for complex parameters u_i $(i = 1, 2, \dots, N)$, we define

$$X(z) := \sum_{i=1}^{N} u_i \widetilde{\Lambda}_i, \tag{5}$$

$$\widetilde{\Lambda}_i := \varphi_-^{(1)}((t/q)^{\frac{1}{4}} z)\, \varphi_-^{(2)}((t/q)^{\frac{3}{4}} z) \cdots \varphi_-^{(i)}((t/q)^{\frac{2i-3}{4}} z)\, \eta^{(i)}((t/q)^{\frac{i-1}{2}} z). \tag{6}$$

The operator $X(z)$ is obtained by so-called the level N representation of Ding-Iohara-Miki algebra. The deformed W_N algebra can be generated by its representation [14]. In the $N = 1$ case, the coefficient in front of z^0 of $X(z) = \eta(z)$ is Macdonald's difference operator and its eigenfunctions are the Macdonald symmetric functions. The generalized Macdonald functions are the eigenfunctions of the operator

$$X_0 := \oint \frac{dz}{2\pi\sqrt{-1}z} X(z) \tag{7}$$

in the general N case.

The generalized Macdonald symmetric functions are parametrized by N-tuples of partitions $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$, where a partition $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots)$ is a sequence of non-negative integers such that $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdots \geq 0$. For each partition λ and an N-tuple of partitions

$\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$, we use the symbols : $|\lambda| := \sum_{k \geq 1} \lambda_k$, $\ell(\lambda) := \#\{k \mid \lambda_k \neq 0\}$ and $|\vec{\lambda}| := \sum_{i=1}^{N} |\lambda^{(i)}|$. We write $\vec{\lambda} \geq^{\mathrm{L}} \vec{\mu}$ (resp. $\vec{\lambda} \geq^{\mathrm{R}} \vec{\mu}$) if and only if $|\vec{\lambda}| = |\vec{\mu}|$ and

$$|\lambda^{(N)}| + \cdots + |\lambda^{(j+1)}| + \sum_{k=1}^{i} \lambda_k^{(j)} \geq |\mu^{(N)}| + \cdots + |\mu^{(j+1)}| + \sum_{k=1}^{i} \mu_k^{(j)} \tag{8}$$

$$\left(\text{resp.} \quad |\lambda^{(1)}| + \cdots + |\lambda^{(j-1)}| + \sum_{k=1}^{i} \lambda_k^{(j)} \geq |\mu^{(1)}| + \cdots + |\mu^{(j-1)}| + \sum_{k=1}^{i} \mu_k^{(j)}\right) \tag{9}$$

for all $i \geq 1$ and $1 \leq j \leq N$. Then "$\geq^{\mathrm{L}}$" and "$\geq^{\mathrm{R}}$" are generalized dominace partial orderings on the N-tuples of partitions. By using these partial orderings we can triangulate the representation matrix of X_0.

Fact 2.1 ([4])**.** Let $m_{\vec{\lambda}}$ be the product of monomial symmetric functions $\prod_{i=1}^{N} m_{\lambda^{(i)}}^{(i)}$ ($m_{\lambda^{(i)}}^{(i)}$ is an usual monomial symmetric function of variables $\{x_n^{(i)} \mid n\}$). Then

$$X_0 m_{\vec{\lambda}} = \sum_{\vec{\mu} \leq^{\mathrm{L}} \vec{\lambda}} c_{\vec{\lambda}\vec{\mu}} m_{\vec{\mu}}, \quad c_{\vec{\lambda}\vec{\mu}} \in \mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}}, u_1, \ldots, u_N) \tag{10}$$

and the eigenvalues are

$$c_{\vec{\lambda}\vec{\lambda}} = \sum_{i=1}^{N} u_i \, \epsilon_{\lambda^{(i)}}, \quad \epsilon_\lambda = 1 + (t-1) \sum_{k=1}^{\ell(\lambda)} (q^{\lambda_k} - 1) t^{-k}. \tag{11}$$

Thus we have the following existence theorem.

Fact 2.2 ([4])**.** For each N-tuple of partitions $\vec{\lambda}$, there exists a unique symmetric function $P_{\vec{\lambda}}$ satisfying the following two conditions:

$$P_{\vec{\lambda}} = m_{\vec{\lambda}} + \sum_{\vec{\mu} <^{\mathrm{L}} \vec{\lambda}} d_{\vec{\lambda}\vec{\mu}} m_{\vec{\mu}}, \qquad d_{\vec{\lambda}\vec{\mu}} \in \mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}}, u_1, \ldots, u_N); \tag{12}$$

$$X_0 P_{\vec{\lambda}} = e_{\vec{\lambda}} P_{\vec{\lambda}}, \qquad e_{\vec{\lambda}} \in \mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}}, u_1, \ldots, u_N). \tag{13}$$

In the case $N = 1$, the symmetric functions $P_{\vec{\lambda}}$ are the usual Macdonald symmetric functions. Hence we call these symmetric functions $P_{\vec{\lambda}}$ the generalized Macdonald symmetric functions.

The partial ordering "$\geq^{\mathrm{R}}$" gives the similar existence theorem of the dual symmetric functions $P^*_{\vec{\lambda}}$ for the adjoint operator X_0^* of X_0 with respect to Macdonald's scalar product $\langle -, - \rangle_{q,t}$. The scalar product $\langle -, - \rangle_{q,t}$ is defined by

$$\langle p_{\vec{\lambda}}, p_{\vec{\mu}} \rangle_{q,t} = \delta_{\vec{\lambda},\vec{\mu}} \prod_{i=1}^{N} z_{\lambda^{(i)}} \prod_{k=1}^{\ell(\lambda^{(i)})} \frac{1 - q^{\lambda_k^{(i)}}}{1 - t^{\lambda_k^{(i)}}}, \qquad z_{\lambda^{(i)}} := \prod_{k \geq 1} k^{m_k} \, m_k!, \tag{14}$$

where m_k is the number of entries in $\lambda^{(i)}$ equal to k and $p_{\vec{\lambda}} := \prod_{i=1}^{N} p_{\lambda^{(i)}}^{(i)} := \prod_{i=1}^{N} \prod_{k \geq 1} p_{\lambda_k^{(i)}}^{(i)}$. When q and t are generic, the eigenvalues $e_{\vec{\lambda}}$ of the generalized Macdonald symmetric functions are non-degenerate, that is

$$\vec{\lambda} \neq \vec{\mu} \quad \Rightarrow \quad e_{\vec{\lambda}} \neq e_{\vec{\mu}}. \tag{15}$$

Therefore we have the following orthogonality of them.

Fact 2.3 ([4])**.** If $\vec{\lambda} \neq \vec{\mu}$ then

$$\left\langle P^*_{\vec{\lambda}}, P_{\vec{\mu}} \right\rangle_{q,t} = 0. \tag{16}$$

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3. Limit to β deformation

In this section, we set $u_i = q^{u'_i}$ $(i = 1, \dots, N)$, $t = q^\beta$, $q = e^\hbar$ and take the limit $\hbar \to 0$ with β fixed in order to consider the specialization to β-deformation. Since $(1-t^{-n})(1-t^n q^{-n})/n = \mathcal{O}(\hbar^2)$ and $(1-t^{-n})/n$, $(1-q^n) = \mathcal{O}(\hbar)$,

$$\begin{aligned}\oint \frac{dz}{2\pi\sqrt{-1}z}\widetilde{\Lambda}_i(z) =& 1 + \sum_{n=1}^{\infty}\left(-\frac{(1-t^{-n})(1-q^n)}{n} p_n^{(i)} \frac{\partial}{\partial p_n^{(i)}}\right) \\ &+ \sum_{k=1}^{i-1}\sum_{n=1}^{\infty}\left(-\frac{(1-t^{-n})(1-t^n q^{-n})(1-q^n)}{n}(t/q)^{-\frac{(i-k-1)n}{2}} p_n^{(k)}\frac{\partial}{\partial p_n^{(i)}}\right) \\ &+ \frac{1}{2}\sum_{n,m}\left(-\frac{(1-t^{-n})(1-t^{-m})(1-q^{n+m})}{n\,m} p_n^{(i)} p_m^{(i)} \frac{\partial}{\partial p_{n+m}^{(i)}}\right) \\ &+ \frac{1}{2}\sum_{n,m}\left(\frac{(1-t^{-n-m})(1-q^n)(1-q^m)}{(n+m)} p_{n+m}^{(i)} \frac{\partial}{\partial p_n^{(i)}}\frac{\partial}{\partial p_m^{(i)}}\right) + \mathcal{O}(\hbar^4).\end{aligned} \tag{17}$$

Hence the $\hbar$ expansion is

$$\begin{aligned}\oint \frac{dz}{2\pi\sqrt{-1}z}\widetilde{\Lambda}_i(z) =& 1 + \hbar^2\left\{\beta\sum_{n=1}^{\infty} n\, p_n^{(i)}\frac{\partial}{\partial p_n^{(i)}}\right\} + \hbar^3\left\{\beta(1-\beta)\sum_{k=1}^{i-1}\sum_{i=1}^{\infty} n^2 p_n^{(k)}\frac{\partial}{\partial p_n^{(i)}}\right. \\ &+ \frac{\beta^2}{2}\sum_{n,m}(n+m)p_n^{(i)}p_m^{(i)}\frac{\partial}{\partial p_{n+m}^{(i)}} + \frac{\beta}{2}\sum_{n,m} n\,m\, p_{n+m}^{(i)}\frac{\partial^2}{\partial p_n^{(i)}\partial p_m^{(i)}} \\ &\left. + \frac{\beta(1-\beta)}{2}\sum_{n=1}^{\infty} n^2 p_n^{(i)}\frac{\partial}{\partial p_n^{(i)}}\right\} + \mathcal{O}(\hbar^4).\end{aligned} \tag{18}$$

Thus we get

$$\begin{aligned}u_i\oint \frac{dz}{2\pi\sqrt{-1}z}\widetilde{\Lambda}_i(z) =& 1 + u'_i\,\hbar + \hbar^2\left\{\beta\sum_{n=1}^{\infty} n\, p_n^{(i)}\frac{\partial}{\partial p_n^{(i)}} + \frac{1}{2}u_i'^2\right\} \\ &+ \hbar^3\left\{\beta\,\mathcal{H}_\beta^{(i)} + \beta\sum_{k=1}^{i-1}\mathcal{H}_\beta^{(i,k)} + \frac{u_i'^3}{6}\right\} + \mathcal{O}(\hbar^4),\end{aligned} \tag{19}$$

where

$$\mathcal{H}_\beta^{(i)} := \frac{1}{2}\sum_{n,m}\left(\beta(n+m)p_n^{(i)}p_m^{(i)}\frac{\partial}{\partial p_{n+m}^{(i)}} + n\,m\,p_{n+m}^{(i)}\frac{\partial^2}{\partial p_n^{(i)}\partial p_m^{(i)}}\right) + \sum_{n=1}^{\infty}\left(u'_i + \frac{1-\beta}{2}n\right) n p_n^{(i)}\frac{\partial}{\partial p_n^{(i)}}, \tag{20}$$

$$\mathcal{H}_\beta^{(i,k)} := (1-\beta)\sum_{n=1}^{\infty} n^2 p_n^{(k)}\frac{\partial}{\partial p_n^{(i)}}. \tag{21}$$

For $k = 0, 1, 2, \dots$, we define operators H_k by

$$X_0 =: \sum_{k=0}^{\infty}\hbar^k H_k. \tag{22}$$

All homogeneous symmetric functions of the same degree belong to the same eigenspace of H_0, H_1 and H_2. Even without the operators H_0, H_1 and H_2, the eigenfunctions of X_0 do not change. In addition, we have

$$\lim_{\hbar\to 0}\left(\frac{X_0-(H_0+\hbar H_1+\hbar^2 H_2)}{(t-1)(q-1)^2}\right)=\mathcal{H}_\beta+\frac{1}{6\beta}\sum_{i=1}^N u_i'^3, \tag{23}$$

$$\mathcal{H}_\beta:=\sum_{i=1}^N \mathcal{H}_\beta^{(i)}+\sum_{i>j}\mathcal{H}_\beta^{(i,j)}. \tag{24}$$

Consequently the limit $q\to 1$ of the generalized Macdonald functions are eigenfunctions of the differential operator $\mathcal{H}_\beta$. $\mathcal{H}_\beta$ plus the momentum $(\beta-1)\sum_{i=1}^N\sum_{n=1}^\infty np_n^{(i)}\frac{\partial}{\partial p_n^{(i)}}$ corresponds to the differential operator of [9, 11], the eigenfunctions of which are called generalized Jack symmetric functions.

As the Fact 2.1, we can triangulate $\mathcal{H}_\beta$ similarly. Moreover if $\vec{\lambda}\geq^{\mathrm{L}}\vec{\mu}$ and β is generic, then $e'_{\vec{\lambda}}\neq e'_{\vec{\mu}}$. ($e'_{\vec{\lambda}}$, $e'_{\vec{\mu}}$ are eigenvalues of $\mathcal{H}_\beta$.) Therefore we get the existence theorem of the generalized Jack symmetric functions.

Proposition 3.1. There exists a unique symmetric function $J_{\vec{\lambda}}$ satisfying the following two conditions:

$$J_{\vec{\lambda}}=m_{\vec{\lambda}}+\sum_{\vec{\mu}<^{\mathrm{L}}\vec{\lambda}}d'_{\vec{\lambda}\vec{\mu}}m_{\vec{\mu}},\qquad d'_{\vec{\lambda}\vec{\mu}}\in\mathbb{Q}(\beta,u_1',\dots,u_N'); \tag{25}$$

$$\mathcal{H}_\beta J_{\vec{\lambda}}=e'_{\vec{\lambda}}J_{\vec{\lambda}},\qquad e'_{\vec{\lambda}}\in\mathbb{Q}(\beta,u_1',\dots,u_N'). \tag{26}$$

From the above argument and the uniqueness in this proposition we get the following important result.

Proposition 3.2. The limit of the generalized Macdonald symmetric functions $P_{\vec{\lambda}}$ to β-deformation coincide with the generalized Jack symmetric functions $J_{\vec{\lambda}}$. That is

$$P_{\vec{\lambda}}\ \underset{\substack{\hbar\to 0,\\ u_i=q^{u_i'},\,t=q^\beta,\,q=e^\hbar}}{\longrightarrow}\ J_{\vec{\lambda}}. \tag{27}$$

Remark 3.3. For the dual functions $P^*_{\vec{\lambda}}$ and $J^*_{\vec{\lambda}}$, the similar proposition holds.

By Fact 2.3, Proposition 3.2 and the fact that the scalar product $\langle -,-\rangle_{q,t}$ reduces to the scalar product $\langle -,-\rangle_\beta$ which is defined by

$$\langle p_{\vec{\lambda}},p_{\vec{\mu}}\rangle_\beta=\delta_{\vec{\lambda},\vec{\mu}}\prod_{i=1}^N z_{\lambda^{(i)}}\beta^{-\ell(\lambda^{(i)})}, \tag{28}$$

we obtain the orthogonality of the generalized Jack symmetric functions.

Proposition 3.4. If $\vec{\lambda}\neq\vec{\mu}$, then

$$\left\langle J^*_{\vec{\lambda}},J_{\vec{\mu}}\right\rangle_\beta=0. \tag{29}$$

By this proposition, we can prove the chauchy formula of generalized Jack symmetric functions in the usual way. For example, in the $N=2$ case, we have

$$\sum_{\vec{\lambda}}\frac{J_{\vec{\lambda}}(x^{(1)},x^{(4)})\,J^*_{\vec{\lambda}}(x^{(2)},x^{(3)})}{v_{\vec{\lambda}}}=\exp\left(\beta\sum_{n\geq 1}\frac{1}{n}p_n^{(1)}p_n^{(2)}\right)\exp\left(\beta\sum_{n\geq 1}\frac{1}{n}p_n^{(3)}p_n^{(4)}\right), \tag{30}$$

where $v_{\vec{\lambda}}:=\langle J^*_{\vec{\lambda}},J_{\vec{\lambda}}\rangle_\beta$. This is the essential formula used in the scenario of proof of the AGT conjecture [9].

4. Example

We give examples of Proposition 3.2 in the case $N = 2$. The generalized Macdonald symmetric functions of level 1 and 2 have the forms:

$$\begin{pmatrix} P_{(0),(1)} \\ P_{(1),(0)} \end{pmatrix} = M^1_{q,t} \begin{pmatrix} m_{(0),(1)} \\ m_{(1),(0)} \end{pmatrix}, \qquad M^1_{q,t} := \begin{pmatrix} 1 & (t/q)^{\frac{1}{2}} \frac{(t-q)u_2}{t(u_1-u_2)} \\ 0 & 1 \end{pmatrix}, \tag{31}$$

$$\begin{pmatrix} P_{(0),(2)} \\ P_{(0),(1,1)} \\ P_{(1),(1)} \\ P_{(2),(0)} \\ P_{(1,1),(0)} \end{pmatrix} = M^2_{q,t} \begin{pmatrix} m_{(0),(2)} \\ m_{(0),(1,1)} \\ m_{(1),(1)} \\ m_{(2),(0)} \\ m_{(1,1),(0)} \end{pmatrix}, \qquad M^2_{q,t} := \tag{32}$$

$$\begin{pmatrix} 1 & \frac{(1+q)(t-1)}{qt-1} & \frac{(t/q)^{-\frac{1}{2}}(1+q)(q-t)(t-1)u_2}{(1-qt)(u_1-qu_2)} & \frac{(q-t)((1-q^2)tu_1-q(t^2-q(1+q)t+q)u_2)u_2}{qt(qt-1)(u_1-u_2)(u_1-qu_2)} & \frac{(1+q)(q-t)(t-1)((q-1)tu_1+q(q-t)u_2)u_2}{qt(qt-1)(u_1-u_2)(u_1-qu_2)} \\ 0 & 1 & (t/q)^{\frac{1}{2}} \frac{(t-q)u_2}{t(tu_1-u_2)} & \frac{(q-t)u_2}{q(tu_1-u_2)} & \frac{(q-t)(qu_2-t((t-1)u_1+u_2))u_2}{qt(u_1-u_2)(tu_1-u_2)} \\ 0 & 0 & 1 & (t/q)^{\frac{1}{2}} \frac{(t-q)u_2}{t(qu_1-u_2)} & (t/q)^{\frac{1}{2}} \frac{(q-t)((1+q+(q-1)t)u_1-2tu_2)}{t(qu_1-u_2)(-u_1+tu_2)} \\ 0 & 0 & 0 & 1 & \frac{(1+q)(t-1)}{qt-1} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Also the generalized Jack symmetric functions have the forms:

$$\begin{pmatrix} J_{(0),(1)} \\ J_{(1),(0)} \end{pmatrix} = M^1_{\beta} \begin{pmatrix} m_{(0),(1)} \\ m_{(1),(0)} \end{pmatrix}, \qquad M^1_{\beta} := \begin{pmatrix} 1 & \frac{1-\beta}{-u'_1+u'_2} \\ 0 & 1 \end{pmatrix}, \tag{33}$$

$$\begin{pmatrix} J_{(0),(2)} \\ J_{(0),(1,1)} \\ J_{(1),(1)} \\ J_{(2),(0)} \\ J_{(1,1),(0)} \end{pmatrix} = M^2_{\beta} \begin{pmatrix} m_{(0),(2)} \\ m_{(0),(1,1)} \\ m_{(1),(1)} \\ m_{(2),(0)} \\ m_{(1,1),(0)} \end{pmatrix}, \tag{34}$$

$$M^2_{\beta} := \begin{pmatrix} 1 & \frac{2\beta}{1+\beta} & \frac{2\beta(1-\beta)}{(1+\beta)(1-u'_1+u'_2)} & \frac{(1-\beta)(2+\beta-\beta^2-2u'_1+2u'_2)}{(1+\beta)(u'_1-u'_2)(-1+u'_1-u'_2)} & \frac{2\beta(2-3\beta+\beta^2)}{(1+\beta)(u'_1-u'_2)(-1+u'_1-u'_2)} \\ 0 & 1 & \frac{1-\beta}{-\beta-u'_1+u'_2} & \frac{1-\beta}{\beta+u'_1-u'_2} & \frac{-1+3\beta-2\beta^2}{(u'_1-u'_2)(-\beta-u'_1+u'_2)} \\ 0 & 0 & 1 & \frac{1-\beta}{-1-u'_1+u'_2} & \frac{2(1-\beta)(-1+\beta-u'_1+u'_2)}{(-1-u'_1+u'_2)(\beta-u'_1+u'_2)} \\ 0 & 0 & 0 & 1 & \frac{2\beta}{1+\beta} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

If we take the limit $q \to 1$ of $M^i_{q,t}$, then M^i_{β} appears.

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