Nonlinear Schrödinger Equation, 2D $\mathcal{N} = (2,2)^*$ Topological Yang-Mills-Higgs Theory and Their Gravity Dual

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Abstract. The duality between the N-particle sector of quantum nonlinear Schrödinger equation and the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory was found by Gerasimov and Shatashvili some time ago. At the large N and large 't Hooft coupling limit, the gravity dual of the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory can be constructed. Consequently, as a first example, one can formulate a triangle relation between integrable model, gauge theory and gravity. We present the results of the gravity dual in this paper, and make some checks at classical level between the gravity dual and the nonlinear Schrödinger equation. This paper is based on the talk given by the author at the 24th International Conference on Integrable Systems and Quantum Symmetries, and more details can be found in Ref. [1].

1. Introduction

Many interesting and profound relations between integrable models and gauge theories have been revealed in recent years [2, 3, 4]. The integrable models are defined in (1+1)D, and they can be nonlinear partial differential equations or lattice spin models. The corresponding 2D gauge theories have $\mathcal{N} = (2,2)^*$ supersymmetry.

Among these relations, the simplest example is the one between the nonlinear Schrödinger equation and the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory found by Gerasimov and Shatashvili [5, 6]. From the wave function of the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory one can reproduce the wave function of the quantum nonlinear Schrödinger equation in the N-particle sector.

The 2D $\mathcal{N} = (2,2)^*$ $U(N)$ Yang-Mills-Higgs theory was constructed in Ref. [7]. It can be viewed as the dimensional reduction of 4D topologically twisted $\mathcal{N} = 2$ $U(N)$ super Yang-Mills theory with a deformation term, which breaks 8 supercharges into 4 supercharges. Alternatively, it can also be viewed as the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ super Yang-Mills theory with some supersymmetry exact terms, which do not modify the theory at quantum level. Hence, the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory is equivalent to the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ super Yang-Mills theory, and we can study the latter one instead.

Based on the principle of gauge/gravity correspondence, we can construct the gravity dual of the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ super Yang-Mills theory in the large N and large 't Hooft coupling limit. Together with the relation between the nonlinear Schrödinger equation and the 2D

$\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory, now we have a triangle relation among three theories (see Fig. 1).

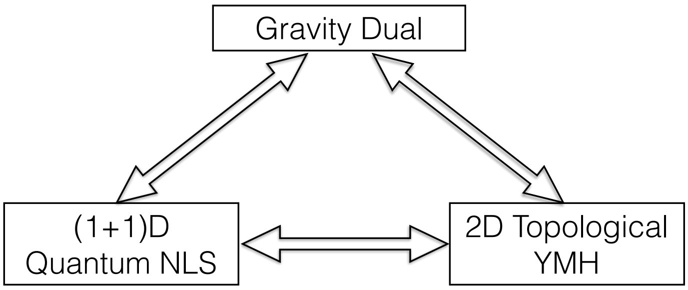

Figure 1. The relation between different theories

By setting up this triangle relation, in principle we can study the integrability of the nonlinear Schrödinger equation on the gravity side, and at the same time study some properties of the gravity on the integrable model side. For the moment, we can first take the classical limit in both the gravity dual and the nonlinear Schrödinger equation, and we can immediately find the correspondence between N D-branes in the supergravity and N solitons in the classical nonlinear Schrödinger equation. The correspondence at quantum level will be investigated in the future.

This paper is based on the talk given by the author at the 24th International Conference on Integrable Systems and Quantum Symmetries, and more details can be found in Ref. [1]. The paper is organized as follows. In Section 2 the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory, the nonlinear Schrödinger equation and the relation between them will be reviewed. In Section 3, we discuss the construction of the gravity dual of the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ super Yang-Mills theory, and the triangle relation demonstrated in Fig. 1 will be set up. Finally, in Section 4 the prospect for the future research will be discussed.

2. Gerasimov-Shatashvili Duality

In this section, we briefly review the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory, the nonlinear Schrödinger equation and the relation between them.

2.1. 2D Topological Yang-Mills-Higgs Theory

The 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory was first constructed in Ref. [7]. It is given by the path integral

$$Z_{YMH}(\Sigma_h) = \frac{1}{\mathrm{Vol}(\mathcal{G}_{\Sigma_h})} \int D\varphi_0\, D\varphi_\pm\, DA\, D\Phi\, D\psi_A\, D\psi_\Phi\, D\chi_\pm\, e^S\,, \tag{1}$$

where

$$S = S_0 + S_1 \tag{2}$$

with

$$S_0 = \frac{1}{2\pi}\int_{\Sigma_h} d^2z \left[\mathrm{Tr}\left(i\varphi_0(F(A) - \Phi\wedge\Phi) - c\,\Phi\wedge *\Phi\right) + \varphi_+\nabla_A^{(1,0)}\Phi^{(0,1)} + \varphi_-\nabla_A^{(0,1)}\Phi^{(1,0)}\right], \tag{3}$$

$$S_1 = \frac{1}{2\pi}\int_{\Sigma_h} d^2z\,\mathrm{Tr}\left[\frac{1}{2}\psi_A\wedge\psi_A + \frac{1}{2}\psi_\Phi\wedge\psi_\Phi + \chi_+\left[\psi_A^{1,0},\,\Phi^{(0,1)}\right] + \chi_-\left[\psi_A^{(0,1)},\,\Phi^{(1,0)}\right] + \chi_+\nabla_A^{(1,0)}\psi_\Phi^{(0,1)} + \chi_-\nabla_A^{(0,1)}\psi_\Phi^{(1,0)}\right). \tag{4}$$

In the absence of the deformation term $c\,\mathrm{Tr}(\Phi\wedge *\Phi)$ the theory preserves $\mathcal{N} = (4,4)$ supersymmetry. For generic values of $c \neq 0$ the theory preserves $\mathcal{N} = (2,2)$ supersymmetry, and the $\mathcal{N} = (2,2)$ supersymmetry transformations are given by

$$QA = i\psi_A\,, \quad Q\psi_A = -D\varphi_0\,, \quad Q\varphi_0 = 0\,, \tag{5}$$

$$Q\Phi = i\psi_\Phi\,, \tag{6}$$

$$Q\psi_\Phi^{(1,0)} = [\Phi^{(1,0)},\,\varphi_0] + c\Phi^{(1,0)}\,, \quad Q\psi_\Phi^{(0,1)} = [\Phi^{(0,1)},\,\varphi_0] + c\Phi^{(0,1)}\,, \tag{7}$$

$$Q\chi_\pm = i\varphi_\pm\,, \quad Q\varphi_\pm = [\chi_\pm,\,\varphi_0] \pm c\chi_\pm\,. \tag{8}$$

This theory can also be understood as the dimensional reduction of the 4D topologically twisted $\mathcal{N} = 2$ $U(N)$ super Yang-Mills theory with a deformation term. The details will be discussed in Ref. [1].

From the supersymmetric transformations (5) $\sim$ (8), one can show that the 2D $\mathcal{N} = (2,2)^*$ Yang-Mills-Higgs theory can be written as the 2D $\mathcal{N} = (2,2)^*$ super Yang-Mills theory with a supersymmetry exact deformation as follows:

$$S_{YMH} = S_{YM} + \left[Q,\,\int_{\Sigma_h} d^2z\,\mathrm{Tr}\left(\frac{1}{2}\Phi\wedge\psi_\Phi + \chi_+\nabla_A^{(1,0)}\Phi^{(0,1)} + \chi_-\nabla_A^{(0,1)}\Phi^{(1,0)}\right)\right]\,. \tag{9}$$

In Ref. [5] the supersymmetry exact deformation term was changed without modifying the theory at quantum level. Based on this fact, in the following, when we construct the gravity dual of the 2D $\mathcal{N} = (2,2)^*$ Yang-Mills-Higgs theory, we will consider the gravity dual of the 2D $\mathcal{N} = (2,2)^*$ super Yang-Mills theory instead.

2.2. Nonlinear Schrödinger Equation

The (1+1)D nonlinear Schrödinger equation is

$$i\partial_t\phi = -\frac{1}{2}\partial_x^2\phi + 2c(\phi^*\phi)\phi\,. \tag{10}$$

The Hamiltonian of the theory is given by

$$\mathcal{H} = \int dx\,\left[\frac{1}{2}\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x} + c\,(\phi^*\phi)^2\right], \tag{11}$$

where the field ϕ has the Poisson structure

$$\{\phi^*(x),\,\phi(x')\} = \delta(x - x')\,. \tag{12}$$

In (1+1)D, this theory is integrable both at the classical level and at the quantum level.

For the (1+1)D quantum nonlinear Schrödinger equation, if we consider the N-particle sector in the domain $x_1 \leq x_2 \leq \cdots \leq x_N$, the N-particle wave function satisfies the equation

$$\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\right)\Phi_\lambda(x) = 2\pi^2\left(\sum_{i=1}^{N}\lambda_i^2\right)\Phi_\lambda(x)\,, \tag{13}$$

where λ_i denotes the momentum of the i-th particle. The normalized wave function is given by

$$\Phi_\lambda(x) = \sum_{\omega\in W}(-1)^{l(w)}\prod_{i<j}\left(\frac{\lambda_{\omega(i)}-\lambda_{\omega(j)}+ic\,\mathrm{sgn}(x_i-x_j)}{\lambda_{\omega(i)}-\lambda_{\omega(j)}-ic\,\mathrm{sgn}(x_i-x_j)}\right)^{\frac{1}{2}}\exp\left(2\pi i\sum_i\lambda_{\omega(k)}\,x_k\right)\,, \tag{14}$$

where λ_i obey the Bethe Ansatz equation:

$$e^{2\pi i\lambda_j}\prod_{k\neq j}\frac{\lambda_k-\lambda_j-ic}{\lambda_k-\lambda_j+ic} = 1\,,\quad j=1,\cdots,N. \tag{15}$$

2.3. The Duality

Using the technique of cohomological localization, one can compute the partition function of the 2D $\mathcal{N}=(2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory exactly:

$$Z_{YMH}(\Sigma_h) = e^{(1-h)\,a(c)}\sum_{\lambda\in\mathcal{R}_N}D_\lambda^{2-2h}\,e^{-\sum_{k=1}^{\infty}t_k\,p_k(\lambda)}\,, \tag{16}$$

where $p_k(\lambda)$ is defined as

$$\frac{1}{(2\pi)^k}\,\mathrm{Tr}\,\varphi^k\,\Psi_\lambda(x_1,\cdots,x_N) = p_k(\lambda)\,\Psi_\lambda(x_1,\cdots,x_N)\,, \tag{17}$$

and the factor D_λ is given by

$$D_\lambda = \mu(\lambda)^{-1/2}\prod_{i<j}(\lambda_i-\lambda_j)\,\left(c^2+(\lambda_i-\lambda_j)^2\right)^{1/2}\,, \tag{18}$$

while $\mathcal{R}_N$ denotes the set of λ_i's satisfying the same Bethe Ansatz equation:

$$e^{2\pi i\lambda_j}\prod_{k\neq j}\frac{\lambda_k-\lambda_j+ic}{\lambda_k-\lambda_j-ic} = 1\,,\quad k=1,\cdots,N. \tag{19}$$

From this analysis, we see the equivalence between the wave function of the 2D $\mathcal{N}=(2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory and the wave function of the (1+1)D quantum nonlinear Schrödinger equation in the N-particle sector. Hence, the duality between these two theories at quantum level is implied.

3. Gravity Dual

Now we turn to the construction of the gravity dual of the 2D $\mathcal{N}=(2,2)^*$ $U(N)$ super Yang-Mills theory. We follow the same way how the gravity dual of 4D $\mathcal{N}=2^*$ super Yang-Mills theory was constructed [8, 9]. In that case, one can start with the 5D $\mathcal{N}=8$ gauged supergravity, which is a consistent truncation of the 10D type IIB supergravity on $AdS_5\times S^5$. After turning on additional scalar fields and choosing an appropriate scalar potential, one can break the supersymmetry from

$\mathcal{N} = 4$ to $\mathcal{N} = 2$, which corresponds to a mass deformation on the field theory side. Finally, the solution in the 5D $\mathcal{N} = 8$ gauged supergravity can be uplifted to 10D.

In our case, the gravity dual of the 2D $\mathcal{N} = (4, 4)$ $U(N)$ super Yang-Mills theory was constructed in Ref. [10] using the brane construction, and it was shown that by changing variables the solutions of the gravity dual can also be obtained from the 5D $\mathcal{N} = 2$ gauged supergravity, which is a further consistent truncation of the 5D $\mathcal{N} = 8$ gauged supergravity. Therefore, following the same logic of Ref. [8, 9], we can turn on an additional scalar field and choose an appropriate scalar potential to break the supersymmetry from $\mathcal{N} = (4, 4)$ to $\mathcal{N} = (2, 2)$ in the 5D $\mathcal{N} = 2$ gauged supergravity. After that, the 5D gravity solution can also be uplifted to 10D and rewritten into an expression, from which the brane construction is more transparent.

3.1. Gravity Dual of 2D $\mathcal{N} = (4, 4)$ Super Yang-Mills Theory

As explained in the beginning of this section, to construct the gravity of the 2D $\mathcal{N} = (2, 2)^*$ $U(N)$ super Yang-Mills theory, we start with the known gravity dual of the 2D $\mathcal{N} = (4, 4)$ $U(N)$ super Yang-Mills theory, which was constructed in Ref. [10]. Let us briefly review the construction in this subsection.

To realize the $\mathcal{N} = (4, 4)$ supersymmetry, one considers a D3-brane wrapped on the 2-cycle of a CY 2-fold, which can be seen from the following table:

	$\mathbb{R}^{1,1}$		S^2		N_2		$\mathbb{R}^4$			
D3	−	−	○	○	·	·	·	·	·	·

Locally, the CY 2-fold is $S^2 \times N_2$. From the brane construction, one can propose an Ansatz of the metric in 10 type IIB supergravity:

$$\begin{aligned} ds^2 = & H^{-\frac{1}{2}} \left[dx_{1,1}^2 + \frac{z}{m^2} \left(d\theta^2 + \sin^2\theta \, (d\phi)^2 \right) \right] \\ & + H^{\frac{1}{2}} \left[\frac{1}{z} d\sigma^2 + \frac{\sigma^2}{z} \left(d\psi + \cos\theta \, d\phi \right)^2 + d\rho^2 + \rho^2 d\Omega_3^2 \right] , \end{aligned} \tag{20}$$

where

$$0 \leq \theta \leq \pi \, , \quad 0 \leq \phi, \, \psi < 2\pi \, , \quad 0 \leq \rho, \, \sigma < \infty \, . \tag{21}$$

In addition, the RR 5-form in the 10D type IIB supergravity is given by

$$F_5 = \mathcal{F}_5 + * \mathcal{F}_5 \, , \tag{22}$$

where $\mathcal{F}_5 = d\mathcal{C}_4$ with

$$\mathcal{C}_4 = g(\rho, \, \sigma) \, \omega_3 \wedge (d\psi + \cos\theta \, d\phi) \tag{23}$$

and ω_3 is the volume element of the 3-sphere, i.e.,

$$\omega_3 = \sin^2\alpha_1 \, \sin\alpha_2 \, d\alpha_1 \wedge d\alpha_2 \wedge \alpha_3 \, . \tag{24}$$

The constant m in the metric (20) is fixed by the quantization condition of the RR 5-form F_5:

$$\frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_5} F_5 = N_c \, T_3 \tag{25}$$

with

$$2\kappa_{10}^2 = (2\pi)^7 \, g_s^2 \, (\alpha')^4 \, , \quad T_3 = \frac{1}{(2\pi)^3 \, g_s \, (\alpha')^2} \, . \tag{26}$$

After some analyses, one finds that the constant m is fixed by

$$\frac{1}{m^2} = \sqrt{4\pi g_s\, N_c \alpha'}\,, \tag{27}$$

where g_s and α' are the string coupling constant and the Regge slope respectively.

From the metric (20) and the flux (22), one can write down the BPS equations and try to solve them. It turns out that the BPS equations can be solved by using the results from the 5D $\mathcal{N}=2$ gauged supergravity discussed in Ref. [11]. This is due to the fact that the metric (20) can also be constructed from the 5D $\mathcal{N}=2$ gauged supergravity.

The bosonic part of the 5D $\mathcal{N}=2$ gauged supergravity is given by

$$\mathcal{L} = R - \frac{1}{2}(\partial_\mu\phi_1)^2 - \frac{1}{2}(\partial_\mu\phi_2)^2 + 4\sum_{i=1}^{3} e^{\alpha_i} - \frac{1}{4}\sum_{i=1}^{3} e^{2\alpha_i}\, F^i_{\mu\nu}F^{i,\mu\nu} + \frac{1}{4}\epsilon^{\mu\nu\alpha\beta\rho}F^1_{\mu\nu}F^2_{\alpha\beta}A^3_\rho\,, \tag{28}$$

where

$$\alpha_1 = \frac{\phi_1}{\sqrt{6}} + \frac{\phi_2}{\sqrt{2}}\,, \quad \alpha_2 = \frac{\phi_1}{\sqrt{6}} - \frac{\phi_2}{\sqrt{2}}\,, \quad \alpha_3 = -\frac{2}{\sqrt{6}}\phi_1\,. \tag{29}$$

For the 5D gauged supergravity compactified on a surface H^2, there is the following condition:

$$a_1 + a_2 + a_3 = 1\,. \tag{30}$$

It was explained in Ref. [11] that one can obtain the solution for the compactification on S^2 by replacing $\theta \to i\theta$.

By choosing $a_I = (0,\, 0,\, 1)$, it was constructed in Ref. [10] the gravity solution with $\mathcal{N}=(4,4)$ supersymmetry in the 5D $\mathcal{N}=2$ gauged supergravity:

$$ds_5^2 = e^{2f(r)}\left(dx_{1,1}^2 + dr^2\right) + \frac{e^{2g(r)}}{m^2}\left[d\theta^2 + \sin^2\theta\,(d\phi)^2\right]\,. \tag{31}$$

The gauge fields are chosen to be

$$A^1 = 0\,, \quad A^2 = 0\,, \quad A^3 = \frac{1}{m}\cos\theta\, d\phi\,. \tag{32}$$

This solution can be uplifted to 10D in the following way:

$$ds_{10}^2 = \sqrt{\Delta} ds_5^2 + \frac{3}{m^2\sqrt{\Delta}}\sum_{i=1}^{3} X_i\left[d\mu_i^2 + \mu_i^2\left(d\phi^i + mA^i\right)^2\right]\,, \tag{33}$$

where

$$\Delta = \sum_{i=1}^{3} X^i \mu_i^2\,, \tag{34}$$

and

$$\sum_{i=1}^{3}\mu_i^2 = 1\,. \tag{35}$$

One can parametrize μ_i's as follows:

$$\mu_1 = \cos\widetilde{\theta}\sin\widetilde{\psi}\,, \quad \mu_2 = \cos\widetilde{\theta}\cos\widetilde{\psi}\,, \quad \mu_3 = \sin\widetilde{\theta}\,, \tag{36}$$

where $0 \le \widetilde{\theta} \le \pi/2$ and $0 \le \widetilde{\psi} \le 2\pi$. The quantities X_i and X^i are defined by

$$X_i = \frac{1}{3}\left(e^{\varphi},\, e^{\varphi},\, e^{-2\varphi}\right)\,, \quad X^i = \left(e^{-\varphi},\, e^{-\varphi},\, e^{2\varphi}\right)\,, \tag{37}$$

It was shown in Ref. [10] that indeed the metric (33) can be rewritten into the expression of the metric (20) discussed before by changing variables.

3.2. Gravity Dual of 2D $\mathcal{N}=(2,2)^*$ Super Yang-Mills Theory

Let us consider the gravity dual of the 2D $\mathcal{N}=(2,2)^*$ $U(N)$ super Yang-Mills theory. As mentioned in the beginning of this section, we can start with the the gravity dual of the 2D $\mathcal{N}=(4,4)$ $U(N)$ super Yang-Mills theory discussed in the previous section, and break the supersymmetry from $\mathcal{N}=(4,4)$ to $\mathcal{N}=(2,2)$ in the framework of the 5D $\mathcal{N}=2$ gauged supergravity [11]. To break supercharges in the 5D $\mathcal{N}=2$ gauged supergravity, one can simply choose the parameters a_I to be

$$a_I = (\widetilde{c},\, 0,\, 1-\widetilde{c}) \tag{38}$$

with $0 \leq \widetilde{c} < 1$. When $\widetilde{c}=0$, it returns to the case analyzed in the previous subsection, which preserves $\mathcal{N}=(4,4)$ supersymmetry. When $0<\widetilde{c}<1$, the gravity solution preserves $\mathcal{N}=(2,2)$ supersymmetry. The gauge fields now become

$$A^1 = \frac{\widetilde{c}}{m}\cos\theta\, d\phi\,, \quad A^2 = 0\,, \quad A^3 = \frac{1-\widetilde{c}}{m}\cos\theta\, d\phi\,. \tag{39}$$

Similar to Ref. [11], one can also solve the BPS equations near the boundary $r=0$ (see Ref. [1] for more details). The asymptotic solutions are

$$g(r) = -\log(r) + \frac{7}{36}r^2 + \cdots\,, \tag{40}$$

$$f(r) = -\log(r) - \frac{1}{18}r^2 + \cdots\,, \tag{41}$$

$$\phi_1(r) = -\frac{1-3a_3}{\sqrt{6}}r^2\log(r) + \cdots\,, \tag{42}$$

$$\phi_2(r) = -\frac{a_1-a_2}{\sqrt{2}}r^2\log(r) + \cdots\,, \tag{43}$$

which suggest that the scalar ϕ_2 corresponds to the dimension-2 operator $c\,\mathrm{Tr}(\Phi\wedge *\Phi)$ in the field theory. In the absense of this mass deformation, the field theory has $\mathcal{N}=(4,4)$ supersymmetry, and the scalar ϕ_2 is turned off on the gravity side. When the mass deformation is turned on, the scalar ϕ_2 should also be turned on correspondingly. Hence, at least for small values of c and $\widetilde{c}$, there should be

$$c \propto \widetilde{c}\,. \tag{44}$$

Based on our previous analyses, the 10D metric preserving $\mathcal{N}=(2,2)^*$ supersymmetry can be given in the following way:

$$\begin{aligned} ds_{10}^2 = \sqrt{\Delta}&\left[e^{2f}(dx_{1,1}^2+dr^2) + \frac{e^{2g}}{m^2}\left(d\theta^2+\sin^2\theta\,(d\phi)^2\right)\right] \\ &+\frac{1}{m^2\sqrt{\Delta}}\Bigg[e^{\varphi_1+\varphi_2}d\mu_1^2 + e^{\varphi_1-\varphi_2}d\mu_2^2 + e^{-\varphi_2}d\mu_3^2 + e^{\varphi_1+\varphi_2}\cos^2\widetilde{\theta}\sin^2\widetilde{\psi}\left(d\phi^1+\widetilde{c}\cos\theta\, d\phi\right)^2 \\ &\qquad + e^{\varphi_1-\varphi_2}\cos^2\widetilde{\theta}\cos^2\widetilde{\psi}(d\phi^2)^2 + e^{-2\varphi_1}\sin^2\widetilde{\theta}\left(d\phi^3+(1-\widetilde{c})\cos\theta\, d\phi\right)^2\Bigg]\,, \end{aligned} \tag{45}$$

where $\varphi_1=\phi_1/\sqrt{6}$, $\varphi_2=\phi_2/\sqrt{2}$, and Δ has the same definition as Eq. (34).

In principle, this 10D metric (45) can be rewritten into the form similar to the one given by Eq. (20), from which the brane construction is more transparent. However, for generic values of $\widetilde{c}$, the explicit form of the metric can be very complicated. To simplify our discussions, let us consider a special case $\widetilde{c} = 1/2$. For this case, the metric (45) can be rewritten into a form similar to Eq. (20) by changing variables (for more details see Ref. [1]):

$$\begin{aligned} ds^2 = & H^{-\frac{1}{2}} \left[dx_{1,1}^2 + \frac{z}{m^2} \left(d\theta^2 + \sin^2\theta\, (d\phi)^2 \right) \right] \\ & + H^{\frac{1}{2}} \left[\frac{1}{\sqrt{z}} d\sigma^2 + \frac{\sigma^2}{\sqrt{z}} \left(d\widetilde{\psi}^2 + \sin^2\widetilde{\psi} \left(d\phi^3 + \frac{1}{2}\cos\theta\, d\phi \right)^2 + \cos^2\widetilde{\psi} \left(d\phi^1 + \frac{1}{2}\cos\theta\, d\phi \right)^2 \right) \right. \\ & \left. \qquad + d\rho^2 + \rho^2 d\psi^2 \right] . \end{aligned} \tag{46}$$

From this metric one can read off that the brane construction for this case ($\widetilde{c} = 1/2$) becomes

	$\mathbb{R}^{1,1}$		S^2		N_4				$\mathbb{R}^2$	
D3	$-$	$-$	$\bigcirc$	$\bigcirc$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Hence, now the solutions describes a D3-brane wrapped on the 2-cycle of a CY 3-fold, which is locally $S^2 \times N_4$. This configuration realizes $\mathcal{N} = (2,2)$ supersymmetry. By turning on the mass deformation on the field theory side, we see a change of topology on the gravity side, i.e., from a CY 2-fold at $\widetilde{c} = 0$ becomes a CY 3-fold at $\widetilde{c} \neq 0$, and consequently the supersymmetry is broken from $\mathcal{N} = (4,4)$ to $\mathcal{N} = (2,2)$.

To have the complete gravity solution in 10D type IIB supergravity, we still need to specify the RR 5-form F_5. It is given by

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5 \,, \tag{47}$$

where

$$\mathcal{F}_5 = \sum_{i=1}^{3} \left[2X^i (X^i \mu_i^2 - \Delta) \epsilon_5 + \frac{1}{2(X^i)^2} d(\mu_i^2) \left((d\phi_i + A^i) \wedge *_5 F^i + X^i *_5 dX^i \right) \right] , \tag{48}$$

and ϵ_5 and $*_5$ are the volume form of ds_5^2 and the Hodge dual in ds_5 respectively, while $F^i = dA^i$ are the field strengths of the gauge fields given by Eq. (39). Similar to the $\mathcal{N} = (4,4)$ case, the quantization condition of F_5 will fix the constant m as in Eq. (27).

There is a subtle point that we would like to emphasize. In the original construction in Ref. [11], the gravity solution has a 2-form field strength, which should satisfy a quantization condition when integrated on the Riemann surface. This condition imposes the constraint on the parameters a_I's that they should have rational instead of real values. However, in our discussion above, we only have a flux quantization condition on the 5-form F_5, which fixes the constant m, hence, the a_I's can be real in our solutions.

3.3. The Triangle Relation

After finding the gravity dual, we have set up a triangle relation among the N-particle sector of quantum nonlinear Schrödinger equation, the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory and their gravity dual. This relation should hold at quantum level in the limit $N \to \infty$, $g_{YM}^2 \to 0$ and $g_{YM}^2 N \to \infty$. As a first check, let us take the classical limit $\hbar \to 0$.

It was known in the literature that when $N \to \infty$ the N-particle sector of the quantum nonlinear Schrödinger equation with an attractive potential becomes the N-soliton sector of the classical nonlinear Schrödinger equation. Therefore, in the classical limit we can identify the N-solitons in the nonlinear Schrödinger equation with the N D3-branes from the brane construction on the gravity side, while N becomes the rank of the gauge group on the gauge theory side.

4. Discussions

In this paper, we illustrate the way of setting up a triangle relation between three theories by finding the gravity dual of the 2D $\mathcal{N} = (2,2)^*$ $U(N)$ topological Yang-Mills-Higgs theory. More details of this approach can be found in Ref. [1]. We believe this result opens up a new direction of research, and a lot of interesting questions should be addressed and clarified. Hopefully, it can provide some deep insights into the connections of integrable models, gauge theories and gravities.

An immediate generalization is to add matters in the fundamental representation of the gauge group, i.e., add flavors in the gravity dual. This will allow us to find the gravity dual of the more general duality found by Nekrasov and Shatashvili, and consequently to study the more general triangle relation. Another related question is to study the integrability on the gravity side both at the classical level and at the quantum level. We would like to investigate these perspectives in the future research.

A little unexpected relation comes from the boson/vortex duality discussed in Refs. [12, 13] and recently revisited in Refs. [14, 15]. Using this duality, one can show that in (3+1)D nonlinear Schrödinger equatoin can be mapped into an effective string theory. Correspondingly, different solutions to the (3+1)D nonlinear Schrödinger equatoin (vortex lines, vortex rings, dark solitons) can be mapped into some configurations in the effective string theory (open string, closed strings, D-branes). To relate this duality to the (1+1)D nonlinear Schrödinger equatoin, which is an integrable model, one can either perform a dimensional reduction or apply the duality map directly in (1+1)D. In this way a lot of interesting features emerge, and many apparently different theories are related in a larger duality web. We would like to present this work elsewhere.

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